Example: Let $\Sigma = \{e, a, a^{-1}, b, b^{-1}\}$ and let $L = \text{all reduced words in } \Sigma$.

Then it is easy to see that $L$ has finitely many coset types, so $L$ is regular.

The word problem:

Let $G$ be a group and $\Sigma = \Sigma^{-1} \subseteq G$ a symmetric generating set. We define $\pi: \Sigma^* \to G$ be the canonical semigroup homomorphism, where the operation on $\Sigma^*$ is concatenation.

Assume $\Sigma$ is finite.

We let $WP(G, \Sigma) = \{ \omega \in \Sigma^* | \pi(\omega) = e \}$

Theorem: If $G$ is a finitely generated group, $\Sigma$ a finite, symmetric, generating set.

Then $L = WP(G, \Sigma)$ is regular if and only if $G$ is finite.

Proof: Note that if $\omega, \sigma \in \Sigma^*$, then:

$\text{coset}_L(\omega) = \text{coset}_L(\sigma) \iff \pi(\omega) = \pi(\sigma)$. $\Box$
Normal Forms

Suppose \( \Sigma \) is a finite, symmetric, generating set for a group \( G \). Let \( \pi : \Sigma^* \to G \) be the canonical subgroup homomorphism.

**Definition.** A normal form \( L \subseteq (G, \Sigma) \) is a language \( L \subseteq \Sigma^* \) such that \( \pi : L \to G \) is bijective.

**Examples.**
- The freely reduced words are a regular normal form for \( \Sigma_2 \) (since there are only finitely many such words).
- Let \( \Sigma^2 \) be the free abelian group on generators \( a \) and \( b \). Then \( L = \{ a^i b^j : i, j \in \Sigma^2 \} \) is a regular normal form.

**Proposition.** Suppose \( \Sigma \) and \( \Lambda \) are finite, symmetric, generating sets for a group \( G \). Then \( G \) has a regular normal form \( \Sigma \) if and only if it has a regular normal form \( \Lambda \).

**Theorem.** Suppose \( L \subseteq \Sigma^* \) is a regular normal form.

For every \( s \in \Sigma \) let \( w_s \in \Lambda^* \) be a rewriting of \( s \) in base \( \Lambda \). So if \( \mathcal{H} \) is an FSA accepting \( L \), we can build a generalized FSA by replacing labels \( s \) on \( \mathcal{H} \) by new labels \( w_s \).

The language \( L' \subseteq \Lambda^* \) accepted is a regular normal form \( L' \in G \) w.r.t \( \Lambda \).
Example: \(D_{10} = \mathbb{Z}_2 \ast \mathbb{Z}_2\) is generated by two reflections \(a\) and \(b\), but it is also generated by a reflection \(a\) and a translation \(T = ab\).
Since \(a = s^{-1}\), \(b = t^{-1}\), \(S = \{s, t\}\), \(\Lambda = \{s, t, T\}\) are symmetric generating sets.

Also, \(L = \frac{1}{2} (ab)^n, (ab)^n a, b(ab)^n \) for \(n < 0\) is a regular normal form:

\[
\begin{array}{c}
\xrightarrow{a} \circ \xrightarrow{b} \circ \\
\xrightarrow{b} \circ \xrightarrow{a} \circ
\end{array}
\]

Now \(b = at\), so \(L = \frac{1}{2} (aat)^n, (aat)^n a, at (aat)^n \) for \(n < 0\) is a regular normal form:

\[
\begin{array}{c}
\xrightarrow{a} \circ \xrightarrow{r} \circ \\
\xrightarrow{a} \circ \xrightarrow{r} \circ
\end{array}
\]

Proposition: If \(G, H\) have regular normal forms,
then so do \(G \ast H\) and \(G \ast H\). In particular,
if \(G, H\) are finite, then \(G \ast H\) has a regular
normal form.

Proof: If \(L, K\) are regular normal forms for \(G\) and
\(H\) respectively, then \(L, K\) and
\((L, K)^* \cup (K, L)^* \cup K(LK)^*\) are regular.
normal forms for \( G \oplus H \) and \( G \rtimes H \) respectively.

(\text{Thm \textsc{R. Gilman}})

Suppose \( G \) is an infinite group with a regular normal form. Then \( G \) has an element of infinite order.

Let \( \Sigma \) be the finite symmetric generating set and \( L \subseteq \Sigma^* \) the regular normal form. Then by the pumping lemma, which applies since \( L \) is infinite, there is some \( w \beta \in L \) s.t. \( w \neq \varepsilon \) and \( \omega w^n \beta \in L \) for all \( n \geq 0 \). Since \( L \) is a normal form, \( \pi(\omega w^n \beta) \neq \pi(\omega w^m \beta) \) for \( n \neq m \), whence

\[
\pi(L)^n = \pi(\pi^{-1}(\pi(\omega w^n \beta) \pi(\pi^{-1}(\pi(\beta))) = \pi(\pi(\omega)^m)
\]

for all \( m \geq 0 \).

---

Finitely generated subgroups of free groups

Suppose \( G \) is a group with a finite, symmetric, generating set \( \Sigma \subseteq G \). A subgroup \( H \subseteq G \) is said to be an image of a regular language over \( \Sigma \) if there is a regular language \( L \subseteq \Sigma^* \) such that \( \pi(L) = H \).

[\boxed{\text{\qed}}]
Theorem: Suppose \( G \) is a group with a finite symmetric generating set \( \Sigma \subseteq G \).

Then, a subgroup \( H \leq G \) is the image of a regular language \( L \) over \( \Sigma \) if and only if \( H \) is finitely generated.

Proof: If \( H \) is finitely generated, let \( S \) be a generating set for \( H \), \( S = S^{-1} \). Also, for every \( s \in S \), let \( w_s \in \Sigma^* \) be a word with \( \pi(w_s) = s \).

Then \( L = \{ w_s \mid s \in S \} \subseteq \Sigma^* \) is regular and \( \pi(L) = H \).

Conversely, suppose \( L \) is a regular language in \( \Sigma^* \) with \( \pi(L) = H \). Pick a finite selector \( T_0 \subseteq \Sigma^* \) of all non-empty cone types of \( L \), i.e., for every \( w \in \Sigma^* \) with \( \text{cone}(w) \neq \emptyset \), there is \( \sigma \in T_0 \) with \( \text{cone}(\sigma) = \text{cone}(w) \). Let \( n = \max \{ |w| : w \in T_0 \} \) and set

\[
S = \pi\left( L \cap \Sigma^{\leq n} \right) \cup \left\{ \pi(w) \pi(\sigma)^{-1} \mid \sigma \in \Sigma^{n+1}, w \in T_0 \right\}.
\]

First, to see that \( S \subseteq H \), note that if \( \sigma \in \Sigma^*, w \in T_0 \) and \( \text{cone}(w) = \text{cone}(\sigma) \), then, in particular, \( \text{cone}(w) \neq \emptyset \). So for some \( \beta \in \Sigma^* \),
\[ \omega, \sigma \in L, \text{ i.e., } \pi(\omega) \pi(\sigma) \pi(\omega') \in H, \]
whence
\[ \pi(\omega) \pi(\sigma)^{-1} = \pi(\omega) \pi(\sigma) \pi(\sigma)^{-1} \in H. \]
Now, we claim that if \( \omega \in H, 1|\omega| > n \), then there
is \( \beta \in L, |\beta| < |\omega| \), such that
\[ \pi(\alpha) \in S^{-1} \pi(\beta). \]
To see this, let \( \alpha = \sigma \delta \), \( |\alpha| = n+1 \), and find
\( \omega \in H \). Since \( \omega \) is \( \text{cone} \) of \( \alpha \), then \( \pi(\omega) \pi(\sigma)^{-1} \in S \)
and \( \beta = \omega \delta \in L \), whence
\[ \pi(\omega) \pi(\sigma)^{-1} \pi(\alpha) = \pi(\omega) \pi(\sigma)^{-1} \pi(\sigma) \pi(\delta) = \pi(\omega \delta) = \pi(\beta). \]
It then follows that \( H = \left< S \right> \). □

Proposition. Suppose \( \Sigma \) is a finite symmetric subset
of a group \( G \) and \( L \) is a regular language
over \( \Sigma \). Then \( L R \) is the set of all words
obtained by freely reducing a word in \( L \).
\( R \) is regular.

Proof. Suppose \( L \) is the language accepted by some
finite automaton \( M \). Let \( M' \) be the automaton
obtained from \( M \) as follows: If there is a path
from state \( s_1 \) to state \( s_2 \), labeled \( \Sigma \), \( \Sigma' \),
add an edge from \( s_1 \) to \( s_2 \), labeled \( \Sigma \). Then
if \( S \) is the language accepted by \( M' \), a freely
reduced word in \( S \) belongs to \( S \) if and only
if it belongs to \( R \). So \( R \) is the intersection of
\( S \) with the regular language of freely reduced words. □
Howson's Theorem. The intersection $H \cap K$ of the finitely generated subgroups $H, K$ of $F_n$ is finitely generated.

Proof. Let $\Sigma = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$ be the basis for $F_n$. Since $H$ and $K$ are finitely generated, there are regular languages $L_H, L_K \subseteq \Sigma^*$ such that $H = \pi(L_H)$ and $K = \pi(L_K)$. Let $R_H, R_K$ be the regular languages obtained by freely reducing words in $L_H$ and $L_K$ respectively.

Then since $H, K \subseteq F_n$ and $\Sigma$ is the symmetric basis for $F_n$, $\pi : R_H \rightarrow H$ and $\pi : R_K \rightarrow K$ are bijections, whence $\pi : R_H \cap R_K \rightarrow H \cap K$ is a bijection too.

Since $R_H \cap R_K$ is regular, $H \cap K$ is finitely generated. \[ \square \]

Automata on pairs of strings

Suppose $\Sigma$ and $\Lambda$ are finite alphabets, and $\$ is a symbol not occurring in any of $\Sigma, \Lambda$. For $(\omega, \sigma) \in \Sigma^* \times \Lambda^*$, let
\[
(\omega, \sigma)^{\$} = \begin{cases} 
(\omega, \sigma) & \text{if } |\omega| = |\sigma| \\
(\omega, \sigma^{\$n}) & \text{if } |\omega| = |\sigma| + n \\
(\sigma^{\$n}, \sigma) & \text{if } |\omega| + n = |\sigma|.
\end{cases}
\]