EXERCISES FOR METAMATH I

1. Exercise

An DFA $\mathscr{M}$ over a finite alphabet Σ is said to be *minimal* if there is no DFA $\mathscr{N}$ with $L(\mathscr{N}) = L(\mathscr{M})$ having fewer states than $\mathscr{M}$. Given a regular language L, what is the connection between the number of states in a minimal automaton accepting L and the number of cone types of L?

2. Exercise

Let L be a language accepted by a DFA $\mathscr{M}$ having n states. Show that for any $x \in \Sigma^*$, if $\mathrm{cone}_L(x) \neq \emptyset$, then there is some $y \in \Sigma^*$ with $|y| < n$ such that $xy \in L$.

3. Exercise

Suppose $L, R \subseteq \Sigma^*$ are languages. We define the *quotient* of R by L as follows

$$R/L = \{x \in \Sigma^* \mid \exists y \in L \;\; xy \in R\}.$$

(a) Show that if R is regular, then so is R/L.

(b) Show that if R is regular, then so is

$$\mathrm{prefix}(R) = \{x \in \Sigma^* \mid \exists y \;\; xy \in R\}.$$

4. Exercise

Suppose Σ and Λ are finite alphabets. A *homomorphism* from Σ^* to Λ^* is simply a homomorphism of monoids, i.e., $\pi\colon \Sigma^* \to \Lambda^*$ is a homomorphism if $\pi(\epsilon) = \epsilon$ and $\pi(xy) = \pi(x)\pi(y)$. Note that any homomorphism is completely determined by its images of the letters of Σ.

(a) Show that if $\pi\colon \Sigma^* \to \Lambda^*$ is a homomorphism and $L \subseteq \Sigma^*$ is regular, then so is $\pi(L) \subseteq \Lambda^*$.

(b) Show that if $\pi\colon \Sigma^* \to \Lambda^*$ is a homomorphism and $L \subseteq \Lambda^*$ is regular, then so is $\pi^{-1}(L) \subseteq \Sigma^*$.

5. Exercise

Find a language L over the alphabet $\{0,1\}$ such that for any distinct $x, y \in \{0,1\}^*$, $\mathrm{cone}_L(x) \neq \mathrm{cone}_L(y)$.