Theorem Suppose A is a class and H is a class function of one variable defined on the class of all H-inductive functions $f : \alpha \to X$, where α is an ordinal and X is a subset of A. Assume also that H only takes values in A.
Then there is a unique class function F, i.e., given by a formula of set theory, such that

- F is defined on Ord
- $\forall \alpha \quad F(\alpha) = H(F\restriction_\alpha)$

Proof The class function F is defined by

$y = F(\alpha) \iff$ there is an H-inductive function $f : \alpha \to X$, $X \subseteq A$, and $y = H(f)$.

$\square$

Stratified or ranked classes

A class W is said to be <u>stratified</u> or <u>ranked</u> if there is a class function ρ with domain W and taking values in Ord such that for any ordinal α, the class

$$W_\alpha = \{x \mid W(x) \ \& \ \rho(x) < \alpha\}$$

is a set.

<u>Thm</u> Suppose W is a stratified class with correspon-ding stratification ρ. Let also M be the class of all functions with domain W_α for some α and $H(x,f)=y$ a class function with domain $W \times M$.

Then there is a unique class function F with domain W such that for any a in W:

$$F(a) = H\left(a, F \upharpoonright W_{\rho(a)}\right).$$

<u>Note</u> that for a in W, $a \in W_{\rho(a)+1} \setminus W_{\rho(a)}$.

<u>Proof</u> Set $W'_\alpha = \{x \mid W(x) \ \& \ \rho(x) = \alpha\}$ and note that W'_α is a set and

$$W_\alpha = \bigcup_{\beta<\alpha} W'_\beta.$$

By induction on Ord, i.e., by applying the preceding theorem, we find a class function G defined on ordinals s.t. for every α,

$$G(\alpha) = \text{the function } \phi \text{ with domain } W'_\alpha \text{ s.t.}$$

$$\phi(x) := H\left[x, \bigcup_{\beta<\alpha} G(\beta)\right] \quad \text{for all } x \in W'_\alpha .$$

(Note that $G(\alpha)$ is written as a function of $G\restriction_\alpha$, so the theorem applies.)

Also, for any a in W,

$$G(\rho(a)) = H\left[\cdot, \bigcup_{\beta<\rho(a)} G(\beta)\right] \quad \text{with domain } W'_{\rho(a)}$$

We can now write

$$F(a) = b \iff W(a) \ \& \ b = G(\rho(a))(a) .$$

To see that this works, note that

$$F\restriction_{W_\alpha} = \bigcup_{\beta<\alpha} G(\beta) \quad ; \text{ for let } a \in W_\alpha \text{, say}$$

$a \in W'_\beta$ for some $\beta = \rho(a) < \alpha$ and so

$$F(a) = G(\beta)(a) .$$

It thus follows that for a in W

$$F(a) = G(\rho(a))(a) = H\left[a, F\restriction_{W_{\rho(a)}}\right] .$$

 Uniqueness is an exercise. $\square$

(8) Axiom of choice (AC)

For any set X and $A \subseteq \mathcal{P}(X)$ set of pairwise disjoint non-empty subsets of X, there is a set $T \subseteq X$ such that

$\forall a \in A$ $T \cap a$ contains exactly one elt.

—

Define also the statements

(AC') For any set X there is a function

$\pi : \mathcal{P}(X) \setminus \{\emptyset\} \longrightarrow X$ s.t. $\pi(a) \in a$ for every $a \subseteq X$, $a \neq \emptyset$.

(AC'') If $(X_i)_{i \in I}$ is an indexed family of non-empty sets, then $\prod_{i \in I} X_i \neq \emptyset$.

Prop AC $\Leftrightarrow$ AC' $\Leftrightarrow$ AC'' (given the background theory of axioms (1) – (7))

Theorem (Zermelo) Every set can be well-ordered.

Pf Suppose X is a set and let

$\pi : \mathcal{P}(X) \setminus \{\emptyset\} \longrightarrow X$ be a choice function, i.e.,

$\pi(a) \in a$ for $a \subseteq X$, $a \neq \emptyset$.

Proof Assume not.

Let H be the class function defined by

$$H(f) = y \iff f \text{ is a function with } \operatorname{dom}(f) = \alpha \text{ an ordinal, } \operatorname{Im}(f) \not\subseteq X \text{ and } y = \pi(X \setminus \operatorname{Im}(f))$$

(*) H is defined on the class of H-inductive functions. Also, H-inductive functions are injective.

For if $f : \alpha \to X$ is H-inductive, then for any $\beta < \alpha$, $f(\beta) = H(f\restriction_\beta) \in X \setminus \operatorname{Im}(f\restriction_\beta)$, whence for any $\gamma < \beta$, $f(\gamma) \neq f(\beta)$.

It thus follows that f is injective.

If also f were surjective, then this would induce a well-ordering of X, contradicting our assumption.

By (*), we know that there is an H-inductive class function $F : \mathrm{Ord} \to X$, which is injective by (*). But then since $\operatorname{Im}(F)$ is a set, $F^{-1} : \operatorname{Im}(F) \to \mathrm{Ord}$ is a function from a set onto Ord, which is impossible. $\square$

Note Any well-ordered set admits a choice function.

Theorem (Zorn)

Suppose $(X, \leq)$ is a partially ordered set all of whose linearly ordered subsets admits an upper bound. Then $(X, \leq)$ has a maximal element, i.e., there is $y \in X$ such that $\forall x \in X \; y \not< x$.

Proof

Let $\mathbb{A} = \{ Y \subseteq X \mid \exists x \in X \; \forall y \in Y \; y < x \}$ and let $\pi : \mathcal{P}(X) \setminus \{\emptyset\} \to X$ be a choice function. Define $\rho : \mathbb{A} \to X$ by

$$\rho(Y) = \pi\big(\{x \in X \mid \forall y \in Y \; y < x\}\big)$$

Define a class function H by

$H(f) = y \iff f$ is a function with $\mathrm{dom}(f)$ an ordinal and $\mathrm{Im}(f) \in \mathbb{A}$

& $y = \rho(\mathrm{Im}(f))$.

(*) Any H-inductive function $f : \alpha \to X$ is strictly increasing, i.e.,

$$\xi < \zeta < \alpha \implies f(\xi) < f(\zeta)$$

It follows from (*) that the image of any H-inductive function $f : \alpha \to X$

linearly ordered and hence has an upper bound $x_f \in X$, i.e., $\forall \beta < \alpha \; f(\beta) < x_f$.

Now, if $f : \alpha \to X$ is H-inductive, but H is not defined on f, then $\mathrm{Im}(f)$ has no strict majorant, whence $f(\beta) = x_f$ for some $\beta < \alpha$ and x_f thus is the maximum in $\mathrm{Im}(f)$ and a maximal element of X.

If, on the other hand, H is defined on the class of H-inductive functions, then we can find an H-inductive class function $F : \mathrm{Ord} \to X$, which by $(*)$ is strictly increasing. But then F would define an injection of a proper class into the set X, which is impossible. $\square$

<u>Note</u> Zorn's Lemma $\Rightarrow$ (AC):

Suppose A is a collection of pairwise disjoint subsets of a set X and let

$\mathcal{B} = \{ T \subseteq X \mid T$ is a partial transversal for A, i.e., $\forall a \in A \; T \cap a$ has at most a single element $\}$.

We order $\mathbb{B}$ by inclusion and note that any linearly ordered subset of $\mathbb{B}$ has an upper bound. So, by Zorn's Lemma, $\mathbb{B}$ has a maximal element, which is easily seen to be a transversal for A, i.e., intersects every element of A in a singleton.

Ordinal arithmetic

Recall that if α is an ordinal, the successor of α, i.e., the smallest ordinal strictly larger than α, is $\alpha + 1 = \alpha \cup \{\alpha\}$.

Definition An ordinal β is a successor ordinal if $\beta = \alpha + 1$ for some α.

Let also 0 denote the smallest ordinal, i.e., $0 = \emptyset$.

β is a limit ordinal if $\beta \neq 0$ and β is not a successor.

β is a natural number or a finite ordinal

$\forall \alpha \leq \beta \ (\alpha = 0 \ \vee \ \alpha$ is a successor$)$.

Let also $n = \underbrace{((0+1)+1)+\dots+1)}_{n \text{ times}}$

Note that if $(X, <_X)$ and $(Y, <_Y)$ are well-ordered sets, then we can define well-orderings on the sets

$$X \times \{0\} \cup Y \times \{1\} \quad \text{and} \quad X \times Y$$

by

$$(a, i) < (b, j) \iff \begin{cases} i = j = 0 \text{ and } a <_X b \\ i = j = 1 \text{ and } a <_Y b \\ i < j \end{cases}$$

and

$$(x_0, y_0) < (x_1, y_1) \iff \begin{cases} y_0 < y_1 \\ y_0 = y_1 \ \& \ x_0 < x_1 . \end{cases}$$

The first ordering, i.e., on $X \times \{0\} \cup Y \times \{1\}$, is said to be the <u>sum</u> of $(X, <_X)$ and $(Y, <_Y)$, while the latter is the <u>product</u>

<u>Definition</u> For ordinals α, β

- $\alpha + \beta$ denotes the unique ordinal γ that is order-isomorphic to the sum of α and β.

- $\alpha\beta$ denotes the unique ordinal γ that is order-isomorphic to the product of

α with β.

Lemma $+$ is associative and 0 is a two-sided additive identity.

$\cdot$ is associative, $\alpha \cdot 0 = 0$, $\alpha \cdot 1 = 1 \cdot \alpha = \alpha$, $\alpha \cdot (\beta + \gamma) = \alpha\beta + \alpha\gamma$, and if λ is a limit then $\alpha\lambda = \sup_{\beta < \lambda} \alpha\beta$.

For all we know till now, multiplication could be commutative, but this fails after the next axiom.

(9) Axiom of infinity

There is an ordinal which is not a natural number, i.e., an infinite ordinal,

$$\exists \alpha \, (\alpha \text{ an ordinal } \& \ \exists \beta \leq \alpha \, (\beta \neq 0 \ \& \ \beta \text{ is not a successor})).$$

Note that the natural numbers is an initial segment of the ordinals. So if we let ω denote the smallest infinite ordinal, we see that ω is a limit ordinal.

Remark $\omega \cdot 2 = \omega + \omega \neq 2 \cdot \omega = \omega$.

$\omega + 2 \neq \omega = 2 + \omega$.

Exponentiation: We define α^β by recursion on β uniformly in α by

$$\alpha^0 = 1$$

$$\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$$

$$\alpha^\beta = \sup_{\xi < \beta} \alpha^\xi \text{, whenever } \beta \text{ is a limit.}$$

Note Formally, we can see ordinal exponentiation as being given by a class function $Exp : Ord \times Ord \to Ord$ defined by the first order formula

$Exp(\alpha, \beta) = \gamma \iff$ there is a function $f : \beta + 1 \to Ord$ such that $\forall \zeta \leq \beta$

- $f(\beta) = \gamma$
- $\zeta = 0 \Rightarrow f(\zeta) = 1$
- $\zeta = \xi + 1 \Rightarrow f(\zeta) = f(\xi) \cdot \alpha$
- ζ limit $\Rightarrow f(\zeta) = \sup_{\xi < \zeta} f(\xi)$

The proof of uniqueness and existence of f follows along the same lines as the general proof for inductive definitions (or can be deduced from it directly).

Cardinals and their arithmetic

Definition

Given a set X, we define the cardinality of X, denoted $|X|$ or $\mathrm{card}(X)$, to be the smallest ordinal κ s.t. X can be well-ordered in ordertype κ, i.e., s.t. there is a bijection of X and κ.

Two sets are equipotent if there is a bijection between them. So X and Y are equipotent if and only if $|X| = |Y|$.

Thm Suppose X and Y are non-empty sets. TFAE

(i) there is an injection from X into Y

(ii) there is a surjection from Y onto X

(iii) $|X| \leq |Y|$.

Note $|X|$ and $|Y|$ are defined using (AC), (i) $\Rightarrow$ (ii) does not require (AC), while (ii) $\Rightarrow$ (i) does.

Thm (Cantor – Schröder – Bernstein)

X and Y are equipotent if and only if X injects into Y and vice versa.

This is trivial given (AC). But it is possible to give a proof without.

Theorem (Cantor) For any set X,

$$|X| < |\mathcal{P}(X)|.$$

Pf Suppose towards a contradiction that there is a surjection $\pi : X \twoheadrightarrow \mathcal{P}(X)$ and define $Y = \{x \in X \mid x \notin \pi(x)\}$. Then if $Y = \pi(y)$, we have

$$y \in \pi(y) \iff y \in Y \iff y \notin \pi(y)$$

which is impossible. $\square$

Definition An ordinal number κ is said to be a cardinal if $|\kappa| = \kappa$.

Corollary The class of cardinal numbers is a proper class.

Pf Suppose towards a contradiction that A were the set of all cardinal numbers. Then $\xi = \sup A$ is an ordinal and

$$\kappa = |\mathcal{P}(\xi)| > \xi \geq \lambda \quad \text{for any } \lambda \in A.$$

Since κ is a cardinal, this is impossible. $\square$

Definition
A set X is finite if $|X|$ is a finite ordinal and is infinite otherwise.
So X is infinite if and only if ω injects into X.

Proposition (Galileo) A set X is infinite if and only if it properly injects into itself.

Pf For one direction it suffices to note that $\alpha \longmapsto \alpha + 1$ is a proper injection of ω into itself.
For the converse, one shows by induction on finite ordinals that they are cardinal numbers. $\square$

The $\aleph$ - function :

The class of infinite cardinals, being cofinal in the proper class of ordinals, is itself a proper class well-ordered by the usual ordering of ordinals. It thus follows that there is a uniquely defined class function

$\aleph$: Ord $\longrightarrow$ Class of infinite cardinals

preserving the ordering.

For example, $\aleph_0 = \omega$ and $\aleph_{\alpha+1}$ is the smallest cardinal number larger than $\aleph_\alpha$.

Since the cardinal numbers are well-ordered and unbounded, for any cardinal number κ there is a smallest cardinal $> \kappa$, which we denote κ^+.

So, e.g., $n^+ = n+1$ for n finite and $\aleph_\alpha^+ = \aleph_{\alpha+1}$.

Note For any ordinal ξ, we have

$$|\xi| \leq \xi < |\xi|^+ .$$

For if $|\xi|^+ \leq \xi$, then $|\xi|^+ \subseteq \xi$ and thus

44 $|\xi|^+$ would inject into the smaller

cardinal $|\xi|$, which is impossible.

<u>Definition</u> Cardinals of the form κ^+ are called <u>successors</u>, while non-zero, non-successors are called <u>limit</u> cardinals.

<u>Proposition</u> The function $\aleph$ is continuous wrt the order-topology, that is, for any limit ordinal λ,

$$\aleph_\lambda = \sup_{\xi<\lambda} \aleph_\xi .$$

<u>Pf</u> Set $\gamma = \sup_{\xi<\lambda} \aleph_\xi$. Then $|\gamma| \leq \gamma < |\gamma|^+$.

Now, if $\gamma < \aleph_\lambda$, then there is some $\xi_0 < \lambda$ such that $\aleph_{\xi_0} = |\gamma|$, whence

$$\sup_{\xi<\lambda} \aleph_\xi = \gamma < |\gamma|^+ = \aleph_{\xi_0+1} \leq \sup_{\xi<\lambda} \aleph_\xi$$

which is impossible. So

$$\sup_{\xi<\lambda} \aleph_\xi \leq \aleph_\lambda \leq \sup_{\xi<\lambda} \aleph_\xi . \qquad \square$$

<u>Definition</u> A set X is <u>countable</u> if $|X| \leq \aleph_0$ and <u>uncountable</u> otherwise.

The <u>Continuum Hypothesis</u> (CH) is the statement

$$|\mathcal{P}(\omega)| = \aleph_1 .$$

Cardinal arithmetic

Definition For cardinal numbers κ and λ we set

$$\kappa \otimes \lambda = |\kappa \times \lambda|$$

$$\kappa \oplus \lambda = |\kappa \times \{0\} \cup \lambda \times \{1\}|$$

Note that both cardinal multiplication and addition are commutative and associative.

Theorem If κ is an infinite cardinal number, then

$$\kappa \otimes \kappa = \kappa$$

Pf This is by induction on κ (i.e., by induction on α in $\kappa = \aleph_\alpha$).

So suppose that for all $\beta < \kappa$,

$$|\beta \times \beta| = |\beta| \otimes |\beta| = |\beta|$$

and define a well-ordering $\prec$ on $\kappa \times \kappa$ by $(\alpha, \beta) \prec (\gamma, \eta)$

$$\iff \begin{cases} \max(\alpha,\beta) < \max(\gamma,\eta) \\ \max(\alpha,\beta) = \max(\gamma,\eta) \ \& \ \alpha < \gamma \\ \max(\alpha,\beta) = \max(\gamma,\eta) \ \& \ \alpha = \gamma \ \& \ \beta < \eta. \end{cases}$$

Since $\kappa\times\kappa$ is the increasing union of $\xi\times\xi$ for $\xi<\kappa$, each $\xi\times\xi$ is an initial segment of $(\kappa\times\kappa,\prec)$ and $(\xi\times\xi,\prec)$ is isomorphic to some ordinal γ with $|\gamma|=|\xi\times\xi|=|\xi|<\kappa$, and thus $\gamma<\kappa$, we see that $(\kappa\times\kappa,\prec)$ is order isomorphic to κ with the usual order. So $|\kappa\times\kappa|=\kappa$. $\square$

<u>Cor</u> $\kappa\oplus\kappa=\kappa$ for all infinite κ.

<u>Pf</u> $\kappa\oplus\kappa=|\kappa\times 2|\le|\kappa\times\kappa|=\kappa\otimes\kappa=\kappa$ $\square$

<u>Definition</u> For cardinal numbers κ and λ let $\kappa^\lambda=|\{f\mid f \text{ is a function from } \lambda \text{ to } \kappa\}|$.

Thus, $2^\kappa=|\mathcal{P}(\kappa)|$ by identifying a set with its characteristic function.

<u>Lemma</u> If $\lambda\ge\omega$ and $2\le\kappa\le\lambda$, then $\kappa^\lambda=2^\lambda$.

<u>Pf</u> Note that $2^\lambda=2^{\lambda\otimes\lambda}=(2^\lambda)^\lambda\geq\lambda^\lambda\geq\kappa^\lambda$ $\square$

<u>Definition</u> A function $f : \alpha \to \beta$ between ordinals α and β is said to be <u>cofinal</u> if $\mathrm{Im}(f)$ is unbounded in β, i.e., $\forall \xi < \beta \; \exists \eta < \alpha : f(\eta) \geq \xi$.

The <u>cofinality</u>, $\mathrm{cof}(\beta)$, of an ordinal β is the smallest ordinal α that maps cofinally into β. Thus $\mathrm{cof}(\beta) \leq \beta$.

<u>Note</u> $\mathrm{cof}(\beta)$ is always a cardinal.

<u>Observation</u> There is a strictly increasing cofinal map $h : \mathrm{cof}(\beta) \to \beta$.

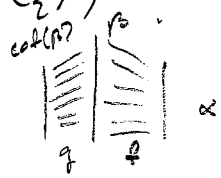
<u>Pf</u> If $f : \mathrm{cof}(\beta) \to \beta$ is cofinal, define

$$h(\xi) = \max\left(f(\xi), \sup_{\eta < \xi} (h(\eta) + 1)\right) \qquad \boxed{}$$

<u>Note</u> If α and β are limit ordinals and $f : \alpha \to \beta$ is a strictly increasing cofinal map, then $\mathrm{cof}(\alpha) = \mathrm{cof}(\beta)$.

<u>Pf</u> $\mathrm{cof}(\beta) \leq \mathrm{cof}(\alpha)$ is clear. Conversely, if $g : \mathrm{cof}(\beta) \to \beta$ is a cofinal map, then we can define $h : \mathrm{cof}(\beta) \to \alpha$ by

$h(\xi) = \min(\eta < \alpha \mid f(\eta) > g(\xi))$. Then h is cofinal in α.



$\square$

<u>Cor</u> For α limit, $\operatorname{cof}(\aleph_\alpha) = \operatorname{cof}(\alpha)$.

<u>Cor</u> For any β, $\operatorname{cof}(\beta) = \operatorname{cof}(\operatorname{cof}(\beta))$.

So the cofinality operation is idempotent, so has plenty of fixed points.

<u>Definition</u> An ordinal β is said to be <u>regular</u> if $\operatorname{cof}(\beta) = \beta$ and β is a limit.

<u>Note</u> Regular ordinals are cardinals and the first regular cardinal is ω.

<u>Lemma</u> κ^+ is regular for $\kappa \geq \omega$.

<u>Pf</u> If $f : \alpha \to \kappa^+$ is a cofinal map, then $\kappa^+ = \sup_{\xi < \alpha} f(\xi) = \bigcup_{\xi < \alpha} f(\xi)$.

Since each ordinal $f(\xi)$ is a set of cardinality $< \kappa^+$, i.e., $\leq \kappa$, we see that

$$\kappa^+ = \left| \bigcup_{\xi < \alpha} f(\xi) \right| \leq |\alpha \times \kappa| = \max\{|\alpha|, |\kappa|\}$$

and so $|\alpha| = \kappa^+$, whence $\kappa^+ \leq \alpha$. $\square$

Note. Since for a limit ordinal α, we have $\mathrm{cof}(\aleph_\alpha) = \mathrm{cof}(\alpha)$, if $\aleph_\alpha$ is regular, then

$$\aleph_\alpha = \mathrm{cof}(\aleph_\alpha) = \mathrm{cof}(\alpha) \leq \alpha \leq \aleph_\alpha, \quad \text{so } \aleph_\alpha = \alpha.$$

Definition A cardinal κ is weakly inaccessible if κ is a regular limit cardinal $> \omega$.
Moreover, κ is (strongly) inaccessible if $\kappa > \omega$, κ is regular and for any $\lambda < \kappa$, $2^\lambda < \kappa$.

One cannot prove with the axioms given (nor with the axiom of foundation) that weakly inaccessible cardinals exist.

Lemma (König)
If κ is an infinite cardinal and $\lambda \geq \mathrm{cof}(\kappa)$, then $\kappa^\lambda > \kappa$.

Pf Fix a cofinal map $f: \lambda \to \kappa$ and consider any function $G: \kappa \to \kappa^\lambda$.
We have to show that G is not surjective.
We think of κ^λ as the set of functions from λ to κ. So define $h: \lambda \to \kappa$

by $h(\xi) = \min\left(\kappa \setminus \{G(\alpha)(\xi) \mid \alpha \leq f(\xi)\}\right)$.

Note then that if $h = G(\alpha)$ for some $\alpha < \kappa$, pick $\xi < \lambda$ s.t. $f(\xi) \geq \alpha$. Then

$h(\xi) \neq G(\alpha)(\xi)$, which is a contradiction. $\square$

Corollary If $\lambda \geq \omega$, then $\mathrm{cof}(2^\lambda) > \lambda$.

Pf $(2^\lambda)^\lambda = 2^{\lambda \otimes \lambda} = 2^\lambda$, so setting $\kappa = 2^\lambda$, if $\lambda \geq \mathrm{cof}(2^\lambda)$, then the lemma would give

$2^\lambda = (2^\lambda)^\lambda > 2^\lambda$ ↯. $\square$

(9) The axiom of foundation (AF)

The axiom of foundation states that $\in$ admits no infinite descending chains or equivalently

$$\forall x \left(x \neq \emptyset \Rightarrow \exists y \in x \ \forall z \in y \ z \notin x\right).$$

Note that if $(x_n)_{n \in \omega}$ were an infinite sequence, i.e., formally a function with domain ω, st $x_{n+1} \in x_n$ for every n, then the set $x = \{x_n\}_{n \in \omega}$ would contradict (AF).

Note (AF) $\Rightarrow \forall x \ x \notin x$.

Without assuming the axiom of foundation we can define a class function $V : \mathrm{Ord} \to \mathcal{U}$ by transfinite induction:

$$V_\beta := \bigcup_{\alpha<\beta} \mathcal{P}(V_\alpha).$$

Thus, $V_0 = \emptyset$ and $\alpha \le \beta \Longrightarrow V_\alpha \subseteq V_\beta$.

From this it follows that $V_{\beta+1} = \bigcup_{\alpha \le \beta} \mathcal{P}(V_\alpha) = \mathcal{P}(V_\beta)$. So for λ limit, we get $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$.

Let also V be the class defined by

$$V(x) \iff \exists \alpha \text{ ordinal } x \in V_\alpha.$$

<u>Definition</u> For every set x in V, we let $\mathrm{rank}(x) = \mathrm{rk}(x) = \min(\alpha \mid x \in V_\alpha)$. Note that $\mathrm{rg}(x)$ is always a successor ordinal.

<u>Lemma</u> $V(x) \iff \forall y \in x \; V(y)$.

Also, if $V(x)$, then $\mathrm{rk}(y) < \mathrm{rk}(x)$ for all $y \in x$.

Pf Suppose $V(x)$ and $rk(x) = \beta+1$. Then $x \in V_{\beta+1} = \mathcal{P}(V_\beta)$ and so $x \subseteq V_\beta$. Thus $y \in x$ gives $y \in V_\beta$ and $rk(y) \leq \beta < rk(x)$.

Conversely, if $\forall y \in x\ V(y)$, then class function $rk : V \to \mathrm{Ord}$ will be bounded on the set x, whence for some β, $x \subseteq V_\beta$. Thus, $x \in V_{\beta+1}$ and thus $V(x)$. $\square$

One easily checks that:

Lemma For every ordinal α, $V(\alpha)$ and $rk(\alpha) = \alpha+1$.

Theorem AF $\iff$ $\forall x\ V(x)$.

Pf Suppose $\forall x\, V(x)$ and let a be any non-empty set. Let $b \in a$ be a set of minimal rank. Then for any $c \in b$, $rk(c) < rk(b)$, so $c \notin a$, showing AF.

For the converse we need the following definition.

Definition For any set X, define a function f with domain ω by $f(0) = X$, $f(n+1) = \bigcup_{x \in f(n)} x = \bigcup f(n)$. We let $cl(X) = \bigcup_{n \in \omega} f(n)$.

Then clearly $X \subseteq cl(X)$ and $cl(X)$ is transitive. Moreover, if Z is any transitive set containing X, then $Z \supseteq cl(X)$. So $cl(X)$ is the unique <u>transitive</u> <u>closure</u> of X.

—

Now suppose X does not belong to V, but that AF holds. Thus $X \subseteq cl(X)$ and we claim that $Y = \{ y \in cl(X) \mid \neg V(y) \}$ is non-empty. For if not, then $cl(X)$ and hence also X would be a subset of V and thus belong to V itself, which is not the case.

On the other hand, if $y \in Y$, then $\neg V(y)$ and hence y cannot be a subset of V. Thus, for some $z \in y$, $\neg V(z)$. Since $cl(X)$ is transitive, also $z \in cl(X)$ and thus $z \in Y$ too. It follows that Y is a counter-example to AF. ▨

<u>Proposition</u> $$cl(X) = X \cup \bigcup_{y \in X} cl(y)$$

<u>Proof</u> $X \subseteq cl(X)$ and the latter is ~~transitive~~, so $y \subseteq cl(X)$ for all $y \in X$. Since $cl(y)$

is the minimal transitive set containing y, also $cl(y) \subseteq cl(X)$ for any $y \in X$. So

$$cl(X) \supseteq X \cup \bigcup_{y \in X} cl(y).$$

For the other direction it suffices to note that the right hand side is transitive. $\square$

<u>Definition</u> The theory ZF^- consists of all axioms except for AC and AF. Moreover,

$$ZF = ZF^- + AF$$
$$ZFC = ZF + AC = ZF^- + AF + AC.$$

<u>Definition</u> A set X is <u>extensional</u> if

$$\forall x, y \in X \ (x \cap X = y \cap X \to x = y).$$

That is, $(X, \in) \models$ axiom of extensionality.

Note that every transitive set is extensional simply because the axiom of extensionality holds.

<u>Theorem</u> (The Mostowski collapse) (ZF).

For any extensional set X there is a unique isomorphism

$$\pi : (X, \in) \to (Y, \in)$$

onto a transitive set Y.

Proof Uniqueness follows by induction on the rank of elements of X.

Existence: We define $\pi(x)$ by induction on the rank of $x \in X$.

$$\pi(x) = \{\pi(a) \mid a \in x \cap X\}.$$

(Formally, this is induction on a stratified class).

That is, note that by extensionality of X, there is a unique element $a_0 \in X$ of minimal rank. Set $\pi(a_0) = \emptyset$.

Now, if $x \in X$ and $\pi(a)$ has been defined for all $a \in X$ with $rk(a) < rk(x)$, we set $\pi(x) = \{\pi(a) \mid a \in x \cap X\}$.

By induction on the rank, we see that π is injective and also that $\pi[X] = Y$ is a transitive set.

Finally, π is an isomorphism; for if $x, y \in X$ and $x \in y$, then $x \in y \cap X$ and so

$$\pi(x) \in \{\pi(a) \mid a \in y \cap X\} = \pi(y);$$

and, conversely, if $x, y \in X$ and $\pi(x) \in \pi(y)$ then $\pi(x) = \pi(a)$ for some $a \in y \cap X$, so by injectivity $x = a \in y$. $\square$

Relativisation

Suppose C is a class. We define for every formula $\phi(\bar{x}, \bar{a})$ in parameters $\bar{a} = (a_1, \dots, a_n)$, a_i belonging to C, the relativised formula $\phi^C(\bar{x}, \bar{a})$ by induction on ϕ:

(1) If ϕ is quantifier-free then $\phi^C = \phi$.

(2) $(\neg\phi)^C = \neg\phi^C$, $(\phi \vee \psi)^C = \phi^C \vee \psi^C$

(3) $(\exists y\, \phi)^C = \exists y\, (C(y) \wedge \phi^C)$

(4) $(\forall y\, \phi)^C = \forall y\, (C(y) \rightarrow \phi^C)$

So ϕ^C simply relativises all quantifications to C. So for any $\phi(\bar{x})$ w/o parameters and any parameters $\bar{a}$ in C we have

$$(\mathcal{U}, \in) \models \phi^C(\bar{a}) \iff (C, \in) \models \phi(\bar{a}).$$

Consistency of the axiom of foundation

Something that should be on our mind is the consistency of the axioms we have stated so far. So proofs of relative consistency are highly important. It follows from Gödel's incompleteness Theorem that we cannot prove the consistency of ZFC from ZFC itself, so the only results we can hope for are relative consistency results such as

Theorem If ZF^- is consistent, then so is $ZF = ZF^- + AF$.

In fact, we shall prove the following more precise statement.

Theorem Suppose $\mathcal{U}$ is a universe of sets satisfying the theory ZF^-. Then the class V constructed in $\mathcal{U}$ will be a universe of sets in which ZF holds.

<u>Proof</u> Recall that ZF^- is axiomatised by

(i) the axiom of extensionality

(ii) the union axiom

(iii) the powerset axiom

(iv) the axiom of replacement

(v) set existence

(vi) the axiom of infinity

We thus have to prove that if $\mathcal{U}$ is a universe of sets and V is the class of sets defined by $V = \bigcup_\alpha V_\alpha$,

then $(V, \in) \models$ (i) – (vi), i.e., for any axiom ϕ among (i) – (vi), ϕ^V holds.

(i): Suppose x, y belong to V and that for any z in V, $z \in x \Leftrightarrow z \in y$.
Then as x, y are subsets of V, we have $\forall z\, (z \in x \leftrightarrow z \in y)$, so since

extensionality holds in $\mathcal{U}$, we have $x = y$, and thus $(x = y)^V$ too.

(ii) Suppose x belongs to V. Then $\cup x = \{z \mid \exists y \in x \;\; z \in x\}$ is a subset of V and hence itself belongs to V.

(iii) If x belongs to V, then so does $\mathcal{P}(x)$; for any subset of x will be a subset of V and thus belong to V, whence $\mathcal{P}(x)$ is a subset of V and thus an element of V.

(iv) Suppose $\phi(x, y)$ is a formula with parameters in V, that defines a class function in V, i.e.,

$$\left(\forall x \, \exists^{\leq 1} y \; \phi(x,y)\right)^V$$

or more explicitly

$$\forall x \left(V(x) \rightarrow \exists^{\leq 1} y \left(V(y) \mathbin{\&} \phi^V(x,y) \right) \right).$$

Then the formula

$$\psi(x,y) := V(x) \mathbin{\&} V(y) \mathbin{\&} \phi^V(x,y)$$

defines a functional relation in $\mathcal{U}$ and thus for any set A there is by replacement in $\mathcal{U}$ a set B s.t.

$$y \in B \iff \exists x \in A \ \psi(x,y)$$
$$\iff \left(\exists x \in A \ \phi^{V}(x,y)\right) \ \& \ V(y)$$

Since B is a subset of V, B actually belongs to V and is the image of A in V of the class function defined by ϕ.

(v) is trivial since $\emptyset$ belongs to V.

(vi) We only have to show that ω belongs to V, but actually we may show that all ordinals belong to V.

This is by induction.

First $0 = \emptyset$ belongs to V.

Now suppose ξ belongs to V for all $\xi < \alpha$.

Since $\alpha = \{\xi \mid \xi < \alpha\}$, α is a subset of V and hence belongs to V, finishing the induction.

Finally, to see that AF holds in V, suppose that a is a non-empty set belonging to V. Let $b \in a$ have minimal rank. Then since a is also a subset of V, b belongs to V too. Moreover, if $c \in b$, then $rk(c) < rk(b)$ and so $c \notin a$. It follows that $b \cap a = \emptyset$, so AF holds in V. $\boxed{V}$

Inaccessible cardinals and models of ZFC

Suppose ZFC holds in our universe $\mathcal{U}$. By induction on ordinals, we see that $\xi \subseteq V_\xi$ for any ordinal ξ.

Lemma If κ is a strongly inaccessible cardinal, then $|V_\kappa| = \kappa$ and for any $a \subseteq V_\kappa$,

$$a \in V_\kappa \iff |a| < \kappa$$

Proof since $\kappa \subseteq V_\kappa$, clearly $|V_\kappa| \geq \kappa$. Conversely, by induction we show that for any $\xi < \kappa$, $|V_\xi| < \kappa$, which implies that $|V_\kappa| = \kappa$.

For the induction, note that if $|V_\xi| < \kappa$, then $|V_{\xi+1}| = |\mathcal{P}(V_\xi)| = 2^{|V_\xi|} < \kappa$. And if $|V_\xi| < \kappa$ for all $\xi < \lambda < \kappa$, λ limit, then $|V_\lambda| = \left|\bigcup_{\xi<\lambda} V_\xi\right| \leq \sup_{\xi<\lambda} |V_\xi| < \kappa$, by regularity of κ.

Thus, if $a \subseteq V_\kappa$, $|a| < \kappa$, then $rk : a \to \kappa$ cannot be cofinal in κ, since κ is regular. So for some $\beta < \kappa$, $a \subseteq V_\beta$, whence $a \in V_{\beta+1} \subseteq V_\kappa$. $\Box$

Lemma If κ is inaccessible, then V_κ satisfies ZFC.

Pf One can check extensionality, union, powerset, set existence, axiom of infinity and AF in the structure $(V_\kappa, \in)$.

So let us check AC and replacement.

If $a \in V_\kappa$ is a family of pairwise disjoint non-empty sets, then we know that there is some T which is a transversal for a, i.e., s.t. $\forall b \in a \; T \cap b$ is a singleton, and $T \subseteq \bigcup a$. Since a is a subset of V_κ, so are all its subsets, whence $T \subseteq V_\kappa$. Moreover, since $a \in V_\kappa$, $|T| = |a| < \kappa$, whence $T \in V_\kappa$. So AC holds in V_κ.

For replacement, suppose $\phi(x,y)$ is a formula with parameters in V_κ defining a class function in V_κ, i.e.,

$$\forall x \in V_\kappa \; \exists^{\leq 1} y \in V_\kappa \quad \phi^{V_\kappa}(x,y),$$

and that $a \in V_\kappa$.

Then $\psi(x,y)$ given by

$$x \in V_\kappa \;\&\; y \in V_\kappa \;\&\; \phi^{V_\kappa}(x,y)$$

defines a function f with domain contained in V_κ. So $f[a]$ is a subset of V_κ of size $< \kappa$, whence $f[a] \in V_\kappa$, verifying replacement. $\square$

Theorem If ZFC is consistent, then so is ZFC + "there are no strongly inaccessible cardinals".

Remark We note that this is a statement of the metatheory, that is, not a statement of the first order language of set theory. We are claiming, that if there is no way of obtaining a contradiction from ZFC, then the same holds for ZFC + "there are no strongly inaccessible cardinals."

Proof Suppose $\mathcal{U}$ is a universe of set theory satisfying ZFC. If there are no inaccessible cardinals in $\mathcal{U}$, we are done. So suppose there are and let κ be the minimal of these.

We will now show that

$$(\text{"there are no strongly inaccessible cardinals"})^{V_\kappa}$$

holds.

Note first that since AF holds, a set α is an ordinal if and only if α is transitive and totally ordered by $\in$, i.e.,

(*) $\forall x, y \in \alpha \, (x \in y \vee y \in x \vee x = y) \;\&\; \forall x \, (x \in \alpha \rightarrow x \subseteq \alpha)$.

- The ordinals in V_κ are simply the ordinals below κ, i.e.,

$$\mathrm{Ord}^{V_\kappa} = \kappa.$$

Pf Note that $\kappa \subseteq V_\kappa$. So if $\alpha < \kappa$, then $\alpha \in V_\kappa$ and $(*)$ holds for α. But then also $(*)^{V_\kappa}$ holds, whence α is an ordinal in V_κ, i.e., α belongs to Ord^{V_κ}.

Conversely, if α belongs to Ord^{V_κ}, then $\alpha \in V_\kappa$ and is transitive and totally ordered by $\in$, whence α is an ordinal, $|\alpha| < \kappa$, i.e., $\alpha \in \kappa$. $\triangle$

- The cardinals in V_κ are the cardinals below κ.

Pf If λ is a cardinal in V_κ, then it is an ordinal in V_κ and thus $\lambda \in \kappa$. To see that λ is also an actual cardinal, note that if $f : \lambda \leftrightarrow \alpha$ were a bijection of λ with an ordinal $\alpha < \lambda$, then $f \subseteq V_\kappa$, $|f| < \kappa$, and thus $f \in V_\kappa$. It follows that f would be a bijection in V_κ between λ and a smaller ordinal, contradicting that λ is a cardinal in V_κ.

And similarly, if $\lambda < \kappa$ is a cardinal, then $\lambda \in V_\kappa$ and λ is an ordinal in V_κ. Moreover, any bijection in V_κ between λ and a smaller ordinal would also be a bijection in $\mathcal{U}$ between λ and a smaller ordinal, contradicting that λ is a cardinal.

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- No cardinal $\lambda \in V_\kappa$ is strongly inaccessible in V_κ.

<u>Pf</u> If $\lambda \in V_\kappa$ is a cardinal in V_κ, then $\lambda < \kappa$ is also a cardinal in $\mathcal{U}$, but not strongly inaccessible. So either

(i) $\lambda \leq \omega$, whence also $(\lambda \leq \omega)^{V_\kappa}$,

(ii) $\lambda \leq 2^\eta$ for some cardinal $\eta < \lambda < \kappa$, whence $\eta, 2^\eta \in V_\kappa$ and also $(\eta < \lambda \leq 2^\eta)^{V_\kappa}$, or

(iii) there is a cofinal function $f : \alpha \to \lambda$ from an ordinal $\alpha < \lambda$, whence again $\alpha, f \in V_\kappa$ and thus λ is singular in V_κ.

So no cardinal in V_κ is inaccessible. $\square$

The reflection scheme

Definition Suppose C is a class and $\phi(x_1, \dots, x_n)$ is a formula all of whose parameters belong to C.

We say that $\phi(\bar{x})$ is absolute for C if for all $a_1, \dots, a_n$ in C,

$$\phi(a_1, \dots, a_n) \iff \phi^C(a_1, \dots, a_n).$$

I.e., if and only if

$$\forall x_1, \dots, x_n \, \big(C(x_1) \,\&\, \dots \,\&\, C(x_n) \to \big(\phi(x_1, \dots, x_n) \leftrightarrow \phi^C(x_1, \dots, x_n) \big) \big).$$

Since the relativization ϕ^C of a quantifier-free formula is ϕ itself, any quantifier-free ϕ is absolute for C.

A formula $\phi(x_1, \ldots, x_n)$ is said to be in *prenex-form* if $\phi = Q_1 y_1 Q_2 y_2 \cdots Q_m y_m \psi$, where Q_i are quantifiers and ψ is quantifier-free.

Obs The class of formulas absolute for C is closed under *logical* equivalence and Boolean combinations. That is, if $\vdash \psi(\bar{x}) \longleftrightarrow \phi(\bar{x})$, then ψ is absolute for C if and only if ϕ is absolute for C. This follows from:

$$\vdash (\psi \to \phi) \implies \vdash (\psi^C \to \phi^C),$$

which can easily be proved by induction on proofs or by model-theoretic considerations.

Since every formula is logically equivalent to one in prenex-form, when dealing with absoluteness it suffices to consider formulas in prenex-form.

Lemma Suppose $\phi(x_1, \dots, x_n)$ is a formula w/o parameters in prenex-form and that $(X_n)_{n\in\omega}$ is an increasing sequence of sets.

If ϕ and all its subformulas are absolute for every X_n, then ϕ and all of its subformulas are absolute for $X = \bigcup_{n\in\omega} X_n$.

Proof The result is proved by induction on the length of the prefix of ϕ.

If ϕ is quantifier-free, then ϕ is absolute for any class or set, so the result is trivial.

Suppose now that the result is true for $\psi(y, x_1, \dots, x_n)$ and let $\phi(\bar{x}) = \exists y\, \psi(y, \bar{x})$.

Then for any $c_1, \dots, c_n \in X$, choose $k < \omega$ s.t. $c_1, \dots, c_n \in X_k$.

Now, if $\phi(\bar{c})$ holds, then since ϕ is absolute for X_k, also $\phi^{X_k}(\bar{c})$ holds, so for some $b \in X_k$, $\psi^{X_k}(b, \bar{c})$ holds. As ψ is absolute for X_k, we get that

$\psi(b,\bar{c})$, and as ψ is absolute for X we get that $\psi^X(b,\bar{c})$.

Thus, finally, $\exists b \in X\, \psi^X(b,\bar{c})$, i.e., $\phi^X(\bar{c})$.

Conversely, if $\phi^X(\bar{c})$, then for some $b \in X$, $\psi^X(b,\bar{c})$. Since ψ is absolute for X, also $\psi(b,\bar{c})$ and so $\exists y\, \psi(y,\bar{c})$, i.e., $\phi(\bar{c})$.

Universal quantification is proved similarly. $\square$

<u>Theorem</u> (The reflection scheme) (ZF).

Suppose $\phi(\bar{x})$ is a formula w/o parameters. Then for every α there is a limit ordinal $\beta > \alpha$ such that ϕ is absolute for V_β.