Rabin automata

Let $T = \{0, 1\}^*$ denote the set of all finite binary strings. For $x, y \in T$, we let

$x \sqsubseteq y \iff x$ is a prefix of $y$.

Then $T$ is a tree with the empty string $\varepsilon$ and root $\varepsilon$.

**Definition**

A $\Sigma$-valuation on $T$ is a function $v : T \to \Sigma$. Denote by $V_{\Sigma}$ the set of all $\Sigma$-valuations on $T$.

A language is any subset $L \subseteq V_{\Sigma}$.

**Note**

We can represent $\Sigma$-valuations simply by labelings of the infinite binary tree $T$. 

\[
\begin{align*}
&\varepsilon \\
&0 \quad 1 \quad 00 \quad 01 \quad 010 \quad 011 \quad 10 \quad 11 \\
&000 \quad 001 \quad 010 \quad 011 \quad 100 \quad 101 \quad 110 \quad 111
\end{align*}
\]
**Definition** A **Rabin automaton** over \( \Sigma \) is a tuple \( \mathcal{A} = (S, \mathcal{I}, M, \mathcal{F}) \) where

- \( S \) is a finite set of states
- \( \mathcal{I} \subseteq S \) is the set of initial states
- \( M \) is a map from \( S \times \Sigma \) to \( \mathcal{P}(S \times S) \)
  called the **transition table**
- \( \mathcal{F} \subseteq \mathcal{P}(S) \) are the designated subsets of \( S \).

We shall now define computations of Rabin automata on \( \Sigma \)-valuations. Internally, the automaton \( \mathcal{A} \) will non-deterministically do computations along each branch of the tree representing

**Definition** A **run** or computation of \( \mathcal{A} \) on a \( \Sigma \)-valuation \( \nu \) is a function \( \sigma: T \to S \) such that

\[ \sigma(\epsilon) \in \mathcal{I} \]

and for all \( x \in T \)

\[ (\sigma(x_0), \sigma(x_1)) \in M(\sigma(x), \nu(x)) \]
In other words, \( \sigma \) produces a labeling (or computation) \( \sigma \) of \( T \) by the states \( S \) by first labeling \( \varepsilon \) by some \( s_0 \in I \). Also, if \( \sigma \) has labeled \( x \in T \) by \( s \in S \), the \( \sigma \) has to label the pair of nodes \( (x_0, x_1) \) by some pair of states \( (t_0, t_1) \) belonging to \( \mu(s, \nu(x)) \).

**Example**

Consider \( \sigma \varepsilon = (s, s, \varepsilon, \overline{\varepsilon}, \mu, \overline{\nu}) \) on \( \varepsilon \in \{a, b\} \) where \( \mu \) is given by

\[
\begin{align*}
\mu(s, a) &= \mu(t, b) = \xi(t, t) \\
\mu(t, a) &= \mu(s, b) = \xi(s, t)
\end{align*}
\]

Then \( \sigma \varepsilon \) is determinate.

![Diagram of the computation process](image)
**Definition** A run of computation $\sigma : T \rightarrow S$ of $O_\Sigma$ on $v \in V_\Sigma$ is said to be **successful** if for any $x \in \{0,1\}^*$ we have

$$\{ s \in S \mid \exists u \in T \ni \sigma(u,v) = s \} \in \mathcal{F}.$$ 

So the run is successful if along each branch of $T$, it satisfies the success criteria of Muller automata.

The language $L \subseteq V_\Sigma$ recognized by a Rabin automaton $O_\Sigma$ is the set of all $v \in V_\Sigma$ on which $O_\Sigma$ has a successful run.

**Definition** The Rabin automaton $O_\Sigma = (S, I, \mathcal{M}, \mathcal{F})$ is complete if $M(s,a) \neq \emptyset$ for all $s \in S$ and $a \in \Sigma$. It is **deterministic** if $T$ is a singleton and $M(s,a)$ is a singleton for all $s \in S$ and $a \in \Sigma$.

**Proposition** For any Rabin automaton $O_\Sigma$ there is a complete Rabin automaton $O_B$ with a single initial state such that

$$L(O_\Sigma) = L(O_B).$$
Proof. To see how to reduce $L$ to a singleton, suppose $\alpha = (S, \Sigma, M, \delta, F)$ in $L$ and let $s_0 \in S$. We let $\alpha_0 = (S, \Sigma, M', \delta', F')$ where

$$ M'(s, a) = \bigcup_{s \in S} M(s', a) $$

and

$$ M'(s_0, a) = M(s_0, a) \quad \text{for } a \in \Sigma. $$

We leave it as an exercise how to make the automaton complete. \(\square\)

Proposition. If $L_1, L_2 \subseteq V_\Sigma$ are both recognizable, then so is $L_1 \cup L_2$.

Proof. Suppose $\alpha_1 = (S_1, I_1, M_1, \delta_1, F_1)$ and $\alpha_2 = (S_2, I_2, M_2, \delta_2, F_2)$ where

$$ S = S_1 \cup S_2, $$

$$ I = I_1 \cup I_2, $$

$$ M(s, a) = \begin{cases} M_1(s, a) & \text{if } s \in S_1, \\ M_2(s, a) & \text{if } s \in S_2. \end{cases} $$

$$ F = F_1 \cup F_2 \subseteq \mathcal{P}(S_1 \cup S_2). $$

It follows that $L(S) = L(\alpha_1) \cup L(\alpha_2) = L_1 \cup L_2$. \(\square\)
Proposition 2. If $L_1, L_2 \subseteq V_2$ are recognized by Rabin automata $\mathcal{A}_1$ and $\mathcal{A}_2$, as before, also $L_1 \cap L_2$ is recognized by a Rabin automaton.

Proof. Let $\mathcal{A} = (\mathcal{A}_1 \times \mathcal{A}_2, \mathcal{I}_1 \times \mathcal{I}_2, \mathcal{M}, \mathcal{F})$, where

$$(e_1, e_2) \in M((q_1, q_2), a) \iff (e_1(q_1, a) \circ (e_2(q_2, a)))$$

and for $F \subseteq S_1 \times S_2$ we have

$$F \in \mathcal{F} \iff \text{proj}_1(F) \in \mathcal{F}_1 \land \text{proj}_2(F) \in \mathcal{F}_2.$$ 

Again, $L(\mathcal{A}) = L_1 \cap L_2$.

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**Game automata**

In order to show that Rabin recognizable languages are closed under complementation, we will make an appeal to another game-theoretic model of computation.

The idea is that to see if a Rabin automaton has a successful run $\sigma : T \rightarrow E$ on an input $w : T \rightarrow E$ we should just be able to inductively show how $\sigma$ is constructed along branches of $T$. 

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Definition

A game automaton is a tuple $A = (S, I, R, F)$ where

- $S$ is a finite set of states
- $I \subseteq S \times S$ is the set of initial states
- $R \subseteq S \times \{0, 1\} \times S \times S$ is the set of rules
- $F \subseteq P(S)$ is the set of winning conditions.

Given a game automaton $A = (S, I, R, F)$ and a $S$-valued map $v : T \to S$ we can define the game $G_{v}$ between players I and II by

$\begin{align*}
I & \rightarrow s_{1} \ s_{2} \ \cdots \\
II & \rightarrow d_{1} \ d_{2} \ d_{3} \ \cdots
\end{align*}
$

where $s_{i} \in S$ and $d_{i} \in \{0, 1\}$ are subject to the condition:

\begin{align*}
(\forall t \in \mathbb{N}) \left\{ \\
( v(e), s_{0} ) & \in I \\
( s_{n}, d_{n+1}, v(d_{1}d_{2}\cdots d_{n+1}), s_{n+1} ) & \in R \text{ for all } n \geq 0
\right. \\
\end{align*}

We say that I wins a run of the
game $G_v$ if $i \in S \mid \exists_\infty s_i = s \in \Sigma$.

Otherwise, $I$ wins the game.

**Definition** A (legal) position of $G_v$ is a finite sequence $p = s_0 \cdot d_1 \cdot s_1 \cdot d_2 \cdot s_2 \cdots d_n \cdot s_n$ or $p = s_0 \cdot d_1 \cdots d_n$ such that $(\ast)$ holds.

A strategy for $I$ in $G_v$ is a function $\phi$ associating to every legal position $p = s_0 \cdot d_1 \cdots d_n$ of even length some $\phi(p) \in S$ such that $s_0 \cdot d_1 \cdots d_n \cdot \phi(p)$ is a legal position.

Similarly, a strategy for $I$ is a function $\psi$ associating to every legal position $p = s_0 \cdot d_1 \cdots d_n \cdot s_n$ of odd length some $\psi(p) \in \Sigma^0,1$.

A run of the game $G_v$: $I \quad s_0 \quad s_1 \quad s_2 \quad \cdots$

is consistent with a strategy $\phi$ for $I$ in case $s_n = \phi(s_0 \cdot d_1 \cdots d_n)$ for all $n$.

A strategy $\phi$ for $I$ is winning if every run consistent with $\phi$ is winning for $I$.

Similarly for $I$.

**Definition** A game automaton $A$ accepts a $\Sigma$-valuation $v \in V_\Sigma$ if $I$ has a winning strategy $\phi$ in $G_v$. 


Let \( L(\mathcal{A}) = \bigcup \{ v \in \mathcal{L} \mid v \text{ is accepted by } \mathcal{A} \} \).

**Theorem** For any Rabin automaton \( \mathcal{A} \), there is a game automaton \( \mathcal{B} \) with \( L(\mathcal{A}) = L(\mathcal{B}) \).

**Proof** Wlog, we can assume that \( \mathcal{A} = (s_2, s_0, \delta, s_0, \mathcal{F}) \) is a complete Rabin automaton with a single initial state. We construct the game automaton \( \mathcal{B} = (s_1, I_1, R_1, F_1) \) as follows:

\[
S_1 = S \times S \times S
\]

\[
I_1 = \{ (a, (t_0, t_1, t_2)) \mid (t_0, t_1) \in M(s_0, a) \}
\]

\[
((t_0, t_1, t_2), d, a, (q_0, q_1, q_2)) \in R
\]

\[\Leftrightarrow (q_0, q_1) \in M(t_2, a) \quad \text{and} \quad q_2 = t_0 d \]

\[\text{and} \quad \mathcal{F} \subseteq S_1 = S \times S \times S \]

\[F \in F_1 \Leftrightarrow \text{proj}_3(F) \in \mathcal{F}\]

Now, do any such \( L(\mathcal{A}) = L(\mathcal{B}) \), suppose first that \( v \in L(\mathcal{A}) \) and let \( \sigma : T \to S \) be a successful run at \( \mathcal{A} \) on \( v \).
We shall use $\sigma$ to construct a winning strategy $\phi$. Let $I \in \mathcal{F}^v$.

First, as $\sigma$ is successful, we have for any $x \in \Delta$, $\exists i \in I$ such that $\exists e \in E \setminus \Delta \setminus \mathcal{F}^v \ni x_i = e \land \sigma(e_i) = x \in \mathcal{F}^v$.

Now let $p = q_0 d_1 q_1 d_2 \cdots q_{n-1} d_n$, $q_i \in S_i$, $d_i \in \sigma_i(q_i)$, is any legal position in $\mathcal{F}^v$, we let

$$\phi(p) = (\sigma(d_1 d_2 \cdots d_n 0), \sigma(d_1 d_2 \cdots d_n 1), \sigma(d_1 d_2 \cdots d_n)).$$

To see that this is a legal move, note first that

$$\phi(e) = (\sigma(0), \sigma(1), \sigma(1)) = (\sigma(0), \sigma(1), s_0)$$

and

$$\left(\nu(e), (\sigma(0), \sigma(1), s_0)\right) \in I$$

since

$$\left(\sigma(0), \sigma(1)\right) \in \mathcal{R}(\nu(e), s_0) = \mathcal{R}(\nu(e), \sigma(0)).$$

Also, for any $n \geq 0$, we have

$$\left(q_{n-1} d_{n-1} \nu(d_1 d_2 \cdots d_n), \phi(p)\right)$$

$$= (\sigma(d_1 \cdots d_{n-1} 0), \sigma(d_1 \cdots d_{n-1} 1), \sigma(d_1 \cdots d_{n-1})), d_n, \nu(d_1 d_2 \cdots d_n),$$

$$\left(\sigma(d_1 \cdots d_n 0), \sigma(d_1 \cdots d_n 1), \sigma(d_1 \cdots d_n)\right) \in R.$$
since by assumption on $\sigma$
\[(\sigma(d_1d_2...d_n), \sigma(d_1...d_{n-1})) \in M(v(d_1...d_n), \sigma(d_1...d_n))\]
so $\phi$ is indeed a strategy for $I$ in $G^B$.

We note the function of $\phi$:

For any given position $p = g(d_1...d_n)$,
the last coordinate of $\phi(p)$, i.e., $\sigma(d_1d_2...d_n)$
gives the current state of $G'$ where running
the computation $\sigma$ along $d_1d_2...d_n$ in $T$.

The first two coordinates of $\phi(p)$, namely
$\sigma(d_1...d_{n-1})$ and $\sigma(d_1...d_{n-1})$, give the two
next possible states

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Play at $I$ using $\phi$
against
\[d_1d_2d_3d_4d_5... = 10011...\]
played by $W$:

\[(\sigma(0), \sigma(1), \sigma(2)), (\sigma(10), \sigma(11), \sigma(11))\]

......
To see that $\sigma$ is winning for $I$, suppose
$x = d_1d_2d_3\ldots$ is being played by $II$.
Then using $\sigma$, $I$ responds with
$(\sigma(0), \sigma(1), \sigma(2)), (\sigma(d_0), \sigma(d_1), \sigma(d_2)), \ldots$
and since the sequence of third coordinates of $I$'s play is just
$\sigma(x), \sigma(x|1), \sigma(x|2), \ldots$,
since $\sigma$ is successful,
$\exists x \in \mathbb{S} \mid \exists \omega \in \sigma(x|\omega) = \emptyset$,
showing that $I$ wins the game.
Thus, $\sigma$ is a winning strategy for $I$ in $E^3_0$
and so $\nu \in L(B)$. Hence $L(\nu) \subseteq L(B)$.

The converse implication is similar. For suppose
$\nu \in L(B)$ and let $\sigma$ be a winning strategy
for $I$ in $E^3_0$. Then it $d_1d_2\ldots d_n \in T$
but $sd_1s d_2s\ldots s dn$ is the corresponding
run consistent with $\sigma$. We then set
$\sigma(d_1d_2\ldots d_n)$ = third coordinate of $sn$.
Thus $\sigma$ is successful and so $L(B) \subseteq L(\nu)$. □
Theorem. For any game automaton $\mathcal{A} = (S, \Gamma, R, \mathcal{F})$
there is a Rabin automaton $\mathcal{B}$ with $L(\mathcal{A}) = L(\mathcal{B})$.

Proof.

We let $\mathcal{B} = (S_1, I_1, M_1, \mathcal{F}_1)$ where

- $S_1 = S \times \{0, 1\} \cup \{t_0\}$, where $t_0 \notin S$
- $I_1 = \{t_0\}$
- $M_1(t_0, a) = \{ (s, 0), (s, 1) \mid (a, s) \in I \}$
- $M_1((s, d), a) = \{ ((t, 0), (t, 1)) \mid (s, d, a, t) \in R \}$
- $\mathcal{F}_1 = \{ F \subseteq S \times \{0, 1\} \mid \text{proj}_2(F) \in \mathcal{F} \}$

To see that $L(\mathcal{B}) = L(\mathcal{A})$, suppose first that $w \in L(\mathcal{A})$ and let $\Phi$ be a winning strategy for $I$ in $G^\mathcal{A}_w$.

We define a successful run $\sigma : T \rightarrow S_1$ of $\mathcal{B}$ on $w$ as follows:

- First, $\sigma (\varepsilon) = t_0$,
- $\sigma (0) = (\Phi(0), 0)$, $\sigma (1) = (\Phi(1), 1)$ and
  \[d_0, d_1, d_2, \ldots, d_n \in \{0, 1\}\]
has been played according to $\Phi$, let
\[ \sigma(d_1 d_2 \ldots d_n 0) = (s_n, 0) \]
\[ \sigma(d_1 d_2 \ldots d_n 1) = (s_n, 1) \]
To see that $\sigma$ is successful, note that if $x \in \mathbb{E}_{0,1}^n$, $x = d_1 d_2 d_3 \ldots$ and
\[ s_0 d_1 s_1 d_2 s_2 \ldots \]
is played according to $\Phi$, then
\[ \exists i \in \mathbb{N}_0 \text{ s.t. } \pi_i(\sigma(x_0)) = s_i \]
\[ \Rightarrow \exists i \in \mathbb{N}_0 \text{ s.t. } s_i = s \text{ ? } e \in F \]
when $\sigma$ is successful.

Commonly, suppose $\sigma: T \rightarrow S_2$ is a successful run of $B$ on $v \in V_2$. Then we can define a winning strategy $\Phi \in I$ in $G_{\Sigma^*}$ as follows:

First note that $\sigma(0) = t_0$ and
\[ (\sigma(0), \sigma(1)) \in \mathcal{M}_1(\sigma(0), v(0)) = \mathcal{M}(t_0, v(0)) \]
\[ = \{ ((s_2 0), (s_2 1)) \mid (v(0), s) \in I \} \]
So $\pi_2(\sigma(0)) = \pi_2(\sigma(1))$.

Similarly, $\pi_2(\sigma(x 0)) = \pi_2(\sigma(x 1))$ for all $x \in \{0,1\}^+$.
For $d_1d_2...d_n \in T \times \mathbb{R}$:

$$v(d_1d_2...d_n) = 2$$ 
$$\sigma(d_1d_2...d_n) = (s, d_n)$$

$$\sigma(d_2...d_n 0) = (t, 0)$$ 
$$\sigma(d_2...d_n 1) = (t, 1)$$

where $(s, d_n, a, t) \in \mathbb{R}$, $(a, t) \in \mathbb{I}$.

For $\varepsilon$:

$$v(\varepsilon) = 2$$ 
$$\sigma(\varepsilon) = \varepsilon_0$$

$$\sigma(\varepsilon) = (s, 0)$$ 
$$\sigma(\varepsilon) = (s, 1)$$

Assume $p \in d_1d_2...d_n$ has been played, then we let

$$\Phi(p) = \text{proj}_1(\sigma(d_1d_2...d_n 0)) = \text{proj}_1(\sigma(d_1d_2...d_n 1))$$

Again, $\Phi$ can be seen to be winning in $\mathbb{I}$. $\square$