The McNaughton Theorem

Any Büchi recognizable language is Müller recognisable.

The proof of this theorem will require some amount of preparatory work.

Fix in the following a regular language $W \subseteq \Sigma^*$ but $\alpha \in \Sigma^\omega$.

We say that j is a <u>good point</u> (of α) if there is some $j < i$ such that

(a) $\alpha[1, j[\in W$

(b) $\alpha[1, i[\sim_W \alpha[j, i[$

The smallest such i is said to be the <u>flag point</u> associated with j.

<u>**Lemma**</u> If α has infinitely many good points, then $\alpha \in W^\omega$.

<u>Proof</u> If α has an infinite sequence of good points, then by passing to a subsequence we can find a sequence (j_n) of good points with corresponding flag points (i_n) such

that

$$j_1 < i_1 < j_2 < i_2 < j_3 < i_3 < \ldots$$

By definition, for all n:

$$\alpha[1, j_n[\, \in W$$

and

$$\alpha[1, i_n[\, \sim_W \alpha[j_n, i_n[$$

Thus, since $\alpha[1, i_n[\, ^\frown \alpha[i_n, j_{n+1}[\, = \alpha[1, j_{n+1}[\, \in W$,

also $\alpha[j_n, j_{n+1}[\, = \alpha[j_n, i_n[\, ^\frown \alpha[i_n, j_{n+1}[\, \in W$.

It follows that

$$\alpha = \alpha[1, j_1[\, ^\frown \alpha[j_1, j_2[\, ^\frown \ldots \in W^\omega. \qquad \square$$

Let $W^+ = \{ \alpha \in \Sigma^\omega \mid \alpha$ has infinitely many good points $\}$.

<u>Lemma</u> Suppose $WW \subseteq W$. Then $W^\omega = WW^+$.

<u>Proof</u> We have already seen that $W^+ \subseteq W^\omega$, so also $WW^+ \subseteq W^\omega$.

Conversely, suppose $\alpha \in W^\omega$ and let $j_1 < j_2 < \ldots$ be s.t. $\alpha[1, j_1[\, \in W$ and $\alpha[j_n, j_{n+1}[\, \in W$.

Then, by Ramsey's theorem, we can find an infinite subsequence (k_n) of (j_n) such that

$\alpha[k_n, k_m[\ \sim_W \ \alpha[k_p, k_q[$ whenever $n < m$ and $p < q$.

Write $\alpha = \alpha[1, k_1[\frown \beta$. Then for any n,

$$\beta[1, k_{n+1} - k_1[= \alpha[k_1, k_2[\frown \alpha[k_2, k_3[\frown \cdots \frown \alpha[k_n, k_{n+1}[$$

$$\in W^* \subseteq W,$$

$$\beta[1, k_{n+2} - k_1[\ = \ \alpha[k_1, k_{n+2}[\ \sim_W \alpha[k_{n+1}, k_{n+2}[$$

$$= \beta[k_{n+1} - k_1, k_{n+2} - k_1[.$$

Thus, $(k_{n+1} - k_1)_{n=1}^{\infty}$ is an infinite sequence of good points of β ($k_{n+1} - k_1$ is good as witnessed by $k_{n+2} - k_1$). Since also $\alpha[1, k_1[\in W$, $\alpha \in WW^+$. $\square$

<u>Lemma</u> For any regular language $W \subseteq \Sigma^*$, the language W^+ is Müller recognizable.

<u>Proof</u> Let $V = \{ \alpha[1, i[\mid \alpha \in W^+$ and i is a flag point of α associated with some good point $j < i \}$. Then clearly $\vec{V} = W^+$.

So it suffices to prove that V is regular.

But

$$V = \{ xy \mid x \in W,\ y \in \Sigma^+ \text{ and } y \text{ has minimal length such that } xy \sim_W y \},$$

which is easily seen to be regular. $\square$

Lemma If $W \subseteq \Sigma^*$ is regular and $L \subseteq \Sigma^\omega$ is Müller recognisable, ~~then~~ also WL is Müller recognisable.

Proof Let $\mathcal{A} = (S_0, s_0, T_0, F_0)$ be a deterministic Büchi automaton recognising W and $\mathcal{B} = (S_1, s_1, T_1, \mathcal{F}_1)$ be a Müller automaton recognising L. Since both $\mathcal{A}$ and $\mathcal{B}$ are deterministic, we can think of T_i as functions $T_i : S_i \times \Sigma \to S_i$.

We build another Müller automaton $\mathcal{C} = (S_2, s_2, T_2, \mathcal{F}_2)$ recognising WL as follows:

First let $\#$ be a symbol not in S_1 and $k = |S_1| + 1$.

Put

$$S_2 = \{ (q, t_1, \dots, t_k) \in S_0 \times (S_1 \cup \{\#\})^k \mid \text{if } i \neq j \text{ and } t_i = t_j \text{, then } t_i = t_j = \# \}.$$

So for every $(q, t_1, \dots, t_k) \in S_2$, at least one t_i is $\#$.

Also, set

$$s_2 = (s_0, \#, \#, \dots, \#).$$

Now, suppose $(q, t_1, \dots, t_k) \in S_2$ and $a \in \Sigma$.

We define

$$r_i = \begin{cases} T_1(t_i, a) & \text{if } t_i \neq \# \text{ and for all } j < i \text{ with } t_j \neq \# \\ & T_1(t_j, a) \neq T_1(t_i, a), \\ T_1(s_1, a) & \text{if } q \in F_0, \ i \text{ is minimal with} \\ & t_i = \# \text{ and for all } t_j \neq \#, \\ & T_1(s_1, a) \neq T_1(t_j, a), \\ \# & \text{otherwise.} \end{cases}$$

Then set $T_2((q, t_1, \ldots, t_k), a) = (T_0(q, a), r_1, \ldots, r_k)$.

Finally, for $i = 1, \ldots, k$ let $\pi_k : S_2 \longrightarrow S_1 \cup \{\#\}$ denote the projection onto the $k+1$'st coordinate. Then for $D \subseteq S_2$, we put $D \in \mathcal{F}_2$ if and only if for some $i = 1, \ldots, k$, $\pi_i[D] \in \mathcal{F}_1$.

We claim that $\mathcal{C} = (S_2, s_2, T_2, \mathcal{F}_2)$ recognises WL.

Suppose first that $\alpha \in L(\mathcal{C})$ and let
$(\tau, \sigma_1, \sigma_2, \ldots, \sigma_k) \in S_0^\omega \times (S_1 \cup \{\#\})^\omega \times \ldots \times (S_1 \cup \{\#\})^\omega$
is the successful run of $\mathcal{C}$ on α.

Then, for some $i = 1, \ldots, k$, writing $\sigma_i = p_1 p_2 p_3 \ldots$ we must have

$$\{t \in S_1 \cup \{\#\} \mid \exists^\infty j \quad t = p_j\} \in \mathcal{F}_1$$

In particular, $p_j \neq \#$ for all but finitely many j. So pick j_0 maximal s.t. $p_{j_0} = \#$ and set $\tau = q_1 q_2 q_3 \ldots$ and $\alpha = a_1 a_2 a_3 \ldots$. Then, by the definition of T_2, we must have $q_{j_0} \in F_0$, whence $\alpha[1, j_0[\ \in W$.

Also, letting $\sigma = s_1 p_{j_0+1} p_{j_0+2} p_{j_0+3} \cdots \in S_1^{\omega}$, we see that σ is a successful run of $\mathcal{B}$ on $\alpha[j_0, \infty[$, so $\alpha[j_0, \infty[\ \in L$, whence

$\alpha = \alpha[1, j_0[\,^\frown \alpha[j_0, \infty[\ \in WL$.

Thus, $L(\mathcal{C}) \subseteq WL$.

Conversely, suppose $\alpha \in WL$ and write $\alpha = w\beta$, where $w \in W$, $\beta \in L$.

Let $(\tau, \sigma_1, \ldots, \sigma_k) \in S_0^{\omega} \times (S_1 \cup \{\#\})^{\omega} \times \ldots \times (S_1 \cup \{\#\})^{\omega}$ be the run of $\mathcal{C}$ on $\alpha = w\beta$ and let $\sigma \in S_1^{\omega}$ be the successful run of $\mathcal{B}$ on β.

Then $\tau[|w|+1] \in F_0$, since $w \in L(\mathcal{A}) = W$.

So for some $k \geq i_1 > i_2 > \ldots > i_\ell \geq 1$ and

$|w|+2 = j_1 < j_2 < \ldots < j_\ell$, we have by construction

$\sigma = s_1 \,^\frown \sigma_{i_1}[j_1, j_2[\,^\frown \sigma_{i_2}[j_2, j_3[\,^\frown \ldots \,^\frown \sigma_{i_{\ell-1}}[j_{\ell-1}, j_\ell[\,^\frown \sigma_{i_\ell}[j_\ell, \infty[$.

Now, since σ is successful, we have

$$\pi_{i_\ell}\left(\{ s \in S_2 \mid \exists^\infty j \quad s = (\tau[j], \sigma_1[j], \dots, \sigma_k[j])\}\right)$$

$$= \{ \pi_{i_\ell}(s) \mid s \in S_2 \ \& \ \exists^\infty j \quad s = (\tau[j], \sigma_1[j], \dots, \sigma_k[j])\}$$

$$= \{ p \in S_1 \mid \exists^\infty j \quad p = \sigma[j]\} \in \mathcal{F}_1 .$$

So $(\tau, \sigma_1, \dots, \sigma_k)$ is a successful run of $\mathcal{C}$ on α, whence $\alpha = w\beta \in L(\mathcal{C})$. Thus, $L(\mathcal{C}) = WL$. ■

<u>Theorem</u> (McNaughton)

Any Büchi recognisable language is Müller recognisable.

<u>Proof</u> Suppose $K \subseteq \Sigma^\omega$ is Büchi recognisable. Then there are regular languages V_i and W_i such that $K = \bigcup_{i=1}^{n} V_i W_i^\omega$. Since the Müller recognisable languages form a Boolean algebra, it now suffices to prove that each $V_i W_i^\omega$ is Müller recognisable.

But, as W_i^* is regular and $W_i^* W_i^* \subseteq W_i^*$, we have

$$V_i W_i^\omega = V_i (W_i^*)^\omega = V_i W_i^* (W_i^*)^+$$

which is Müller recognisable by the two preceding lemmas. □

Finally, let us sum up all our results on ω-languages.

<u>Theorem</u> Let Σ be a finite alphabet. TFAE for a language $L \subseteq \Sigma^\omega$.

(a) L is recognized by a Büchi automaton,
(b) L is recognized by a Müller automaton,
(c) L is recognized by a sequential Rabin automaton,
(d) there are regular languages $V_i, W_i \subseteq \Sigma^*$ with
$$L = \bigcup_{i=1}^{n} V_i W_i^\omega,$$
(e) L is a Boolean combination of languages $\vec{W}$, where $W \subseteq \Sigma^*$ are regular.

<u>Theorem</u> A language $L \subseteq \Sigma^\omega$ is recognized by a deterministic Büchi automaton if and only if it is of the form $\vec{W}$ for $W \subseteq \Sigma^*$ regular.

<u>Theorem</u> The class of Büchi recognizable languages form a Boolean algebra.