The McNaughton Theorem

Any Büchi recognizable language in Müller recognizable.

The proof of this theorem will require some amount of preparatory work.

Fix in the following a regular language $W \subseteq \Sigma^*$, but $x \in \Sigma^w$.

We say that $j$ is a good point (of $x$) if there is some $j < i$ such that

(a) $x[j, i] \in W$

(b) $x[j, i] \in W \Rightarrow x[j, i] \in W$

The smallest such $i$ is said to be the flag point associated with $j$.

Lemma: If $x$ has infinitely many good points, then $x \in W^w$.

Proof: If $x$ has an infinite sequence of good points, then by passing to a subsequence we can find a sequence $(j_n)$ of good points with corresponding flag points $(i_n)$ such
\[ f_1 < f_2 < f_3 < f_4 < \ldots \]

By definition, for all \( n \):
\[ xEI, j_n [ \in W \]
and
\[ xEI, j_n [ \wedge \alpha[I, j_n [ = \alpha[I, j_n [ \in W, \]
Thus, since \( xEI, j_n [ \wedge \alpha[I, j_{n+1} [ = \alpha[I, j_{n+1} [ \in W \),
also \( \alpha[I, j_{n+1} [ = \alpha[I, j_n [ \wedge \alpha[I, j_{n+1} [ \in W \).

It follows that
\[ x = xEI, j_n [ \wedge xEI, j_2 [ \in W^\omega \]

Let \( W^+ = \{ x \in \mathbb{Z}^\omega | x \text{ has infinitely many good points} \} \).

**Lemma:** Suppose \( WW \subseteq W \). Then \( W^\omega = WW^+ \).

**Proof:** We have already seen that \( W^+ \subseteq W^\omega \),
so also \( WW^+ \subseteq W^\omega \).

Conversely, suppose \( x \in W^\omega \) and let \( f_1 < f_2 < \ldots \)
in \( xEI, f_1 [ \in W \) and \( xEI, f_n [ \in W \).

Thus, by Ramsey's Theorem, we can find
an infinite subsequence \((k_n)\) of \((f_n)\)
such that
\[ x \in \mathcal{K}_u, k_u \in \mathcal{W} x \in \mathcal{K}^{p}, k_q \in \mathcal{W} \quad \text{whenever} \quad u = w \quad \text{and} \quad p < q \]

Write \( x = x[k_1, k_1] \beta \). Then for any \( v \),

\[ \beta [1, k_{u+1}-k_1] = x[k_1, k_2] \cdots x[k_{u+2}, k_{u+1}] \in \mathcal{W}^* \subseteq \mathcal{W}, \]

\[ \beta [1, k_{u+2}-k_1] = x[k_1, k_{u+2}] \sim_{W} x[k_{u+1}, k_{u+2}] = \beta [k_{u+1}-k_1, k_{u+2}-k_1]. \]

Thus, \( (k_{u+1}-k_1)_{u=1}^{\infty} \) is an infinite sequence of good points of \( \beta \) (\( k_{u+1}-k_1 \) is good as witnessed by \( k_{u+2}-k_1 \)). Since also \( x[k_1, k_1] \in \mathcal{W}^* \), \( \alpha \in \mathcal{W} \mathcal{W}^* \).

\[ \square \]

Lemma: For any regular language \( \mathcal{W} \subseteq \Sigma^* \), the language \( \mathcal{W}^+ \) is Muller recognizable.

Proof: Let \( V = \{ x[l, i] : \alpha \in \mathcal{W} \} \) and \( \bar{V} \) is a flag point of \( \alpha \) associated with some good point \( j < i \). Then clearly \( \bar{V} = \mathcal{W}^+ \).

So it suffices to prove that \( \bar{V} \) is regular.
But

\[ V = \{ x y | x \in W, \ y \in \Sigma^+ \text{ and } y \text{ has minimal length such that } xy \sim wy \} \]

which is easily seen to be regular.
Lemma: If \( W \in \Sigma^* \) is regular and \( L \subseteq \Sigma^* \)

is Müller recognizable, then also \( WL \) is Müller recognizable.

**Proof**

Let \( Q = (S_0, s_0, T_0, F_0) \) be a deterministic

Büchi automaton recognizing \( W \)

and \( Q_0 = (S_1, s_1, T_1, F_1) \) be a Müller automaton recognizing \( L \).

Since both \( Q \) and \( Q_0 \) are deterministic,

we can think of \( T_i \) as functions \( T_i : S_i \times \Sigma \to S_i \).

We build another Müller automaton \( Q_2 = (S_2, s_2, T_2, F_2) \)

recognizing \( WL \) as follows:

First let \( \# \) be a symbol not in \( S_1 \) and \( k = |S_1| + 1 \),

Put

\[
S_2 = \left( q_1, t_1, \ldots, t_k \right) \in S_0 \times (S_1 \cup \{\#\})^k
\]

if \( i \neq j \) and \( t_i = t_j \), then \( t_i = t_j = \# \).

So for every \( (q_1, t_1, \ldots, t_k) \in S_2 \), at least one \( t_i \) is \( \# \).

Also, set

\[
S_2 = (s_0, \#, \#, \ldots, \#).
\]

Now, suppose \( (q_1, t_1, \ldots, t_k) \in S_2 \) and \( w \in \Sigma \).

We define
\[
T_i(t_i, a) \quad \text{if } t_i \neq \# \text{ and for all } j < i \text{ with } t_j \neq \#
\]
\[
T_j(t_j, a) \quad \text{if } j \in F_0, \text{ } i \text{ is minimal with } t_i = \# \text{ and for all } t_j \neq \#, \\
T_1(s_i, a) \quad \text{otherwise.}
\]

Then, the set 
\[
T_2((s_1, t_1, ..., t_k), a) = (T_0(s_1, a), r_1, ..., r_k).
\]

Finally, for \( i = 1, ..., k \) let \( \pi_k : S_2 \rightarrow S_1 \cup \{\#\} \) denote the projection onto the \( k \)-th coordinate. Then for \( D \in S_2 \), we put \( D \in T_2 \) if and only if for some \( i = 1, ..., k \), \( \pi_i[D] \in T_1 \).

We claim that \( \mathcal{C} = (S_2, s_2, T_2, T_2) \) recognizes \( W_L \).

Suppose first that \( x \in L(\mathcal{C}) \) and let 
\[
(t, \sigma_1, \sigma_2, ..., \sigma_k) \in S_0 \times (S_1 \cup \{\#\})^* \times ... \times (S_1 \cup \{\#\})^* 
\]
be the successful run of \( \mathcal{C} \) on \( x \).

Then, for some \( i = 1, ..., k \), writing \( \sigma_i = p_1 p_2 p_3 ..., \)
we must have
\[
\exists t \in S_1 \cup \{\#\}^* \mid \exists^* \ t = p_1^j \in T_1.
\]
In particular, \( p_f \neq \# \) for all but finitely many \( f \).

So pick \( f_0 \) maximal \( \# p_{f_0} = \# \) and set

\[ G = g_1 g_2 g_3 \ldots \quad \text{and} \quad \alpha = a_1 a_2 a_3 \ldots \].

Then, by the definition of \( T_2 \), we must have \( \alpha_{f_0} \in F_0 \),

whence

\[ \alpha \in [1, \alpha_{f_0}] \in W \].

Also, letting \( \sigma = s_p f_{j+1} f_{j+2} f_{j+3} \ldots \in S^w \), we see that \( \sigma \) is a successful run of \( S_0 \)

on \( \alpha \in [1, \alpha_{f_0}] \in W \), so \( \alpha \in [1, \alpha_{f_0}] \in W \), whence

\[ \alpha = \alpha \in [1, \alpha_{f_0}] - \alpha \in [1, \alpha_{f_0}] \in W \).

Thus, \( \alpha \in W \).

Conversely, suppose \( \alpha \in W \) and write \( \alpha = w \beta \),

where \( w \in W \) and \( \beta \in L \).

Let \( (\sigma_0, \sigma_1, \ldots, \sigma_k) \in S^w \times (S_0 \times \#) \times \ldots \times (S_0 \times \#) \)

be the run of \( \beta \) on \( \alpha = w \beta \) and let \( \sigma \in S^w \)

be the successful run of \( S_0 \) on \( \beta \).

Thus \( \sigma [1, |\beta| + 1] \in F_0 \), since \( w \in L(\alpha) = W \).

So for some \( k \geq i_1 > i_2 > \ldots > i_{k+1} \geq 1 \) and

\[ 1, |\beta| + 1 = j_1 < j_2 < \ldots < j_{k+1} \],

we have by construction

\[ \sigma = s_0 \sigma_0 \sigma_1 \sigma_{i_1} e_1 d_1 \sigma_{i_2} e_2 d_2 \sigma_{i_3} e_3 d_3 \ldots \sigma_{i_{k+1}} e_{k+1} d_{k+1} \).

\[ \sigma \in [1, \alpha_{f_0}, \alpha_\infty] \in W \).
Now, since $\sigma$ is successful, we have
\[
\prod_i \left( \{ s \in S_k \mid \exists \sigma^i, s = (\tau[\pi_i], \sigma[\pi_i], \ldots, \sigma_k[\pi_i]) \} \right)
\]
\[
\prod_i \left( \{ s \in S_2 \mid \exists \sigma^i, s = (\tau[\pi_i], \sigma[\pi_i], \ldots, \sigma_{k-1}[\pi_i]) \} \right)
\]
\[
\prod_i \left( \{ p \in S_1 \mid \exists \sigma^i, p = \sigma[\pi_i] \} \right)
\]
So $(t, \sigma_1, \ldots, \sigma_k)$ is a successful run of $L$ on $x$, which $x = w^2 \in L(\Sigma)$. Thus, $L(\Sigma) = W_L$.

**Theorem (McNaughton)**

Any Büchi recognizable language is Müller recognizable.

**Proof** Suppose $K \subseteq \Sigma^\omega$ is Büchi recognizable. Then since all regular languages $V_i$ and $W_i$ such that $K = \bigcup_i V_i W_i^\omega$. Since the Müller recognizable languages form a Boolean algebra, it now suffices to prove that each $V_i W_i^\omega$ is Müller recognizable.

But, as $W_i^*$ is regular and $W_i^* W_i^* = W_i^*$, we have
\[
V_i W_i^\omega = V_i (W_i^*)^\omega = V_i W_i^* (W_i^*)^+
\]
which is Müller recognizable by the two preceding lemmas.
Finally, let us sum up all our results on ω-languages.

**Theorem.** Let $Σ$ be a finite alphabet. TFAE for a language $L \subseteq \Sigma^ω$.

1. $L$ is recognized by a Büchi automaton,
2. $L$ is recognized by a Müller automaton,
3. $L$ is recognized by a sequential Rabin automaton,
4. There are regular languages $V_i, W_i \subseteq \Sigma^*$ with $L = \bigcup_{i \geq 1} V_i W_i^{ω}$,
5. $L$ is a Boolean combination of languages $\overline{W}$, where $W \subseteq \Sigma^*$ are regular.

**Theorem.** A language $L \subseteq \Sigma^ω$ is recognized by a deterministic Büchi automaton if and only if it is of the form $\overline{W}$ for $W \subseteq \Sigma^*$ regular.

**Theorem.** The class of Büchi recognizable languages form a Boolean algebra.