Languages on pairs of strings.

Suppose Σ, Λ are finite alphabets and $\$$ a symbol in neither of Σ and Λ.

For $(x,y) \in \Sigma^* \times \Lambda^*$, let

$$(x,y)^{\$} = \begin{cases} (x,y) & \text{if } |x| = |y| \\ (x\$^n, y) & \text{if } |x|+n = |y| \\ (x, y\$^n) & \text{if } |x| = |y|+n . \end{cases}$$

Also, for $L \subseteq \Sigma^* \times \Lambda^*$, we let $L^{\$} = \{(x,y)^{\$} \mid (x,y) \in L\}$.

By interweaving sequences, we can see $L^{\$}$ as being a language on $(\Sigma \cup \{\$\}) \times (\Lambda \cup \{\$\})$.

<u>Definition</u> A language $L \subseteq \Sigma^* \times \Lambda^*$ is <u>regular</u> if $L^{\$}$ is a regular language on the alphabet $(\Sigma \cup \{\$\}) \times (\Lambda \cup \{\$\})$.

<u>Remark</u> So $L \subseteq \Sigma^* \times \Lambda^*$ is regular if and only if there is a finite automaton whose edge labels are of the form

$$(a,b), (a,\$), (\$,b), (\$,\$), \quad a \in \Sigma, b \in \Lambda,$$

recognizing $L^{\$}$. However, because of the special form of $L^{\$}$, one can suppose that not edge is labeled $(\$,\$)$ and, moreover, we can suppose that if a state has an incoming edge

labeled $(a, \$)$, $a \in \Sigma$, then all outgoing edges are of the form $(b, \$)$, $b \in \Sigma$. Similarly for $(\$, a)$.

<u>Definition</u> Suppose $L \subseteq \Sigma^* \times \Lambda^*$ is a language and $(x, y) \in \Sigma^* \times \Lambda^*$. We define

$$\text{leftcone}_L(x,y) = \{ z \in \Sigma^* \mid (xz, y) \in L \}$$

$$\text{rightcone}_L(x,y) = \{ z \in \Lambda^* \mid (x, yz) \in L \}$$

$$\text{cone}_L(x,y) = \{ (z,u) \in \Sigma^* \times \Lambda^* \mid (xz, yu) \in L \}.$$

<u>Theorem</u> TFAE for a language $L \subseteq \Sigma^* \times \Lambda^*$:

(i) L is regular

(ii) the sets

$$\text{Leftcone}(L) := \{ \text{leftcone}_L(x,y) \mid |x| > |y| \}$$

$$\text{Rightcone}(L) := \{ \text{rightcone}_L(x,y) \mid |x| < |y| \}$$

$$\text{Cone}(L) := \{ \text{cone}_L(x,y) \mid |x| = |y| \}$$

are all finite.

<u>Proof</u> We note that for $|x| = |y|$ and $|z| = |u|$,

$$(x,y) \sim_{L\$} (z,u)$$

$$\Updownarrow$$

$$\text{cone}_L(x,y) = \text{cone}_L(z,u)$$

And if $k, m \geq 1$ and $|x| + k = |y|$, $|z| + m = |u|$, then

$$(x\$^k, y) \sim_{L^\$} (z\$^m, u)$$

$$\Updownarrow$$

$$\forall w \in \Lambda^* \quad (x\$^{k+|w|}, yw) \in L^\$ \longleftrightarrow (z\$^{m+|w|}, uw) \in L^\$$$

$$\Updownarrow$$

$$\forall w \in \Lambda^* \quad (x, yw) \in L \longleftrightarrow (z, uw) \in L$$

$$\Updownarrow$$

$$\text{rightcone}_L(x,y) = \text{rightcone}_L(z,u).$$

Similarly for leftcone_L.

Thus, if L is regular, then $\sim_{L^\$}$ has only finitely many classes and so the three sets are finite.

Conversely, if the three sets are finite, then using the three above equivalences and the special nature of $L^\$$, we see that $\sim_{L^\$}$ has only finitely many classes and so L is regular. $\boxtimes$

Automatic groups

Given a finitely generated group G, a <u>rational structure</u> on G is a pair (L, Σ), where $\Sigma \subseteq G$ is a finite symmetric generating set and $L \subseteq \Sigma^*$ is a regular language such that $\pi : L \to G$ is surjective.

A rational structure (L, Σ) on G is said to be <u>automatic</u> if for every $a \in \Sigma \cup \{\varepsilon\}$, the language

$$L_a = \{ (x,y) \in \Sigma^* \times \Sigma^* \mid x, y \in L,\ \pi(xa) = \pi(y) \}$$

is regular.

<u>Note</u> Every finitely generated group admits a rational structure, but not necessarily an automatic structure.

<u>Definition</u> A finitely generated group is <u>automatic</u> if it admits an automatic structure.

<u>Example</u>: Finitely generated Abelian and non-Abelian groups are automatic.

<u>Theorem</u> If G is an automatic group, then G has a <u>solvable</u> <u>word problem</u>, i.e., there is a finite symmetric generating set $\Sigma \subseteq G$ and an algorithm that decides for any $w \in \Sigma^*$ whether or not $\pi(w) = 1$.

<u>Proof</u> Fix an automatic structure (L, Σ) on G and order Σ^* first by length and then lexicographically. Note that this defines an effective linear ordering $<$ of Σ in ordertype ω.

Pick some $x_0 \in L$ s.t. $\pi(x_0) = 1$.

Given $w \in \Sigma^*$, $w = a_1 \cdots a_n$, we define inductively $w_0, w_1, \ldots, w_n \in L$:

First let $w_0 = x_0$. Now, given $w_i \in L$, we let w_{i+1} be the $<$-minimal element of L such $(w_i, w_{i+1}) \in L_{a_{i+1}}$. Since both L and $L_{a_{i+1}}$ are effectively decided, the choice of w_{i+1} is effective in w_i and w.

So since $(w_i, w_{i+1}) \in L_{a_{i+1}}$ for all i,

$$\pi(w_n) = \pi(w_{n-1} a_n) = \pi(w_{n-1})\pi(a_n) = \pi(w_{n-2} a_{n-1})\pi(a_n)$$

$$= \cdots = \pi(w_0)\pi(a_1)\pi(a_2)\cdots\pi(a_n)$$

$$= 1 \cdot \pi(a_1 a_2 \cdots a_n) = \pi(w).$$

So to see if $\pi(w) = 1$, it suffices to check whether $(w_n, x_0) \in L_\varepsilon$. $\square$

<u>Definition</u> The <u>Cayley graph</u> $\Gamma(G, \Sigma)$ is the graph on G whose edges are $\{g, ga\}$ for $g \in G$ and $a \in \Sigma$.

We let d denote the distance in the Cayley graph, so

$$d(g,h) = \min\left(|w| \;\middle|\; w \in \Sigma^* \text{ \& } \pi(w) = g^{-1}h\right).$$

For $w \in \Sigma^*$, $w = a_1 a_2 \cdots a_n$ and $m \in \mathbb{N}$, we set

$$w|_m = \begin{cases} a_1 a_2 \cdots a_m & \text{for } m \le n \\ w & \text{otherwise} \end{cases}$$

<u>Definition</u> Suppose Σ is a finite symmetric generating set for G. We say that a language $L \subseteq \Sigma^*$ has the <u>fellow traveler's condition</u> if there is a constant K such that for all $a \in \Sigma \cup \{\varepsilon\}$, $(x,y) \in L_a$ and $m \in \mathbb{N}$,

$$d(\pi(x|_m), \pi(y|_m)) \le K.$$

<u>Theorem</u> Suppose (L, Σ) is a rational structure on a group G. TFAE

(i) (L, Σ) is an automatic structure on G

(ii) L has the fellow traveler's condition.

For the proof we will need the following:

<u>Lemma</u> Suppose $L \subseteq \Sigma^* \times \Lambda^*$ is a regular language. Then there is a constant K such that

(i) if $|x| = |y|$ and $\mathrm{cone}_L(x,y) \neq \emptyset$, then there is $(z,u) \in \mathrm{cone}_L(x,y)$ with $|z|, |u| \leq K$,

(ii) if $|x| > |y|$ and $\mathrm{leftcone}_L(x,y) \neq \emptyset$, then there is $z \in \mathrm{leftcone}_L(x,y)$ with $|z| \leq K$,

(iii) if $|x| < |y|$ and $\mathrm{rightcone}_L(x,y) \neq \emptyset$, then there is $u \in \mathrm{rightcone}_L(x,y)$ with $|u| \leq K$.

<u>Proof</u> Set

$$K = \#\{\mathrm{cone}_L(x,y) \mid |x| = |y|\}$$
$$+ \#\{\mathrm{leftcone}_L(x,y) \mid |x| > |y|\}$$
$$+ \#\{\mathrm{rightcone}_L(x,y) \mid |x| < |y|\}$$

Let us show that (i) holds, (ii), (iii) are similar.

Given $|x| = |y|$ with $\mathrm{cone}_L(x,y) \neq \emptyset$, pick $(z,u) \in \mathrm{cone}_L(x,y)$ with $\max(|z|,|u|)$ minimal.

By symmetry, we can suppose that $|z| \leq |u|$.

Then, by the minimality of $\max(|z|,|u|)$, we see that

$$\mathrm{cone}_L(x,y),\ \mathrm{cone}_L(xz|_1, yu|_1),\ \mathrm{cone}_L(xz|_2, yu|_2),$$

$$\ldots,\ \mathrm{cone}_L(xz, yu|_{|z|})$$ are all distinct

and so $|z| < \#\{\mathrm{cone}_L(v,w) \mid |v| = |w|\}$.

Now, if $|z| = |u|$, we are done. Otherwise, if $|z| < |u|$, we see that

$$\mathrm{rightcone}_L(xz, yu|_{|z|+1}),\ \mathrm{rightcone}_L(xz, yu|_{|z|+2}),$$

$$\ldots,\ \mathrm{rightcone}_L(xz, yu)$$ are all distinct,

and so

$$|u| \leq 1 + \{\mathrm{cone}_L(v,w) \mid |v| = |w|\} + \{\mathrm{rightcone}_L(v,w) \mid |v| < |w|\}$$
$$\leq K. \qquad \square$$

<u>Proof of theorem</u>:

(i) $\Rightarrow$ (ii): Suppose (L, Σ) is an automatic structure on G. Fix $a \in \Sigma \cup \{\varepsilon\}$ and let K be as in the lemma for L_a. Suppose $(x,y) \in L_a$.

Then for all $m \le \min(|x|, |y|)$ there is some $(z, u) \in \mathrm{cone}_{L_a}(x|_m, y|_m)$ s.t. $|z|, |u| \le K$.

Thus,

$$\pi(x|_m) = \pi(x|_m z a)\,\pi(za)^{-1} = \pi(y|_m u)\,\pi(za)^{-1}$$
$$= \pi(y|_m)\,\pi(u)\,\pi(za)^{-1}$$

and hence $d(\pi(x|_m), \pi(y|_m)) \le |u| + |z| + 1 \le 2K+1$.

And if, e.g., $|x| < m \le |y|$, then there is some $u \in \mathrm{rightcone}_L(x, y|_m)$ with $|u| \le K$. Thus,

$$\pi(x|_m) = \pi(xa)\,\pi(a)^{-1} = \pi(y|_m u)\,\pi(a)^{-1} = \pi(y|_m)\,\pi(u)\,\pi(a)^{-1}$$

and hence $d(\pi(x|_m), \pi(y|_m)) \le K+1$.

Similarly for $|y| < m \le |x|$.

Since $a \in \Sigma \cup \{\varepsilon\}$ is arbitrary, it follows that L has the fellow traveler's condition.

(ii) $\Rightarrow$ (i): Suppose L has the fellow traveler's condition with constant K and let $a \in \Sigma \cup \{\varepsilon\}$. We must show that L_a is regular.

For this, we let $B = \{g \in G \mid d(1, g) \le K\}$, which is finite.

Note now that, by the fellow traveler's condition, we have for $|x| = |y|$, if $\mathrm{cone}_{L_a}(x, y) \neq \emptyset$

then $d(\pi(x), \pi(y)) \leq K$ and so $\pi(x)^{-1}\pi(y) \in B$.

Similarly, for $|x| < |y|$, if $\text{rightcone}_{L_a}(x,y) \neq \emptyset$, then $\pi(x)^{-1}\pi(y) \in B$, and for $|x| > |y|$, if $\text{leftcone}_{L_a}(x,y) \neq \emptyset$, then $\pi^{-1}(x)\pi(y) \in B$.

It follows from this that supposing

$$\left. x \sim_L z \ \& \ y \sim_L w \ \& \ \left(\pi(x)^{-1}\pi(y) = \pi(z)^{-1}\pi(w) \text{ or } \pi(x)^{-1}\pi(y),\ \pi(z)^{-1}\pi(w) \notin B \right) \right\} \quad (*)$$

then

$$|x| = |y| \ \& \ |z| = |w| \implies \text{cone}_{L_a}(x,y) = \text{cone}_{L_a}(z,w)$$

$$|x| < |y| \ \& \ |z| < |w| \implies \text{rightcone}_{L_a}(x,y) = \text{rightcone}_{L_a}(z,w)$$

$$|x| > |y| \ \& \ |z| > |w| \implies \text{leftcone}_{L_a}(x,y) = \text{leftcone}_{L_a}(z,w).$$

Since L is regular, $(*)$ defines an equivalence relation on $\Sigma^* \times \Sigma^*$ with only finitely many classes, so L_a is regular. $\square$