Axioms for set theory

A <u>universe</u> of set theory is a non-empty collection of objects $\mathcal{U}$ equipped with a binary relation $\in$ called the membership relation.

For x, y belonging to $\mathcal{U}$ (which we will <u>not</u> denote by $x, y \in \mathcal{U}$ to avoid confusion with the above membership relation), when

$$x \in y$$

we say that x is a <u>member</u> of y or x <u>belongs</u> to y.

The objects in $\mathcal{U}$ are called <u>sets</u>.

Moreover, $\mathcal{U}$ and $\in$ are supposed to satisfy the following list of axioms that we detail individually:

(1) <u>Axiom</u> of <u>extensionality</u>

$$\forall x \forall y \left(\forall z \left(z \in x \leftrightarrow z \in y \right) \rightarrow x = y \right)$$

That is, two sets with the same members are identical.

(2) <u>Pairing axiom</u>

$$\forall x \forall y \exists z \forall u \ (u \in z \leftrightarrow (u = x \ \vee \ u = y))$$

That is, for any two sets x and y there is a set z whose members are exactly x and y. Since, by the axiom of extensionality, this set z is unique, we will denote it by the notation

$$\{x, y\}.$$

Note also that this applies to the case when $x = y$, and we get a set $\{x, x\}$, whose unique member is x.
We will simplify this notation to $\{x\}$ and call it the <u>singleton</u> of x.

When $x \neq y$, we say that $\{x, y\}$ is a <u>pair</u> or a <u>doubleton</u>.
Given sets x and y, we can form the <u>ordered pair</u> or <u>couple</u> $\{\{x\}, \{x, y\}\}$ by repeated application of the pairing axiom. We denote this by (x, y).

Lemma If x, y, a, b are sets and $(x,y) = (a,b)$, then $x = a$ and $y = b$.

Proof If $x = y$, then $(x,y) = \{\{x\}, \{x,y\}\}$ $= \{\{x\}, \{x\}\} = \{\{x\}\}$ has only a single element, implying that also

$$(a,b) = \{\{a\}, \{a,b\}\}$$

has a single element which must be $\{a\} = \{x\}$. Thus, $a = b = x = y$.

If $x \neq y$, then (x,y) has two elements namely a singleton $\{x\}$ and a doubleton $\{x,y\}$. It follows that also (a,b) must contain a unique singleton, namely $\{a\}$, and a unique doubleton, namely $\{a,b\}$. From the uniqueness and the axiom of extensionality, $a = x$ and then also $b = y$. $\square$

Definition For any sets $x_1, x_2, \ldots, x_n$ we can inductively define n-tuples $(x_1, x_2, \ldots, x_n)$ by

$$(x_1, x_2, x_3) := (x_1, (x_2, x_3)),$$

$$(x_1, x_2, x_3, x_4) := (x_1, (x_2, x_3, x_4))$$

etc.

<u>Theorem</u> For any $n \geq 1$, we have

$$(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n)$$
$$\Rightarrow x_1 = y_1 \ \& \ \dots \ \& \ x_n = y_n$$

(3) <u>Union axiom</u>

$$\forall x \, \exists y \, \forall z \, (z \in y \leftrightarrow \exists u \, (z \in u \ \& \ u \in x))$$

That is, for every set x there is a set y whose members are exactly the z that belong to a member of x.

Again, by extensionality, this set y is unique and is denoted

$$\bigcup x \quad \text{or} \quad \bigcup_{u \in x} u .$$

We call it the <u>union</u> over x.

For example, $\bigcup (x,y) = \bigcup \{\{x\}, \{x,y\}\} = \{x, y\}$

So by induction on n, we can for any sets $x_1, \dots, x_n$ construct a set $\{x_1, \dots, x_n\}$ whose members are exactly $\{x_1, \dots, x_n\}$

For this,

$$\{x_1, \dots, x_n\} := \bigcup \{\{x_1\}, \{x_2, \dots, x_n\}\} .$$

Also, if x and y are sets, we denote by

$$x \cup y := \bigcup \{x, y\}$$

and call it the union of x and y

Fact The operation $\cup$ is associative, i.e.,

$$x \cup (y \cup z) = (x \cup y) \cup z$$

so we denote

$$(x_1 \cup (x_2 \cup \ldots\ldots (x_{n-1} \cup x_n) \ldots))$$

unambiguously by $x_1 \cup x_2 \cup \ldots \cup x_n$.

(4) Power set axiom

To simplify notation we write

$$x \subseteq y$$

whenever the following holds:

$$\forall z \, (z \in x \rightarrow z \in y).$$

When $x \subseteq y$, we say that x is a subset of y or is contained/included in y

The power set axiom is then the following statement:

$$\forall x \exists y \forall z \, (z \subseteq x \leftrightarrow z \in y).$$

That is, to every x there is a (unique) set y whose elements are exactly the subsets of x. We call y the <u>power set</u> of x and denote it by $\mathcal{P}(x)$.

<u>Caution</u> One must be careful when understanding the power set axiom. For the variable z only refers to objects in $\mathcal{U}$ and not subsets of x that happen not to be in $\mathcal{U}$. In fact, it is a basic idea in the construction of universes to make judicious choices of which subsets of a set to include in $\mathcal{U}$ and which to leave out. So in such a $\mathcal{U}$, $\mathcal{P}(x)$ will only consist of the subsets of x that are actually in $\mathcal{U}$.

<u>Class relations</u>:

The language of set theory is the usual first-order language including the logical symbol $=$ and the extra-logical symbol $\in$

Now, if $\phi(x,y,z)$ is a formula with at most three free variables x, y, z and possibly having <u>parameters</u> $a_1, \ldots, a_n$, we have a corresponding relation R_ϕ on $\mathcal{U}$ defined by

$R_\phi(a,b,c) \iff \phi$ holds of a,b,c in $\mathcal{U}$.

For example, if ϕ is

$$\forall z \ (z \in x \rightarrow z \in y)$$

then R_ϕ is the inclusion relation $\subseteq$.

The relations so obtained are called <u>class relations</u> and unary class relations are also just called <u>classes</u>.

Later on we shall misuse notation and write $(b_1, \ldots, b_n) \in R_\phi$ to denote that $\phi(b_1, \ldots, b_n)$ holds even though R_ϕ is <u>not</u> in general a set.

<u>Example</u>

$\phi(x)$: $\forall u (u \in x \rightarrow \exists v (v \in x \ \& \ \forall t (t \in v \leftrightarrow t = u \vee t \in u)))$

defines the class of all sets x such that if $u \in x$, then also $u \cup \{u\} \in x$.

<u>Class</u> <u>functions</u> Suppose $\phi(x_1, x_2, \ldots, x_n, y)$ is a formula. We say that ϕ defines a <u>class function</u> R_ϕ if the following holds in $\mathcal{U}$

$$\forall x_1 \forall x_2 \ldots \forall x_n \forall y \forall y'$$

$$(\phi(x_1, \ldots, x_n, y) \ \& \ \phi(x_1, \ldots, x_n, y') \longrightarrow y = y').$$

In this case, the formula $\exists y\, \phi(x_1, \dots, x_n, y)$ in variables $x_1, \dots, x_n$ defines the <u>domain</u> of R_ϕ while $\exists x_1 \exists x_2 \dots \exists x_n\, \phi(x_1, \dots, x_n, y)$ defines the <u>image</u> of R_ϕ.

To simplify notation, we shall often write

$$R_\phi(x_1, \dots, x_n) = y$$

to denote

$$\phi(x_1, \dots, x_n, y).$$

(5) <u>Axiom of replacement or substitution</u>

Suppose $\phi(x, y, a_1, \dots, a_n)$ is a formula with at most free variables x and y and having parameters $a_1, \dots, a_n$ from $\mathcal{U}$. Suppose ϕ defines a class function on $\mathcal{U}$, i.e.,

$$\forall x \forall y \forall y' \left(\phi(x, y, \bar{a}) \,\&\, \phi(x, y', \bar{a}) \rightarrow y = y' \right),$$

then the following is an axiom

$$\forall z \exists u \forall y \left(y \in u \longleftrightarrow \exists x \in z\ \phi(x, y, \bar{a}) \right).$$

That is, for every set z, we can take the image of z under the class function R_ϕ. Note that this does not say that the image under R_ϕ

i.e., that the image of all of U under R_Φ is a set, only that the image of a set z is a set u. As we shall see the unrestricted axiom would lead to contradictions.

Russell's paradox:

Note that $\Phi(x,y)$ given by "$x=y$ & $y \notin y$" is a functional relation. So if we allowed the unrestricted replacement axiom, then there would be a set u s.t.

$$y \in u \leftrightarrow y \notin y.$$

But then $u \in u \rightarrow u \notin u$ and $u \notin u \rightarrow u \in u$, i.e., $u \in u \leftrightarrow u \notin u$, which is a contradiction.

So we must restrict the replacement axiom to a specific set z.

(6) The comprehension scheme

Suppose now $\Psi(x, a_1, \ldots, a_n)$ is a formula with a single free variable x and with parameters $a_1, \ldots, a_n$. From Ψ we construct a new formula $\Phi(x,y,\bar{a})$ by

$$\Phi(x,y,\bar{a}) := \Psi(x,\bar{a}) \ \& \ x=y.$$

Again, $\Phi(x,y,\bar{a})$ defines a class function, so we can apply the replacement axiom to obtain

$$\forall z \exists u \forall v \left(v \in u \leftrightarrow \left(v \in z \ \& \ \psi(v, \bar{a}) \right) \right).$$

For every z this u is unique and will be denoted

$$\{ v \in z \mid \psi(v, \bar{a}) \}.$$

That is, for any set z and class R_ψ (defined possibly with parameters), there is a set $\{ v \in z \mid R_\psi(v) \}$ consisting of the elements of z belonging to the class R_ψ.

(7) Set existence

Since we demand $\mathcal{U}$ to be a non-empty collection, we also include the axiom

$$\exists x \ \ x = x .$$

From this follows that there is a unique set, denoted $\emptyset$, having no members. For let $\psi(x)$ be the formula $x \neq x$ and let z be any set. Then

$$\emptyset = \{ x \in z \mid x \neq x \}$$

has no members and is the unique such set.

Pairing from replacement :

We shall now see how one can get the pairing axiom from replacement, extensionality, the power set axiom and set existence.

First, note that $\mathcal{P}(\emptyset) = \{\emptyset\}$ and $\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$. Now, given any two sets a and b set

$$\psi(x,y,a,b) : (x = \emptyset \ \& \ y = a) \text{ or } (x = \{\emptyset\} \ \& \ y = b).$$

Then $R\psi$ is a class function and so taking $z = \{\emptyset, \{\emptyset\}\}$, the image becomes $\{a, b\}$.

<u>Exercise</u> :

Show how to define (and prove the existence of) the sets

$$a \setminus b \ , \ a \cap b \ , \ a \times b = \{(x,y) \mid x \in a \ \& \ y \in b\}$$

given two sets a and b.

Functions

Suppose $\psi(x, y, \bar{a})$ is a formula defining a class function R_ψ whose domain is not just a class, but a set. I.e., there is a set z s.t.

$$\forall x\left(\exists y\, \psi(x,y,\bar{a}) \longleftrightarrow x \in z\right).$$

Then R_ψ is itself a set, i.e., for some set u,

$$\forall x \forall y\ \left(\psi(x,y,\bar{a}) \longleftrightarrow (x,y) \in u\right).$$

To see this, just let v be the image of z by R_ψ. Then v is the whole image of R_ψ and

$$u = \{ (x,y) \in z \times v \mid R_\psi(x,y) \}$$

is a set.

We say then that u is a <u>function</u> from z to v and denote this by $u : z \to v$.

So functions are always a <u>set</u> of pairs and hence are identified with their graphs.

Definition

A <u>function</u> $f : a \to b$ from a set a to a set b is a subset $f \subseteq a \times b$ satisfying

- $\forall x \left(x \in a \to \exists y \in b \ (x,y) \in f \right)$
- $\forall x \forall y \forall y' \left((x,y) \in f \ \& \ (x,y') \in f \to y = y' \right)$

Families of sets and cartesian products

Suppose $a : I \to X$ is a function.
For simplicity we write a_i for the unique set $x \in X$ such that $(i,x) \in a$.

We then define

$$\bigcup_{i \in I} a_i = \{ z \in \bigcup X \mid \exists i \in I \ \ z \in a_i \}$$

$$\bigcap_{i \in I} a_i = \{ z \in \bigcup X \mid \forall i \in I \ \ z \in a_i \}$$

$\prod_{i \in I} a_i$ = set of all functions $f : I \to \bigcup X$ such that $\forall i \in I \ \ f(i) \in a_i$.

<u>Exercise</u> Show that $\prod_{i \in I} a_i$ exists.

<u>Caution</u> When $I = \emptyset$, then $\bigcap_{i \in I} a_i = \bigcup X$ and hence depends on the choice of X.

Ordinals and cardinals

Suppose R is a binary class relation and C is a class. We say that R defines a <u>strict</u> <u>ordering</u> of C if for all sets x, y, z, we have

$$R(x,y) \longrightarrow C(x) \;\&\; C(y)$$

$$\neg \big(R(x,y) \;\&\; R(y,x) \big)$$

$$R(x,y) \;\&\; R(y,z) \longrightarrow R(x,y).$$

The ordering R is <u>total</u> or <u>linear</u> if moreover

$$\forall x \forall y \ \big(C(x) \;\&\; C(y) \longrightarrow \big(R(x,y) \lor R(y,x) \lor x=y \big)\big).$$

Suppose now R is a strict linear ordering on the class C and X is a <u>set</u> all of whose elements belong to C. We say that X is <u>well-ordered</u> by R if any non-empty subset $Y \subseteq X$ has a smallest element, i.e.

$$\forall Y\,(Y \subseteq X \ \& \ \emptyset \neq Y \to$$
$$\exists y \in Y\ \forall x \in Y\,(x = y \ \vee \ R(y,x)))\,.$$

[Note that if R is a strict ordering on a class C and X is a subset of C, i.e., a set all of whose elements belong to C, then we can identify R's restriction to X with the <u>set</u> $\{(x,y) \in X \times X \mid R(x,y)\}$.]

Now, suppose X is a set well-ordered by R. A subset $Y \subseteq X$ is said to be an <u>initial segment</u> if

$$\forall x,y \in X\,(y \in Y \ \& \ R(x,y) \to x \in Y)\,.$$

Also, for every $x \in X$, let $S_x(X)$ denote the initial segment $S_x(X) = \{y \in X \mid R(y,x)\}$. Note that since R is strict, $x \notin S_x(X)$

<u>Observation</u> $Y \subseteq X$ is an initial segment if and only if $Y = X$ or $Y = S_x(X)$ for some $x \in X$

[Pf: If $Y \neq X$, just let x be the minimum element in $X \setminus Y$.]

Classes and sets

Recall that a <u>class</u> C is simply the collection of all x satisfying some formula $\phi(x,\bar{a})$ with parameters.

Note that we do not give classes any formal existence, in the sense that they do not belong to $\mathcal{U}$ and any statement about the class C is just a shorthand for a more complex statement involving the formula $\phi(x,\bar{a})$.

On the other hand, suppose there is a set z containing all members in C, i.e.,

$$\forall x\ (\phi(x,\bar{a}) \longrightarrow x \in z)$$

then, by comprehension, we can identify C with the <u>set</u>

$$y = \{x \in z \mid \phi(x,\bar{a})\}$$

Similarly, any set Z can be identified with a class, namely the class given by the formula $\phi(x)$: " $x \in Z$ ".

$$\boxed{\begin{array}{l}\underline{\text{Slogan}}: \\ \text{Classes are collections that sometimes} \\ \text{are too large to be sets, while on} \\ \text{the other hand, all sets are classes}\end{array}}$$

__Definition__ A class C is a __proper class__ when it is not a set, i.e., when there is no set whose elements are exactly those belonging to C.

__Examples__

- $\mathcal{U}$ is a proper class given by the formula "$x = x$"
- Russell's class given by the formula "$x \notin x$" is also a proper class.

Notation If $\phi(x,\bar{a})$ is a formula with a single free variable x and parameters $\bar{a}$, we let

$$\{x \mid \phi(x,\bar{a})\}$$

denote the, possibly proper, class defined by $\phi(x,\bar{a})$.

So sets are the special classes given by expressions

$$\{x \in z \mid \phi(x,\bar{a})\}$$

where z is another set.

Wellorderings:

Definition A class relation R defining a strict linear ordering of a class C is said to be a well-ordering if for any x in C, the class initial segment

$$S_x(C) = \{y \mid R(y,x)\}$$

is a set that is wellordered by R.

Ordinals

A set x is said to be **transitive** if $\forall y\,(y \in x \to y \subseteq x)$.

I.e., $z \in y \in x \Rightarrow z \in x$.

An **ordinal** is a transitive set α that is well-ordered by the class relation $\in$.

Lemma Ordinals form a class called Ord.

$$\begin{aligned}
\mathrm{Ord} = \{\alpha \mid\ & \forall y\,(y \in \alpha \to y \subseteq \alpha) \\
& \& \ \forall x \forall y\,(x \in \alpha \ \&\ y \in \alpha \ \&\ x \neq y \to \\
& \qquad\qquad (x \in y \vee y \in x)) \\
& \& \ \forall x\,(x \in \alpha \to x \notin x) \\
& \& \ \forall x\,(x \subseteq \alpha \ \&\ x \neq \emptyset \to \\
& \qquad\qquad \exists y \in x\ \forall z \in x\,(y \in z \vee y = z))\}
\end{aligned}$$

$\square$

Example $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$ are ordinals.

Facts. If α is an ordinal, then the initial segments of α are α itself and the elements of α.

- If α is an ordinal, then so is any $\beta \in \alpha$.
- If α is an ordinal, then $\alpha \notin \alpha$.

  [For if $\xi \in \alpha$, then $\xi \notin \xi$ as otherwise α would not be well-ordered by $\in$].

<u>Lemma</u> If α and β are ordinals, then either $\alpha \in \beta$, $\alpha = \beta$ or $\beta \in \alpha$.

<u>Pf</u> Let $\xi = \alpha \cap \beta$. Then ξ is an initial segment of α, β. For if $x \in \xi$ and $y \in x$, then as α, β are transitive, also $y \in \alpha \cap \beta = \xi$.

So as any initial segment of an ordinal is either the ordinal itself or an elt of it, there are four possible cases

(1) $\xi = \alpha = \beta$

(2) $\xi = \alpha$ & $\xi \in \beta$, whence $\alpha \in \beta$

(3) $\xi = \beta$ & $\xi \in \alpha$, whence $\beta \in \alpha$

(4) $\xi \in \alpha$ & $\xi \in \beta$. But then ξ is an ordinal and $\xi \in \alpha \cap \beta = \xi$, which is impossible. $\boxed{}$

Proposition The class Ord is well-ordered by the class relation $\in$.

Proof We know that $\in$ linearly orders Ord, for if $\alpha \in \beta \in \gamma$, then by transitivity of γ, $\alpha \in \gamma$ and irreflexivity has been checked. Moreover, for any ordinal α,

$$S_\alpha(\text{Ord}) = \{\beta \mid \beta \text{ an ordinal } \& \ \beta \in \alpha\} = \alpha$$

is a set well-ordered by $\in$. $\square$

Lemma Ord is a proper class.

Pf For if Ord was a set a, then a would itself be an ordinal and hence $a \in a$, which is impossible. [To see that a is transitive, note that for $\alpha \in \beta \in a$, since any elt of an ordinal is an ordinal, also $\alpha \in a$]. $\square$

Note For ordinals α, β, $\alpha \subseteq \beta \iff \alpha \in \beta$ or $\alpha = \beta$.

Lemma If α is an ordinal, then so is

$\alpha + 1 := \alpha \cup \{\alpha\}$, and, moreover, $\alpha+1$ is the successor of α in the ordering $\in$.

Pf That $\alpha \cup \{\alpha\}$ is an ordinal is trivial and $\alpha \in \alpha \cup \{\alpha\}$. Also, if $\alpha \in \beta$ for β an ordinal, then $\alpha \subseteq \beta$ and so $\alpha \cup \{\alpha\} \subseteq \beta$, whence $\alpha \cup \{\alpha\} \in \beta$ or $\alpha \cup \{\alpha\} = \beta$. $\square$

Terminology In the following we will always consider the class Ord with the well-ordering $\in$, that we sometimes also write $<$. That is, for ordinals α, β:

$$\alpha < \beta \iff \alpha \in \beta \iff \alpha \subsetneq \beta .$$

So for any ordinal α, $\alpha = \{\xi \mid \xi \text{ an ordinal } \& \ \xi < \alpha\}$.

Lemma If X is a set of ordinals, then $\sup X = \bigcup X$. I.e., $\bigcup X$ is an ordinal greater than or equal to all elements of X.

Proof $\bigcup X = \{x \mid \exists \alpha \in X \ x \in \alpha\}$ is a set of ordinals and hence is well-ordered by $\in$. To see that $\bigcup X$ is transitive, note that if $y \in x \in \bigcup X$, then there is $\alpha \in X$ s.t. $x \in \alpha$, whence by transitivity of α, also $y \in \alpha$, i.e., $y \in \bigcup X$.

Clearly, if $\alpha \in X$, then $\alpha \subseteq \bigcup X$, so $\alpha \leq \bigcup X$. And if $\beta < \bigcup X$, i.e., $\beta \in \bigcup X$, then $\exists \alpha \in X \ \beta \in \alpha$, i.e., β is not an upper bound for X. So $\bigcup X = \sup X$. $\boxtimes$

Lemma Suppose α and β are ordinals and $f : \alpha \to \beta$ is a strictly increasing function, i.e., for $\zeta, \xi < \alpha$, $\zeta < \xi \Longrightarrow f(\zeta) < f(\xi)$. Then $\alpha \leq \beta$ and $\xi \leq f(\xi)$ for all $\xi < \alpha$.

Proof Suppose towards a contradiction that there is some $\xi < \alpha$ s.t. $f(\xi) < \xi$. Take the minimal such ξ.

Then since $f(\xi) < \xi$ and ξ is minimal, $f(f(\xi)) \geq f(\xi)$, but, on the other hand, since f is strictly increasing

$$f(\xi) < \xi \Rightarrow f(f(\xi)) < f(\xi)$$

which is a contradiction.

But then if $\beta < \alpha$, $\beta \leq f(\beta) \in \beta$, i.e., $\beta \leq f(\beta) < \beta$, which is a contradiction again. $\square$

Theorem Suppose $f : \alpha \to \beta$ is a function which is an isomorphism of the ordered sets $(\alpha, <)$ and $(\beta, <)$. Then $\alpha = \beta$ and the isomorphism is unique, i.e., $f = id_\alpha$. In other words, the structure $(\alpha, <)$ is rigid.

Pf $\alpha = \beta$ follows from the lemma applied to f and f^{-1}. Also this gives us for any $\xi < \alpha$

$$\xi \le f(\xi) \quad \text{and} \quad \xi \le f^{-1}(\xi),$$

whence also $f(\xi) \le f(f^{-1}(\xi)) = \xi$, i.e., $\xi = f(\xi)$. ∎

<u>Theorem</u> For any well-ordered set $(X, \prec)$ there is a unique isomorphism onto an ordinal $(\alpha, <)$.

Pf Uniqueness follows from the theorem above.

For existence:

Let $Y = \{x \in X \mid (S_x, \prec)$ is isomorphic to an ordinal $\}$, where $S_x := S_x(X)$.

By the uniqueness part, for any $x \in Y$ there is a unique ordinal $\beta(x)$ isomorphic to $(S_x, \prec)$.

Note that for $x \prec y$ and $y \in Y$, by the isomorphism of S_y with $\beta(y)$, $S_x \subseteq S_y$ is sent to an initial segment of $\beta(y)$, which is itself an ordinal $\beta(x) < \beta(y)$. So $x \in Y$ and Y is an initial segment of X.

Now, $x \mapsto \beta(x)$ is a function defined on Y, so, by the axiom of replacement, $Z = \{\beta(x) \mid x \in Y\}$ is a set. Moreover, Z is an initial segment of the ordinal numbers:

For if $\xi < \beta(x)$ for some $x \in Y$, then the isomorphism from S_x to $\beta(x)$ takes some $y \in S_x$ to ξ, whence S_y is mapped onto the set of ordinals below ξ, i.e., ξ itself. So $\xi = \beta(y)$.

So Z itself, being a set and an initial segment, is an ordinal $Z = \alpha$ and

$x \in Y \mapsto \beta(x) < \alpha$ is an isomorphism from Y to α.

Now, if $Y \subsetneq X$, let x_0 be minimal in $X \setminus Y$, so $S_{x_0} = Y \cong \alpha$, contradicting that $x_0 \notin Y$.

$\square$

There is also a class version of this theorem.

Namely, if $R(x,y)$ is a class relation that well-orders a proper class C, then there is a class function F from C to Ord which is an isomorphism of the orderings R and $\in$:

Just define F by

$F(x) = \alpha \iff$ there is an order-isomorphism between $(S_x(C), R)$ and $(\alpha, \in)$.

<u>Inductive</u> <u>definitions</u>

Suppose $\phi(x)$ is a formula (possibly with parameters). Then

$\phi(\alpha)$ holds for every ordinal α

iff

$$\forall \alpha \left(\forall \beta (\beta < \alpha \rightarrow \phi(\beta)) \rightarrow \phi(\alpha) \right) \qquad (*)$$

For the non-trivial implication, if the second ~~statement~~ (*) holds, but $\phi(\alpha)$ is not true for all ordinals, ~~then~~ one gets a contradiction by looking at the least α such that $\neg\phi(\alpha)$.

Whereas proving $\forall\alpha\, \phi(\alpha)$ by proving (*) constitutes a proof by induction on ordinals, we can also give definitions by induction.

Suppose F is a class function of one variable and a is a subset of the domain of F, i.e., $\forall x \in a\, \exists y\, F(x) = y$. Then we let $F \restriction a$ denote the function obtained by restricting F to a, i.e., for $b = \{F(x) \mid x \in a\}$, which is a set by replacement,

$$F \restriction a = \{(x,y) \in a \times b \mid F(x) = y\}.$$

Now, suppose H is any class function of one variable. We say that a function f is H-inductive if

- $\alpha = \mathrm{dom}(f)$ is an ordinal, and
- $\forall \beta < \alpha$ $f \restriction \beta$ is in the domain of H and $f(\beta) = H(f \restriction \beta)$.

So we can think of H as associating to each function $f : \beta \to X$ defined on an ordinal

β an extension $\tilde{f} : \beta + 1 \to \tilde{X}$, by $\tilde{f}(\beta) = H(f)$. An H-inductive function just satisfies this equation whenever defined.

<u>Lemma</u> For any class function H and ordinal α, there is at most one H-inductive function f with $\mathrm{dom}(f) = \alpha$.

<u>Proof</u> Suppose $f : \alpha \to X$ and $g : \alpha \to Y$ are distinct H-inductive functions and let $\beta < \alpha$ be minimal such that $f(\beta) \neq g(\beta)$. Then $f\restriction_\beta = g\restriction_\beta$ and so

$$f(\beta) = H(f\restriction_\beta) = H(g\restriction_\beta) = g(\beta),$$

which is impossible. $\square$

<u>Lemma</u> Suppose H is a class function and α is an ordinal such that any function $f : \beta \to X$ with domain β will belong to the domain of H. Then there is an H-inductive function $g : \alpha \to Y$.

Proof Let $\tau = \{\beta < \alpha \mid$ there is an H-inductive function $f_\beta : \beta \to X\}$.

Then τ is a set and is easily seen to be an initial segment of α, whence τ is an ordinal $\leq \alpha$. Moreover, by uniqueness, the assignment $\beta < \tau \longmapsto f_\beta$ is well-defined and since for any $\gamma < \beta$, $f_\beta \restriction \gamma$ is also H-inductive, by uniqueness, we have

$$\gamma < \beta < \tau \implies f_\beta \restriction \gamma = f_\gamma .$$

Thus, $f = \bigcup_{\beta < \tau} f_\beta$ is an H-inductive function with domain $\sigma = \sup_{\beta < \tau} \beta = \bigcup_{\beta < \tau} \beta$.

Assume towards a contradiction that $\sigma < \alpha$. Then $\tilde{f}$ defined by $\tilde{f} \restriction \sigma = f$ and $\tilde{f}(\sigma) = H(f)$ is an H-inductive function with domain $\sigma + 1 = \sigma \cup \{\sigma\} \notin \tau$, which is impossible. $\square$

With only slight adjustments it suffices that for some class A, H takes values in A and any function $f : \beta \to X$, where X is a subset of A, belongs to the domain of H.

Theorem Suppose A is a class and M the class of all functions $f : \alpha \to X$ with domain an ordinal and range X a subset of A.

Suppose H is a class function of one variable defined on all of M and with values in A. Then there is a unique class function F, i.e., given by a formula of set theory, such that

- F is defined on Ord
- $\forall \alpha \quad F(\alpha) = H(F\restriction_{\alpha})$

Proof The class function F is defined by

$y = F(\alpha) \iff$ there is an H-inductive function $f : \alpha \to X$, $X \subseteq A$, and $y = H(f)$.

$\square$