EXERCISES FOR METAMATH II: THIRD SET

1. Exercise

Let $\Sigma_k = \{0, \dots, k-1\}$. Suppose $L \subseteq \Sigma_k^*$ is a regular language and define

$$R = \{x \in L \mid x \text{ is the lexicographically largest element of } L \cap \Sigma_k^{|x|}\}.$$

Show that R is regular.

2. Exercise

Let L be the set of all non-square natural numbers written in base 10, i.e.,

$$L = \{2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, 18, \dots\}.$$

Find the (finite) set of minimal strings in L with respect to the subsequence ordering $\preccurlyeq$. I.e., find the smallest finite subset $S \subseteq L$ such that for any $x \in L$ there is some $y \in S$ with $y \preccurlyeq x$.

3. Exercise

Show that the following are both presentations of the trivial group.

- $\langle x, y \mid xyx^{-1} = y^2,\ yxy^{-1} = x^2 \rangle$,
- $\langle x, y, z \mid xyx^{-1} = y^2,\ yzy^{-1} = z^2,\ zxz^{-1} = x^2 \rangle$.

4. Exercise

Suppose Σ is a finite alphabet and $L \subseteq \Sigma^*$ is a regular language. For $x \in \Sigma^*$, define

$$\mathrm{tree}_L(x) = \{y \in \Sigma^* \mid \mathrm{cone}_L(xy) \neq \emptyset\}.$$

So $\mathrm{tree}_L(x)$ is a (possibly empty) tree on Σ^*, i.e., a set of strings closed under taking prefixes. Let $[\mathrm{tree}_L(x)]$ denote the set of all *infinite branches* of $\mathrm{tree}_L(x)$, i.e., the set of $\alpha \in \Sigma^\omega$ all of whose finite prefixes belong to $\mathrm{tree}_L(x)$. We say that $\mathrm{tree}_L(x)$ is *well-founded* in case $[\mathrm{tree}_L(x)] = \emptyset$.

- Show that there is a constant K only depending on L such that for any $x \in \Sigma^*$, if $\mathrm{tree}_L(x)$ is well-founded, then

$$|\mathrm{tree}_L(x)| \leqslant K.$$

5. Exercise

Find a characterisation of all regular languages $L \subseteq \Sigma^*$ with the following property: For some K dependent on L and for all n,

$$|L \cap \Sigma^n| \leqslant K.$$

6. Exercise

Find the $\sim_L$ equivalence classes of the following languages.

- $L = \{w \in \{0,1\}^* \mid |w|_0 = |w|_1\}$,
- $L = \{a^n b^n c^n \mid n \geqslant 1\}$.

7. Exercise

For every $k \geqslant 1$, show how to construct a regular language L_k over a finite alphabet Σ_k such that for all n,

$$|L_k \cap (\Sigma_k)^n| = n^k.$$

8. Exercise

Let $\Sigma = \{1, 2, \dots, n\}$. A string $w = a_1 a_2 \dots a_k$ is said to be *bipartite* if there is a partition $A \sqcup B = \{1, \dots, k\}$ such that

$$\sum_{i \in A} a_i = \sum_{i \in B} a_i.$$

(a) Show that there is a constant K depending only on n such that any bipartite string w has a bipartite substring $v \preccurlyeq w$ with $|v| \leqslant K$.

(b) Show that the language of bipartite strings is regular.