Exercises on Büchi automata:

Exercise 1. Let $\mathcal{A} = (S, I, T, F)$ be a Büchi automaton defined by $S = \{s_0, s_1, s_2\}$, $I = \{s_0\}$, $\Sigma = \{a, b\}$, $T = \{(s_0, a, s_0), (s_0, b, s_1), (s_1, b, s_1), (s_1, a, s_2), (s_2, a, s_2)\}$, $F = \{s_2\}$.

(a) How many runs does $\mathcal{A}$ have on the input $aaabbbbaaaaa\ldots$?

(b) Find an input on which $\mathcal{A}$ has no runs.

(c) Is there an input on which $\mathcal{A}$ has infinitely many runs?

(d) Find $L(\mathcal{A})$.

Exercise 2. Let $\mathcal{A} = (S, I, T, F)$ be a Büchi automaton on the alphabet $\Sigma = \{a, b\}$ defined by $S = \{s_0, s_1, s_2\}$, $I = \{s_0\}$, $F = \{s_2\}$, $T = \{(s_0, a, s_0), (s_0, b, s_1), (s_1, a, s_2), (s_1, b, s_1), (s_2, a, s_0)\}$.

(a) Draw a graph representing $\mathcal{A}$.

(b) Find $L(\mathcal{A})$. 

Exercise 3 \[ Z = \{ a, b \} \]. Construct Büchi automata recognizing the following languages:

(a) \( \{ x \in \Sigma^* \mid a \text{ occurs exactly once in } x \} \)

(b) \( \{ x \in \Sigma^* \mid x \text{ contains no substring } aa \} \)

(c) \( \{ x \in \Sigma^* \mid x \text{ contains infinitely many } b \text{'s} \} \)

(d) \( \{ x \in \Sigma^* \mid \text{after each } a \text{ occurring in } x \)

\[ \text{there is either an even number of } \]

\[ \text{consecutively occurring } b \text{'s or an infinite number} \]

\[ \{ x \in \Sigma^* \mid x = (ab)^n b^n, \ n \geq 1, \ b \in \Sigma^* \} \]

(e) \( \{ x \in \Sigma^* \mid a \text{ occurs infinitely often in even positions} \} \).
Exercise 7

Let \( V, W \subseteq \Sigma^* \) and fix \( m > 1 \).

Let \( L \subseteq \Sigma^* \) be the language consisting of all \( \alpha = x_1x_2x_3\ldots \)

where \( x_i \in V \cup W \) and

(i) if \( x_i \in W \setminus V \), then we do not have

\[ x_i = x_{i+1} = \ldots = x_{i+m} \]

(ii) if \( x_i \in V \setminus W \), then we do not have

\[ x_i = x_{i+1} = \ldots = x_{i+m} \].

Show that if \( V, W \) are regular, then \( L \) is

\textit{both recognizable.}