EXERCISES FOR METAMATH I

1. Exercise

An DFA $\mathcal{M}$ over a finite alphabet $\Sigma$ is said to be minimal if there is no DFA $\mathcal{N}$ with $L(\mathcal{N}) = L(\mathcal{M})$ having fewer states than $\mathcal{M}$. Given a regular language $L$, what is the connection between the number of states in a minimal automaton accepting $L$ and the number of cone types of $L$?

2. Exercise

Let $L$ be a language accepted by a DFA $\mathcal{M}$ having $n$ states. Show that for any $x \in \Sigma^*$, if cone$_L(x) \neq \emptyset$, then there is some $y \in \Sigma^*$ with $|y| < n$ such that $xy \in L$.

3. Exercise

Suppose $L, R \subseteq \Sigma^*$ are languages. We define the quotient of $R$ by $L$ as follows

$$R/L = \{x \in \Sigma^* \mid \exists y \in L \ xy \in R\}.$$ 

(a) Show that if $R$ is regular, then so is $R/L$.

(b) Show that if $R$ is regular, then so is

$$\text{prefix}(R) = \{x \in \Sigma^* \mid \exists y \xy \in R\}.$$ 

4. Exercise

Suppose $\Sigma$ and $\Lambda$ are finite alphabets. A homomorphism from $\Sigma^*$ to $\Lambda^*$ is simply a homomorphism of monoids, i.e., $\pi: \Sigma^* \rightarrow \Lambda^*$ is a homomorphism if $\pi(\epsilon) = \epsilon$ and $\pi(xy) = \pi(x)\pi(y)$. Note that any homomorphism is completely determined by its images of the letters of $\Sigma$.

(a) Show that if $\pi: \Sigma^* \rightarrow \Lambda^*$ is a homomorphism and $L \subseteq \Sigma^*$ is regular, then so is $\pi(L) \subseteq \Lambda^*$.

(b) Show that if $\pi: \Sigma^* \rightarrow \Lambda^*$ is a homomorphism and $L \subseteq \Lambda^*$ is regular, then so is $\pi^{-1}(L) \subseteq \Sigma^*$.

5. Exercise

Find a language $L$ over the alphabet $\{0, 1\}$ such that for any distinct $x, y \in \{0, 1\}^*$, cone$_L(x) \neq \text{cone}_L(y)$. 