1. Exercise

Let $\Sigma_k = \{0, \ldots, k - 1\}$. Suppose $L \subseteq \Sigma_k^*$ is a regular language and define

$$R = \{x \in L \mid x \text{ is the lexicographically largest element of } L \cap \Sigma_k^{\lfloor |x| \rceil}\}.$$ 

Show that $R$ is regular.

2. Exercise

Let $L$ be the set of all non-square natural numbers written in base 10, i.e.,

$$L = \{2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, 18, \ldots\}.$$ 

Find the (finite) set of minimal strings in $L$ with respect to the subsequence ordering $\prec$. I.e., find the smallest finite subset $S \subseteq L$ such that for any $x \in L$ there is some $y \in S$ with $y \prec x$.

3. Exercise

Show that the following are both presentations of the trivial group.

- $\langle x, y \mid xyx^{-1} = y^2, yxy^{-1} = x^2 \rangle$,
- $\langle x, y, z \mid xyx^{-1} = y^2, yzy^{-1} = z^2, zxz^{-1} = x^2 \rangle$.

4. Exercise

Suppose $\Sigma$ is a finite alphabet and $L \subseteq \Sigma^*$ is a regular language. For $x \in \Sigma^*$, define

$$\text{tree}_L(x) = \{y \in \Sigma^* \mid \text{cone}_L(xy) \neq \emptyset\}.$$ 

So $\text{tree}_L(x)$ is a (possibly empty) tree on $\Sigma^*$, i.e., a set of strings closed under taking prefixes. Let $[\text{tree}_L(x)]$ denote the set of all infinite branches of $\text{tree}_L(x)$, i.e., the set of $\alpha \in \Sigma^\omega$ all of whose finite prefixes belong to $\text{tree}_L(x)$. We say that $\text{tree}_L(x)$ is well-founded in case $[\text{tree}_L(x)] = \emptyset$.

- Show that there is a constant $K$ only depending on $L$ such that for any $x \in \Sigma^*$, if $\text{tree}_L(x)$ is well-founded, then

$$|\text{tree}_L(x)| \leq K.$$ 

1
5. Exercise

Find a characterisation of all regular languages \( L \subseteq \Sigma^* \) with the following property: For some \( K \) dependent on \( L \) and for all \( n \),
\[ |L \cap \Sigma^n| \leq K. \]

6. Exercise

Find the \( \sim_L \) equivalence classes of the following languages.
- \( L = \{ w \in \{0, 1\}^* \mid |w|_0 = |w|_1 \} \),
- \( L = \{a^n b^n c^n \mid n \geq 1 \} \).

7. Exercise

For every \( k \geq 1 \), show how to construct a regular language \( L_k \) over a finite alphabet \( \Sigma_k \) such that for all \( n \),
\[ |L_k \cap (\Sigma_k)^n| = n^k. \]

8. Exercise

Let \( \Sigma = \{1, 2, \ldots, n\} \). A string \( w = a_1 a_2 \ldots a_k \) is said to be bipartite if there is a partition \( A \sqcup B = \{1, \ldots, k\} \) such that
\[ \sum_{i \in A} a_i = \sum_{i \in B} a_i. \]

(a) Show that there is a constant \( K \) depending only on \( n \) such that any bipartite string \( w \) has a bipartite substring \( v \preceq w \) with \( |v| \leq K \).

(b) Show that the language of bipartite strings is regular.