Theorem. Suppose $A$ is a class and $H$ is a class function of one variable defined on the class of all $H$-inductive functions $P: x \rightarrow X$, where $x$ is an ordinal and $X$ is a subset of $A$. Assume also that $H$ only takes values in $A$.

Then there is a unique class function $F$, i.e., given by a formula of set theory, such that

1. $F$ is defined on $\text{Ord}$
2. $\forall x \in X \subseteq A$, $F(x) = H(F\mid x)$

Proof. The class function $F$ is defined by

$y = F(x) \iff$ there is an $H$-inductive function $f: x \rightarrow X$, $X \subseteq A$, and $y = H(f)$.
Stratified or ranked classes

A class \( W \) is said to be stratified or ranked if there is a class function \( q \) with domain \( W \) and taking values in \( \text{Ord} \) such that for any ordinal \( x \), the class
\[
W_x = \{ x \in W | W(x) \not\in q(x) \leq x^2 \}
\]
is a set.

Thus suppose \( W \) is a stratified class with corresponding stratification \( q \). Let also \( W' \) be the class of all functions with domain \( W_x \) in range \( x \) and \( H(x,t) = q \) a class function with domain \( W \times H \).

Thus there is a unique class function \( F \) with domain \( W \) such that for any \( a \) in \( W \):
\[
F(a) = H \left( a, F \upharpoonright W_q(a) \right)
\]

(Note that for \( a \) in \( W \), \( a \in W_{q(a)+1} - W_{q(a)} \).

Proof: set \( W_x = \{ x \in W | W(x) \not\in q(x) = x^2 \} \) and note that \( W_x \) is a set and
\[
W_x = \bigcup_{\beta \leq x} W'_\beta
\]
By induction on Ord, i.e., by applying the preceding theorem, we find a class function $G$ defined on ordinals $\alpha$. In every $\alpha$,
\[
G(\alpha) = \text{the function } \phi \text{ with domain } W^1_\alpha \text{ s.t.}
\]
\[
\phi(x) := H[x, U G(\beta)_{\beta < x}] \text{ for all } x \in W^1_\alpha.
\]
(Note that $G(\alpha)$ is written as a function of $G/\alpha$, so our theorem applies.)

Also, for any $a \in W$,
\[
G(\gamma(a)) = H[\alpha, U G(\beta)_{\beta < \gamma(a)}] \text{ with domain } W^1_{\gamma(a)}.
\]

We can now write
\[
F(a) = b \iff W(a) \land b = G(\gamma(a))(a).
\]

To see that this works, note that
\[
F/\alpha' = U G(\beta)_{\beta < \alpha'}, \text{ for } \alpha' \in W_{\alpha'}, \text{ any }
\]
\[
a \in W_{\beta} \text{ for some } \beta = \gamma(a) < \alpha \text{ and so }
\]
\[
F(a) = G(\beta)(a).
\]

It thus follows that for $a \in W$
\[
F(a) = G(\gamma(a))(a) = H[\alpha, F/\alpha' W_{\gamma(a)}].
\]
(8) Axiom of choice (AC)

For any set \( X \) and \( A \subseteq \mathcal{P}(X) \) set of pairwise disjoint non-empty subsets of \( X \), there is a set \( T \subseteq X \) such that

\[ \forall a \in X \text{ } \exists t \in T \text{ } a = t. \]

Define also other statements:

\((AC')\) For any set \( X \) there is a function

\[ \pi : \mathcal{P}(X) \setminus \{\emptyset\} \rightarrow X \text{ } \forall a \text{ } \pi(a) \in a \]

\((AC'')\) If \( \{X_i\}_{i \in I} \) is an indexed family of non-empty sets, then \( \bigcup_{i \in I} X_i \neq \emptyset \).

Prop \( AC \iff AC' \iff AC'' \)
(given some background theory of axioms (1) - (7)).

Theorem (Zermelo) Every set can be well-ordered.

If \( X \) is a set, and \( \pi : \mathcal{P}(X) \setminus \{\emptyset\} \rightarrow X \) be a choice function, i.e.,

\[ \pi(a) \in a \text{ } \forall a \in X, a \neq \emptyset. \]
Proof. Assume not.
Let \( \mathbf{H} \) be the class function defined by
\[
\mathbf{H}(t) = y \iff t \text{ is a function with } \text{dom}(t) = \alpha \text{ an ordinal, } \text{Im}(t) \not\subseteq X \text{ and } y = \pi (X \setminus \text{Im}(t)).
\]

(*) \( \mathbf{H} \) is defined on the class of \( \mathbf{H} \)-inductive functions. Also, \( \mathbf{H} \)-inductive functions are injective.

For \( f : \alpha \to X \) is \( \mathbf{H} \)-inductive, then in any \( \beta < \alpha \), \( f(\beta) = \mathbf{H}(f|_{\beta}) \in X \setminus \text{Im}(f|_{\beta}) \), whence for any \( \gamma < \beta \), \( f(\gamma) \neq f(\beta) \).

It thus follows that \( f \) is injective.

If also \( f \) were surjective, then this would induce a well-ordering of \( X \), contradicting our assumption.

By (*) we know that \( \mathbf{H} \) is an \( \mathbf{H} \)-inductive class function \( F : \text{Ord} \to X \), which is injective by (*). But since \( \text{Im}(F) \) is a set, \( F^{-1} : \text{Im}(F) \to \text{Ord} \) is a function from a set onto \( \text{Ord} \), which is impossible.

\[ \square \]

Note. Any well-ordered set admits a choice function.
**Theorem (Zorn)**

Suppose $(X, \leq)$ is a partially ordered set all of whose linearly ordered subsets admit an upper bound. Then $(X, \leq)$ has a maximal element, i.e., there is $y \in X$ such that $\forall x \in X. y \not\leq x$.

**Proof**

Let $A = \{ y \in X \mid \exists x \in X. \forall y \in y. y < x \}$

and let $\pi : \mathcal{P}(X) \setminus \emptyset \to X$ be a choice function. Define $p : A \to X$ by

$$p(y) = \pi(\{ x \in X \mid \forall y \in y. y < x \})$$

Define a class function $H$ by

$$H(f) = y \iff f \text{ is a function with }$$

$$\text{dom}(f) \text{ an ordinal and } \text{Im}(f) \in A$$

$$\delta \in y \iff p(\text{Im}(f))$$

(*) Any $H$-inductive function $f : X \to X$ is strictly increasing, i.e.,

$$\exists \xi < \delta < \alpha \implies f(\xi) < f(\delta)$$

It follows from (*), that for the image of any $H$-inductive function $f : X \to X$,
Linearly ordered and hence has an upper bound \( x_2 \in X \), i.e., \( \forall \beta \leq \alpha. f(\beta) < x_2 \).

Now, if \( f : X \rightarrow X \) is \( H \)-inductive, but \( H \) is not defined on \( f \), then \( \text{Im}(f) \)
has no strict maximum, whence \( f(\beta) = x_2 \)
for some \( \beta < \alpha \) and \( x_2 \) thus is the
maximum in \( \text{Im}(f) \) and a maximal
element of \( X \).

If, on the other hand, \( H \) is defined
on the class of \( H \)-inductive functions,
then we can find an \( H \)-inductive class
function \( F : \text{Ord} \rightarrow X \), which by (\( \ast \))
is strictly increasing. But then \( F \) would
define an injection of a proper class
into \( X \), which is impossible. \( \square \)

**Note:** Zorn's Lemma \( \Rightarrow (\ast \iota) \):

Suppose \( \mathcal{A} \) is a collection of pairwise dis-
joint subsets of a set \( X \) and let

\[ B = \{ T \in X | T \text{ is a partial transversal for } \mathcal{A} \}, \]

i.e., \( \forall A \in \mathcal{A} \) \( \text{Tr} \alpha \) has at most
a single element?
We order $T_\mathcal{B}$ by inclusion and note that any linearly ordered subset of $T_\mathcal{B}$ has an upper bound. So, by Zorn's Lemma, $T_\mathcal{B}$ has a maximal element, which is easily seen to be a transversal of $A$, i.e., intersects every element of $A$ in a singleton.

**Ordinal arithmetic**

Recall that if $\alpha$ is an ordinal, the **successor** of $\alpha$, i.e., the smallest ordinal strictly larger than $\alpha$, is $\alpha + 1 = \alpha \cup \{\alpha\}$.

**Definition.** An ordinal $\beta$ is a **successor ordinal** if $\beta = \alpha + 1$ for some $\alpha$.

Let also $\emptyset$ denote the smallest ordinal, i.e., $\emptyset = \emptyset$.

$\beta$ is a **limit ordinal** if $\beta \neq 0$ and $\beta$ is not a successor.

$\beta$ is a **natural number** or a **finite ordinal** if $\alpha \leq \beta$ ($\alpha = 0$ or $\alpha$ is a successor).
Let also \( n = \underbrace{(0+1)+\ldots+1}_n \) times.

Note that if \((X, \leq_X)\) and \((Y, \leq_Y)\) are well-ordered sets, then we can define well-ordinings on the sets
\[
x \times \omega \cup Y \times \{1\} \quad \text{and} \quad X \times Y
\]

by
\[
(x, i) < (y, j) \iff \begin{cases}
  i = j = 0 \text{ and } a <_X b \\
  i = j = 1 \text{ and } a <_Y b \\
  i < j
\end{cases}
\]

and
\[
(x_0, y_0) < (x_1, y_1) \iff \begin{cases}
  y_0 < y_1 \\
  y_0 = y_1 \text{ and } x_0 <_X x_1
\end{cases}
\]

The first ordering, i.e., on \(X \times \omega \cup Y \times \{1\}\), is said to be the sum of \((X, \leq_X)\) and \((Y, \leq_Y)\), while the latter is the product.

**Definition** For ordinals \(\alpha, \beta\):

- \(\alpha + \beta\) denotes the unique ordinal \(\gamma\) that is order-isomorphic to the sum of \(\alpha\) and \(\beta\).
- \(\alpha \beta\) denotes the unique ordinal \(\gamma\) that is order-isomorphic to the product of
\( \delta \) with \( \beta \).

**Lemma** + is associative and \( 0 \) is a two-sided additive identity.

* is associative, \( \alpha \cdot 0 = 0 \), \( \alpha \cdot 1 = 1 \cdot \alpha = \alpha \),\n\( \alpha \cdot (\beta + \gamma) = \alpha \beta + \alpha \gamma \), and if \( \gamma \) is a limit \( \alpha \gamma = \sup_{\beta < \gamma} \alpha \beta \).

For all we know still now, multiplication could be commutative, but this fails after the next axiom.

\((9)\) **Axiom of Infinity**

There is an ordinal which is not a natural number, i.e., an infinite ordinal,

\[ \exists \alpha (\alpha \text{ an ordinal } \land \exists \beta < \alpha (\beta \neq 0 \land \beta \text{ is not a successor}) ) \]

Note that the natural numbers is an initial segment of the ordinals. So if we let \( \omega \) denote the smallest infinite ordinal,
we see that \( \omega \) is a limit ordinal.

**Remark**  \( \omega \cdot 2 = \omega + \omega \neq 2 \cdot \omega = \omega \), \( \omega + 2 \neq \omega = 2 + \omega \).

**Exponentiation:** We define \( \alpha^\beta \) by recursion on \( \beta \) uniquely in \( \alpha \) by

\[
\begin{align*}
\alpha^0 &= 1 \\
\alpha^{\beta+1} &= \alpha^\beta \cdot \alpha \\
\alpha^\beta &= \sup_{\gamma < \beta} \alpha^\gamma , \text{ whenever } \beta \text{ is a limit.}
\end{align*}
\]

**Note**  Formally, we can see ordinal exponentiation as being given by a class function \( \text{Exp} : \text{Ord} \times \text{Ord} \rightarrow \text{Ord} \) defined by the first order formula

\[
\text{Exp}(\alpha, \beta) = \gamma \iff \text{there is a function } \begin{array}{c}
f : \beta + 1 \rightarrow \text{Ord} \\
f(0) = \alpha \\
\end{array}
\]

\[
\begin{align*}
\begin{array}{c}
\text{such that } \\
\forall \xi \in \beta, \xi + 1 \rightarrow \alpha, \xi \\
\text{and } \begin{array}{c}
f(\xi) = \gamma \\
\xi = 0 \Rightarrow f(\xi) = 1 \\
\xi = \xi + 1 \Rightarrow f(\xi) = f(\xi) \cdot \alpha \\
\xi \text{ limit } \Rightarrow f(\xi) = \sup_{\zeta \in \xi \cap \beta} f(\zeta) \end{array}
\end{array}
\end{align*}
\]

The proof of uniqueness and existence of \( f \) follows along the same lines as the general.
Proof by inductive definition (or can be deduced from it directly).

Cardinals and Their Arithmetic

**Definition**

Given a set $X$, we define the **cardinality** of $X$, denoted $|X|$ or card$(X)$, to be the smallest ordinal $\alpha$ such that $X$ can be well-ordered in order-type $\alpha$, i.e., $\exists \alpha$ such that there is a bijection of $X$ and $\alpha$.

Two sets are **equivalent** if there is a bijection between them. So $X$ and $Y$ are equivalent if and only if $|X| = |Y|$.

Thus, suppose $X$ and $Y$ are non-empty sets. TFAE

(i) there is an injection from $X$ into $Y$

(ii) there is a surjection from $Y$ onto $X$

(iii) $|X| \leq |Y|$.

**Note** $|X|$ and $|Y|$ are defined using (AC),

(i) $\implies$ (ii) does not require (AC), while

(ii) $\implies$ (i) does.
Thus (Cantor - Schröder - Bernstein)

X and Y are equipotent if and only if X injects into Y and vice versa.

This is obtained given (AC). But it is possible to give a proof without.

**Theorem (Cantor)** For any set X,

\[ |X| \leq |\mathcal{P}(X)|. \]

If we suppose towards a contradiction that there is a surjection \( \pi : X \to \mathcal{P}(X) \) and define

\[ Y = \{ x \in X \mid x \notin \pi(x) \}. \]

Then if \( y = \pi(y) \), we have

\[ y \in \pi(y) \iff y \in Y \iff y \notin \pi(y) \]

which is impossible. \( \square \)

**Definition** An ordinal number \( \kappa \) is said to be a cardinal if \( |\kappa| = \kappa \).

**Corollary** The class of cardinal numbers is a proper class.
Proof towards a contradiction that $A$ were the set of all cardinal numbers. Then $\frac{1}{2} = \sup A$ is an ordinal and

$$\kappa = |\mathcal{P}(\frac{1}{2})| > \frac{1}{2} \geq 2$$

for any $\kappa \in A$.

Since $\kappa$ is a cardinal, this is impossible. \[ \square \]

**Definition**

A set $X$ is finite if $|X|$ is a finite ordinal and is infinite otherwise.

So $X$ is infinite if and only if it injects into $\mathcal{P}(X)$.

**Proposition (Cardinal)** A set $X$ is infinite if and only if it properly injects into itself.

**Proof** For one direction it suffices to note that $x \mapsto x+1$ is a proper injection of $\omega$ into itself.

For the converse, one shows by induction on finite ordinals that they are cardinal numbers. \[ \square \]
The \( \mathfrak{s} \) - function:

The class of infinite cardinals, being cardinal in the proper class of ordinals, is itself a proper class well-ordered by the usual ordering of ordinals. It thus follows that there is a uniquely defined class function

\[ \mathfrak{s} : \text{Ord} \rightarrow \text{class of infinite cardinals} \]

preserving the ordering.

For example, \( \mathfrak{s}_0 = \omega \) and \( \mathfrak{s}_{\omega+1} \) is the smallest cardinal number larger than \( \mathfrak{s}_\omega \).

Since the cardinal numbers are well-ordered and unbounded, for any cardinal number \( \mathfrak{c} \) there is a smallest cardinal \( \mathfrak{c}' \) which we denote \( \mathfrak{c}' \).

So, e.g., \( \mathfrak{c}' = \mathfrak{n} + 1 \) for a finite and \( \mathfrak{c}' = \mathfrak{n} + 1 \) for an infinite.

Note: For any ordinal \( \mathfrak{c} \), we have

\[ |\mathfrak{c}| \leq \mathfrak{c} < |\mathfrak{c}|^+ \]

For it \( |\mathfrak{c}|^\leq \mathfrak{c} \), then \( |\mathfrak{c}|^+ \leq \mathfrak{c} \) and thus \( |\mathfrak{c}|^+ \) would inject into the smaller.
cardinal $\aleph_1$, which is impossible.

**Definition.** Cardinals of the form $\aleph_\alpha$ are called *successors*, while non-zero, non-successors are called *limit* cardinals.

**Proposition.** The function $\aleph_\alpha$ is continuous not the order-topology, that is, for any limit ordinal $\lambda$,

$$\aleph_\lambda = \sup_{\beta < \lambda} \aleph_\beta.$$ 

Let $\gamma = \sup \aleph_\beta$. Then $|\gamma| \leq \gamma < |\gamma|^+$. Now, if $\gamma < \aleph_\lambda$, then there is some $\kappa_0 < \lambda$ such that $\aleph_{\kappa_0} = |\lambda|$, whence

$$\sup_{\beta < \gamma} \aleph_\beta = \gamma < |\gamma|^+ = \aleph_{\kappa_0+1} \leq \sup_{\beta < \lambda} \aleph_\beta$$

which is impossible. So

$$\sup_{\beta < \lambda} \aleph_\beta \leq \aleph_\lambda = \sup_{\beta < \lambda} \aleph_\beta.$$ 

**Definition.** A set $X$ is *countable* if $|X| \leq \aleph_0$ and *uncountable* otherwise.

The **Continuum Hypothesis** (CH) is the statement

$$|\mathcal{P}(\omega)| = \aleph_1.$$

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**Cardinal arithmetic**

**Definition** For cardinal numbers $\kappa$ and $\lambda$ we set

\[
\kappa \cdot \lambda = |\kappa \times \lambda|\\
\kappa + \lambda = |\kappa \times \omega \cup \lambda \times \omega|\]

Note that both cardinal multiplication and addition are commutative and associative.

**Theorem** If $\kappa$ is an infinite cardinal number, then

\[
\kappa \cdot \kappa = \kappa
\]

**Proof** This is by induction on $\kappa$ (i.e., by induction on $\alpha$ in $\kappa = \aleph_\alpha$).

So suppose that for all $\beta < \kappa$,

\[
|\beta \times \beta| = |\beta| \cdot |\beta| = 1\beta
\]

and define a well-ordering $<$ on $\kappa \times \kappa$ by

\[
(\alpha, \beta) < (\gamma, \eta) \iff \begin{cases} \max(\alpha, \beta) < \max(\gamma, \eta) \\ \max(\alpha, \beta) = \max(\gamma, \eta) \Rightarrow \alpha < \gamma \\ \max(\alpha, \beta) = \max(\gamma, \eta) \Rightarrow \alpha = \gamma \Rightarrow \beta < \eta. \end{cases}
\]
Since \( k \times k \) is the increasing union of \( \frac{1}{2} \times \frac{1}{2} \) \( \leq \frac{1}{2} \times k \), each \( \frac{1}{2} \times \frac{1}{2} \) is an initial segment of \( (k \times k, <) \) and \( (\frac{1}{2} \times \frac{1}{2}, <) \) is isomorphic to some ordinal \( \gamma \) with \( |\gamma| = |\frac{1}{2} \times \frac{1}{2}| = |\frac{1}{2} \times \gamma| < k \), and thus \( \gamma < k \), we see that \( (k \times k, <) \) is order-isomorphic to \( k \) with the usual order. So \( |k \times k| = k \). \( \square \)

Cor \( k \oplus k = k \) for all infinite \( k \).

\[ \text{Pf} \quad k \oplus k = |k \times 2| = |k \times k| = k \quad \square \]

**Definition** For cardinal numbers \( k \) and \( \lambda \), let \( k^\lambda = \left| \{ f \mid f \text{ is a function from } \lambda \text{ to } k \} \right| \).

Thus, \( 2^k = |\mathcal{P}(k)| \) by identifying a set with its characteristic function.

**Lemma** If \( \lambda \geq \omega \) and \( 2 \leq k \leq \lambda \), then \( k^\lambda = 2^\lambda \).

\[ \text{Pf} \quad \text{Note that } 2^\lambda = 2^{\lambda \times \lambda} = (2^\lambda)^\lambda \geq \lambda^\lambda \geq k^\lambda \quad \square \]
Definition: A function \( f : \alpha \rightarrow \beta \) between ordinals \( \alpha \) and \( \beta \) is said to be cofinal if \( \text{Im}(f) \) is unbounded in \( \beta \), i.e.,

\[ \forall \xi \in \beta \exists \eta < \alpha : f(\eta) \geq \xi. \]

The cofinality, \( \text{cof}(\beta) \), of an ordinal \( \beta \) is the smallest ordinal \( \alpha \) that maps cofinally into \( \beta \). Thus \( \text{cof}(\beta) \leq \beta \).

Note: \( \text{cof}(\beta) \) is always a cardinal.

Observation: There is a strictly increasing cofinal map \( h : \text{cof}(\beta) \rightarrow \beta \).

If \( f : \text{cof}(\beta) \rightarrow \beta \) is cofinal, define

\[ h(\xi) = \max \{ f(\eta), \sup_{\eta < \xi} (h(\eta) + 1) \} \]

Note: If \( \alpha \) and \( \beta \) are limit ordinals and \( f : \alpha \rightarrow \beta \) is a strictly increasing cofinal map, then \( \text{cof}(\alpha) = \text{cof}(\beta) \).

Prove: \( \text{cof}(\beta) \leq \text{cof}(\alpha) \) is clear. Conversely, if \( g : \text{cof}(\beta) \rightarrow \beta \) is a cofinal map, then we can define \( h : \text{cof}(\beta) \rightarrow \alpha \) by
\[ h(\xi) = \min \{ \eta < \alpha \mid f(\eta) > f(\xi) \} \text{.} \]

Thus, \( h \) is defined in \( \alpha \). \( \square \)

**Case for a limit:** \( \text{col}(\alpha \kappa) = \text{col}(\alpha) \).

**Case for any \( \beta \):** \( \text{col}(\beta) = \text{col}(\text{col}(\beta)) \).

So the colimation operation is idempotent, as it has plenty of fixed points.

**Definition.** An ordinal \( \beta \) is said to be **regular** if \( \text{col}(\beta) = \beta \) and \( \beta \) is a limit.

**Note.** Regular ordinals are cardinals, and the first regular cardinal is \( \omega \).

**Lemma.** \( \kappa^+ \) is regular in \( \kappa > \omega \).

**Proof.** Let \( f : \kappa \to \kappa^+ \) be a cardinal map,

then \( \kappa^+ = \text{sup} \{ f(\xi) \mid \xi < \kappa \} = \bigcup_{\xi < \kappa} f(\xi) \).

Since each ordinal \( f(\xi) \) is a set of cardinality \( < \kappa^+ \), i.e., \( |f(\xi)| \leq \kappa \), we see that

\[ \kappa^+ = |\bigcup_{\xi < \kappa} f(\xi)| \leq |\kappa \times \kappa| = \max\{|\kappa|, |\kappa|^2\} \]

and so \( |\kappa| = \kappa^+ \), whence \( \kappa^+ \leq \kappa \). \( \square \)
Note. Since for a limit cardinal \( \kappa \), we have
\[ \text{cf}(\kappa^\omega) = \text{cf}(\kappa), \]
if \( \kappa \) is regular, then
\[ \kappa^\omega = \text{cf}(\kappa^\omega) = \text{cf}(\kappa) \leq \kappa \leq \kappa^\omega, \quad \kappa = \kappa^\omega. \]

**Definition.** A cardinal \( \kappa \) is **weakly inaccessible** if \( \kappa \) is a regular limit cardinal \( > \omega \).

Moreover, \( \kappa \) is **(strongly) inaccessible** if \( \kappa > \omega \), \( \kappa \) is regular and for any \( \lambda < \kappa \),
\[ 2^\lambda < \kappa. \]

One cannot prove with the axioms given (nor with the axiom of foundation) that weakly inaccessible cardinals exist.

**Lemma.** (König)

If \( \kappa \) is an infinite cardinal and
\[ \lambda \geq \text{cf}(\kappa), \quad \text{then } \kappa^\lambda > \kappa, \]

**Proof.** Fix a cardinal map \( f : \lambda \rightarrow \kappa \) and consider any function \( g : \kappa \rightarrow \kappa^\lambda \).
We have to show that \( g \) is not surjective.
We think of \( \kappa^\lambda \) as the set of functions from \( \lambda \) to \( \kappa \), so define \( h : \lambda \rightarrow \kappa \).
by \( n(x) = \min \{ n : G(x)(x) \mid x < G(x) \} \).

Note then that if \( n = G(x) \) for some \( x < x \),

then \( 2 < x \) and \( f(x) \geq x \). Thus

\( n(x) \neq G(x)(x) \), which is a contradiction. \( \square \)

**Corollary** If \( \lambda > \alpha \), then \( \text{cf}(2^\lambda) > \lambda \).

\[ (2^\lambda)^2 = 2^\lambda \cdot 2^\lambda = 2^{\lambda^2}, \] so setting \( \lambda = 2^{\lambda} \),

if \( \lambda > \text{cf}(2^{\lambda^2}) \), then the lemma would give

\[ 2^{\lambda^2} = (2^\lambda)^2 > 2^{\lambda^2} \]. \( \square \)

(8) The axiom of foundation (AF)

The axiom of foundation states that \( \notin \) admits no infinite descending chains or equivalently

\[ \forall x (x \neq \emptyset \Rightarrow \exists y \in x \forall z \in y \ z \notin x) . \]

Note that \( \not\in \) (\( W \), \( \text{new} \)) once an infinite sequence,

i.e., formally a function with domain \( W \), is given.

With \( \text{new} \) the empty \( W \), then the set \( x = \{ \text{new} \} \)

would contradict (AF).

Note (AF) \( \Rightarrow \forall x x \notin x \).
Without assuming the axiom of foundation we can define a class function \( V : \text{Ord} \to \mathcal{U} \) by transfinite induction:

\[
V_{\beta} = \bigcup_{\alpha < \beta} P(V_{\alpha})
\]

Thus, \( V_0 = \emptyset \) and \( \alpha < \beta \implies V_{\alpha} \subseteq V_{\beta} \).

From this it follows that \( V_{\beta+1} = \bigcup_{\alpha < \beta} P(V_{\alpha}) = P(V_{\beta}) \).

So for \( \lambda \) limit, we get

\[
V_{\lambda} = \bigcup_{\alpha < \lambda} V_{\alpha}
\]

Let also \( \mathcal{V} \) be the class defined by

\[
V(x) \iff \exists \alpha \text{ ordinal } \land x \in V_{\alpha}
\]

**Definition.** For every set \( x \) in \( \mathcal{V} \), we let

\[
\text{rank}(x) = \text{rk}(x) = \min \{ \alpha \mid x \in V_{\alpha} \}.
\]

Note that \( \text{rk}(x) \) is always a successor ordinal.

**Lemma.** \( \forall x \in \mathcal{V} \) \( \forall y \in x \) \( V(y) \).

Also, if \( V(x), \) then \( \text{rk}(y) < \text{rk}(x) \) for all \( y \in x \).
Proof. Suppose \( V(x) \) and \( \kappa(x) = \beta + 1 \). Then \( x \in V_{\beta + 1} = \mathcal{P}(V_\beta) \) and so \( x \in V_\beta \). Thus \( y \in x \) gives \( y \in V_\beta \) and \( \kappa(y) \leq \beta < \kappa(x) \).

Conversely, if \( \forall y \in V(y) \), then class function \( \kappa : V \to \text{Ord} \) will be bounded on any set \( x \), whereas for some \( \beta \), \( x \in V_\beta \).

Thus, \( x \in V_{\beta + 1} \) and from \( V(\kappa) \).

Our result checks that:

**Lemma.** For every ordinal \( \alpha \), \( V(\alpha) \) and \( \kappa(\alpha) = \alpha + 1 \).

**Theorem.** \( \text{AF} \iff \forall x \in V(x) \).

If \( \forall x \in V(x) \), and let \( a \) be any non-empty set. Let \( b \subseteq a \) be a set of minimal rank. Then for any \( c \in b \), \( \kappa(c) \leq \kappa(b) \), so \( c \notin a \), showing \( \text{AF} \).

For the converse, we need the following definition:

**Definition.** For any set \( X \), define a function

\[
\ell : \omega \to \omega, \quad \ell(n + 1) = U_x = \bigcup_{x \in \ell(n)}
\]

Then clearly \( x \in \ell(X) \) and \( \ell(X) \) is...
Moreover, if $Z$ is any transitive set containing $X$, then $Z \subseteq \text{cl}(X)$. So $\text{cl}(X)$ is the unique transitive closure of $X$.

Now suppose $X$ does not belong to $V$, but that $\text{AF}$ holds. Then $X \subseteq \text{cl}(X)$ and we claim that $Y = \{ y \in \text{cl}(X) \mid \neg \forall (y) \}$ is non-empty. For if not, then $\text{cl}(X)$ and hence also $X$ would be a subset of $V$ and hence belong to $V$ itself, which is not the case.

On the other hand, if $y \in Y$, then $\neg \forall (y)$ and hence $y$ cannot be a subset of $V$. Thus, for some $z \in Y$, $\neg \forall (z)$. Since $\text{cl}(X)$ is transitive, also $z \in \text{cl}(X)$ and thus $z \in Y$ too. It follows that $Y$ is a counterexample to $\text{AF}$. \[ \]

**Proposition** \[ \text{cl}(X) = X \cup \bigcup_{y \in X} \text{cl}(y) \]

**Proof** $X \subseteq \text{cl}(X)$ and the latter is transitive, so $y \subseteq \text{cl}(X)$ for all $y \in X$. Since $\text{cl}(y)$
is the minimal transitive set containing $y$, also $c!(y) \subseteq c!(X)$ for any $y \in X$. So
\[ c!(X) \supseteq X \cup \bigcup_{y \in X} c!(y). \]
For the other direction it suffices to note that the right hand side is transitive. \qed

**Definition.** The theory $ZF$ consists of all axioms except for AC and AF. However,
\[ ZF = ZF^- + AF \]
\[ ZFC = ZF + AC = ZF^- + AF + AC \]

**Definition.** A set $X$ is **extensional** if
\[ \forall x, y \in X \ (x \cap X = y \cap X \rightarrow x = y) \ . \]
That is, $(X, \epsilon) \models$ axiom of extensionality.

Note that every transitive set is extensional simply because the axiom of extensionality holds.

**Theorem (The Mostowski collapse) ($ZF$).**
For any extensional set $X$ there is a unique isomorphism
\[ \pi : (X, \epsilon) \rightarrow (Y, \epsilon) \]
onto a transitive set $Y$. 55
Proof: Uniqueness follows by induction on the rank of elements of \( X \).

Existence: We define \( \pi (x) \) by induction on the rank of \( x \in X \).

\[
\pi (x) = \{ \pi (a) \mid a \in x \cap X^2 \}
\]

(Formally, this is induction on a stratified class.)

That is, note that by extensionality of \( X \), there is a unique element \( z \in X \) of minimal rank. Set \( \pi (a) = \emptyset \).

Now, if \( x \in X \) and \( \pi (a) \) has been defined for all \( a \in X \) with \( \text{rk}(x) < \text{rk}(a) \), we set \( \pi (x) = \{ \pi (a) \mid a \in x \cap X^2 \} \).

By induction on the rank, we see that \( \pi \) is injective and also that \( \pi [X] = Y \) is a transitive set.

Finally, \( \pi \) is an isomorphism; for if \( x, y \in X \) and \( x \subseteq y \), then \( x \cap y \subseteq x \) and so

\[
\pi (x) \subseteq \{ \pi (a) \mid a \in y \cap X^2 \} = \pi (y).
\]
and, conversely, if \( x, y \in X \) and \( \pi(x) \in \pi(y) \)
denote \( \pi(x) = \pi(a) \) for some \( a \in y \cap X \), so
by inductivity \( x = a \in y \cdot \)

---

**Relativization**

Suppose \( C \) is a class. We define
for every formula \( \phi(x, \bar{a}) \) in parameters
\( \bar{a} = (a_1, \ldots, a_n) \), all belonging to \( C \), the
relativized formula \( \phi^c(x, \bar{a}) \) by induction
on \( \phi \):

1. If \( \phi \) is quantifier-free then \( \phi^c = \phi \).
2. \( (\neg \phi)^c = \neg \phi^c \), \( (\phi \vee \psi)^c = \phi^c \vee \psi^c \)
3. \( (\exists y \phi)^c = \exists y (C(y) \land \phi^c) \)
4. \( (\forall y \phi)^c = \forall y (C(y) \rightarrow \phi^c) \)

So \( \phi^c \) simply relativizes all quantifications
so \( C \). So for any \( \phi(x) \) w/o parameters
and any parameters \( \bar{a} \) in \( C \) we have
\( (\mathcal{U}, \varepsilon) \models \phi^c(\bar{a}) \iff (C, \varepsilon) \models \phi(\bar{a}) \).
Consistency of the axiom of foundation

Something that should be on our mind is the consistency of the axioms we have stated so far. So proofs of relative consistency are highly important.

It follows from Gödel's incompleteness theorem that we cannot prove the consistency of ZFC from ZFC itself, so the only results we can hope for are relative consistency results such as:

Theorem If ZF\(^-\) is consistent, then so is ZF = ZF\(^-\) + AF.

In fact, we shall prove the following more precise statement:

Theorem Suppose \(U\) is a universe of sets satisfying the theory ZF\(^-\). Then the class \(V\) constructed in \(U\) will
be a universe of sets in which \( ZF \) holds.

**Proof** Recall that \( ZF^- \) is axiomatized by

(i) the axiom of extensionality
(ii) the union axiom
(iii) the power set axiom
(iv) the axiom of replacement
(v) sets existence
(vi) the axiom of infinity

We thus have to prove that if \( U \) is a universe of sets and \( V \) is the class of sets defined by \( V = \bigcup_{x} x \),

then \( (V, \in) \vdash (i)-(vi) \), i.e., for any axiom \( \phi \) among (i)-(vi), \( \phi \) holds.

(i): Suppose \( x, y \) belong to \( V \) and that for any \( z \) in \( V \), \( z \in x \iff z \in y \).

Then as \( x, y \) are subsets of \( V \), we have \( \forall z \left( z \in x \iff z \in y \right) \), so since
extensibility holds in $V$, we have $x = y$, and thus $(x = y)^V$ too.

(iii) Suppose $x$ belongs to $V$. Then $Ux = \{ z \mid \exists y \in x \; z \in x \}$ is a subset of $V$ and hence itself belongs to $V$.

(iv) If $x$ belongs to $V$, then so does $\mathcal{P}(x)$, for any subset of $x$ will be a subset of $V$ and thus belong to $V$, whereas $\mathcal{P}(x)$ is a subset of $V$ and thus an element of $V$.

(iv) Suppose $\phi(x, y)$ is a formula with parameters in $V$, that defines a class function in $V$, i.e.,

$$(\forall x \exists^! y \phi(x, y))^V$$

or more explicitly

$$\forall x (V(x) \rightarrow \exists^! y (V(y) & \phi^V(x, y)))$$

Then the formula

$$\psi(x, y) := V(x) \& V(y) \& \phi^V(x, y)$$
defines a functional relation in $\mathcal{V}$ and thus for any set $A$ there is
by replacement in $\mathcal{V}$ a set $B$ such
\[ y \in B \iff \exists x \in A \forall (x,y) \]
\[ \iff (\exists x \in A \forall (x,y)) \exists y \in \mathcal{V}(y) \]
Since $B$ is a subset of $\mathcal{V}$, $B$ actually belongs to $\mathcal{V}$ and is the image
of $A$ in $\mathcal{V}$ of the class function
defined by $\phi$.

$(\text{V})$ is trivial since $\emptyset$ belongs to $\mathcal{V}$.

$(\text{VI})$ We only have to show that $w$ belongs
to $\mathcal{V}$, but actually we may show
that all ordinals belong to $\mathcal{V}$.

This is by induction.

First $\emptyset = \emptyset$ belongs to $\mathcal{V}$.

Now suppose $\xi$ belongs to $\mathcal{V}$ for all $\zeta < \kappa$.

Since $\kappa = \{ \xi \mid \xi < \kappa \}$, $\kappa$ is a subset
of $\mathcal{V}$ and hence belongs to $\mathcal{V}$, finish-
ing the induction.
Finally, to see that $\text{AF}$ holds in $V$, suppose that $A$ is a non-empty set belonging to $V$, but be $a$ have minimal rank. Then since $a$ is also a subset of $V$, $b$ belongs to $V$ too. Moreover, if $c \in b$, then $\text{rk}(c) < \text{rk}(b)$ and so $c \in a$. It follows that $b \cap a = \emptyset$, so $\text{AF}$ holds in $V$.

\begin{center}
Inaccessible cardinals and models of ZFC
\end{center}

Suppose ZFC holds in our universe $V$. By induction on ordinals, we see that $\xi \subseteq V_{\eta}$ for any ordinal $\xi$.

**Lemma** If $\kappa$ is a strongly inaccessible cardinal, then $\text{V}_{\kappa}$ is a model of ZFC and for any $a \in V_{\kappa}$,

$$a \in V_{\kappa} \iff |a| < \kappa$$
Proof: Since $\kappa \in V_\kappa$, clearly $|V_\kappa| \geq \kappa$.

Conversely, by induction we show that for any $\xi < \kappa$, $|V_\xi| < \kappa$, which implies that $|V_\kappa| = \kappa$.

For the induction, note that if $|V_\xi| < \kappa$, then $|V_{\xi+1}| = |\mathcal{P}(V_\xi)| = 2^{|V_\xi|} < \kappa$. And if $|V_\xi| < \kappa$ for all $\xi < \lambda < \kappa$, then $|V_\lambda| = \sup_{\xi < \lambda} |V_\xi| < \kappa$, by regularity of $\kappa$.

Thus, if $a \in V_\kappa$, $|a| < \kappa$, then $rk: a \to \kappa$ cannot be cofinal in $\kappa$, since $\kappa$ is regular. So in some $\beta < \kappa$, $a \in V_\beta$, whence $a \in V_{\beta+1} \subseteq V_\kappa$.

Lemma: If $\kappa$ is inaccessible, then $V_\kappa$ satisfies ZFC.

Proof: we can include extensionality, union, powerset, sets existence, axiom of infinity and AC in the structure $(V_\kappa, \in)$. 63
So let us check AC and replacement.

If \( a \in V_k \) is a family of pairwise disjoint non-empty sets, then we know that there is some \( T \) which is a transversal for \( a \), i.e., if \( a \in T \) then \( T \) contains a single element of each set in \( a \), and \( T \subseteq Ua \). Since \( a \) is a subset of \( V_k \), so are all its subsets, whence \( T \in V_k \). Moreover, since \( a \in V_k \),

\[ |T| = |a| < \kappa, \text{ whence } T \in V_k. \]

So AC holds in \( V_k \).

For replacement, suppose \( \phi(x, y) \) is a formula with parameters in \( V_k \), defining a class function in \( V_k \), i.e.,

\[ \forall x \in V_k \exists ! y \in V_k \phi^V_k(x, y), \]

and choose \( a \in V_k \).

Then \( \psi(x, y) \) given by

\[ x \in V_k \land y \in V_k \land \phi^V_k(x, y) \]

defines a function \( f \) with domain contained in \( V_k \). So \( f[a] \) is a subset of \( V_k \) of size \( < \kappa \), whence \( f[a] \in V_k \), verifying replacement.
Theorem If ZFC is consistent, then so is ZFC + "there are no strongly inaccessible cardinals".

Remark We note that this is a statement of the metatheory, that is, not a statement of the first order language of set theory. We are claiming, that if there is no way of obtaining a contradiction from ZFC, then the same holds for ZFC + "there are no strongly inaccessible cardinals".

Proof Suppose \( U \) is a universe of set theory satisfying ZFC. If there are no inaccessible cardinals in \( U \), we are done. So suppose there are some and let \( K \) be the minimal of these.

We will now show that

\[
(\text{"there are no strongly inaccessible cardinals"})^K
\]

holds.

Note first that since AF holds, a set \( x \) is an ordinal if and only if \( x \) is transitive and totally ordered by \( \varepsilon \), i.e.,

\[
(\forall x, y \in x) (x \in y \lor y \in x \lor x = y) \land (\forall x)((x \in x) \rightarrow (x \in x)).
\]
The cardinals in \( V_\kappa \) are simply the cardinals below \( \kappa \), \( \text{Ord} V_\kappa = \kappa \).

First note that \( \kappa \in V_\kappa \), so if \( \lambda < \kappa \), then \( \lambda \in V_\kappa \) and \( (*) \) holds for \( \lambda \).

But then also \( (*)^{V_\kappa} \) holds, whence \( \kappa \) is an ordinal in \( V_\kappa \), i.e., \( \kappa \) belongs to \( \text{Ord} V_\kappa \).

Conversely, if \( \kappa \) belongs to \( \text{Ord} V_\kappa \), then \( \kappa \in V_\kappa \) and is transitive and totally ordered by \( \in \), whence \( \kappa \) is an ordinal, \( \forall \lambda \in \kappa, \forall \xi \in \kappa \).

The cardinals in \( V_\kappa \) are the cardinals below \( \kappa \).

If \( \lambda \) is a cardinal in \( V_\kappa \), then it is an ordinal in \( V_\kappa \) and \( \exists \mu \in \kappa \) such that \( \lambda \leq \mu \). To see that \( \lambda \) is also an actual cardinal, note that if \( f: \lambda \rightarrow \kappa \) were a bijection of \( \lambda \) with an ordinal \( \kappa \), then \( f \in V_\kappa \), \( |f| < \kappa \).
and thus \( \lambda \in V_\kappa \). It follows that \( \lambda \) would be a bijection in \( V_\kappa \) between \( \kappa \) and a smaller ordinal, contradicting that \( \kappa \) is a cardinal in \( V_\kappa \).

And similarly, if \( \lambda < \kappa \) is a cardinal, then \( \lambda \in V_\kappa \) and \( \lambda \) is an ordinal in \( V_\kappa \). However, any bijection in \( V_\kappa \) between \( \lambda \) and a smaller ordinal would also be a bijection in \( V_\kappa \) between \( \lambda \) and a smaller ordinal, contradicting that \( \lambda \) is a cardinal.

\[ \Delta \]

No cardinal \( \lambda \in V_\kappa \) is strongly inaccessible in \( V_\kappa \).

\[ \text{Proof:} \] If \( \lambda \in V_\kappa \) is a cardinal in \( V_\kappa \), then \( \lambda < \kappa \) is also a cardinal in \( V_\kappa \), but not strongly inaccessible. So either

(i) \( \lambda < \omega \), whence \( \lambda \in (\lambda < \omega)^{V_\kappa} \),

(ii) \( \lambda < 2^\kappa \) for some cardinal \( \eta < \lambda < \kappa \), whence \( \eta, 2^\kappa \in V_\kappa \) and also

\[ (\eta < \lambda < 2^\kappa)^{V_\kappa} \], as
The reflection scheme

**Definition.** Suppose $C$ is a class and \( \phi(x_1, \ldots, x_n) \) is a formula all of whose parameters belong to $C$. We say that \( \phi(x) \) is **absolute for $C$** if for all $a_1, \ldots, a_n$ in $C$,

\[
\phi(a_1, \ldots, a_n) \iff \phi^C(a_1, \ldots, a_n),
\]

if and only if

\[
\forall x_1, \ldots, x_n \ (C(x_1) \land \cdots \land C(x_n) \implies \\
(\phi(x_1, \ldots, x_n) \iff \phi^C(x_1, \ldots, x_n))
\]

Since the relativization \( \phi^C \) of a quantifier-free formula \( \phi \) itself, any
quantifiers \( \phi \) is absolute for \( C \).

A formula \( \phi(x_1, \ldots, x_n) \) is said to be in

prefix-form if \( \phi = Q_1y_1Q_2y_2 \cdots Q_my_m \Psi \), where

\( Q_i \) are quantifiers and \( \Psi \) is quantifier-free.

**Obs.** The class of formulas absolute for \( C \)

is closed under logical equivalence and Boolean combinations. That is, if \( \vdash \Psi(x) \leftrightarrow \phi(x) \),

then \( \Psi \) is absolute for \( C \) if and only

if \( \phi \) is absolute for \( C \). This follows from:

\[
\vdash (\Psi \rightarrow \phi) \Rightarrow \vdash (\Psi^C \rightarrow \phi^C),
\]

which can easily be proved by induction on

proofs or by model-theoretic considerations.

Since every formula is logically equivalent to

one in prefix-form, when dealing with

absoluteness it suffices to consider formulas

in prefix-form.
Lemma: Suppose \( \phi(x_1, \ldots, x_n) \) is a formula w/o parameters in prenex-form and that

\[ (X_n)_{new} \] is an increasing sequence of sets.

If \( \phi \) and all its subformulas are absolute for every \( X_n \), then \( \phi \) and all of its subformulas are absolute for \( X = \bigcup_{new} X_n \).

Proof: The result is proved by induction on the length of the prenex of \( \phi \).

If \( \phi \) is quantifier-free, then \( \phi \) is absolute for any class or set, so the result is trivial.

Suppose now that the result is true for

\[ \psi(y, x_1, \ldots, x_n) \] and let \( \phi(x) = \exists y \psi(y, x) \).

Then for any \( e_1, \ldots, e_n \in X \), choose \( k \leq n \)

\[ e_1, \ldots, e_n \in X_k. \]

Now, if \( \phi(\overline{e}) \) holds, then since \( \phi \) is absolute for \( X_k \), also \( \phi^{x_k}(\overline{e}) \) holds, so for some \( b \in X_k \), \( \psi^{x_k}(b, \overline{e}) \) holds.

As \( \psi \) is absolute for \( X_k \), we get that
$\psi(b,\bar{z})$, and as $\psi$ is absolute for $X$ we get that $\psi^X(b,\bar{z})$.
Thus, finally, $\exists b \in X \psi^X(b,\bar{z})$, i.e., $\phi^X(\bar{z})$.

Conversely, if $\phi^X(\bar{z})$, then for some $b \in X$, $\psi^X(b,\bar{z})$. Since $\psi$ is absolute for $X$, also $\psi(b,\bar{z})$ and so $\exists y \psi(y,\bar{z})$, i.e., $\phi(\bar{z})$.

Universal quantification is proved similarly. □

**Theorem** (The reflection scheme) (ZF).

Suppose $\phi(\bar{x})$ is a formula w/o parameters. Then for every $\beta$ there is a limit ordinal $\beta > \alpha$ such that $\phi$ is absolute for $\mathcal{V}_\beta$. 
