(iii) there is a cofinal function $f : \alpha \to \lambda$ from an ordinal $\alpha < \lambda$, whence again $\alpha, f \in V_\kappa$ and thus λ is singular in V_κ.

So no cardinal in V_κ is inaccessible. $\square$

The reflection scheme

__Definition__ Suppose C is a class and $\phi(x_1, \ldots, x_n)$ is a formula all of whose parameters belong to C.

We say that $\phi(\bar{x})$ is __absolute__ for C if for all $a_1, \ldots, a_n$ in C,

$$\phi(a_1, \ldots, a_n) \iff \phi^C(a_1, \ldots, a_n).$$

I.e., if and only if

$$\forall x_1, \ldots, x_n \, (C(x_1) \mathbin{\&} \ldots \mathbin{\&} C(x_n) \rightarrow$$
$$(\phi(x_1, \ldots, x_n) \leftrightarrow \phi^C(x_1, \ldots, x_n))).$$

Since the relativization ϕ^C of a quantifier-free formula is ϕ itself, any quantifier-free ϕ is absolute for C.

A formula $\phi(x_1,\dots,x_n)$ is said to be in <u>prenex-form</u> if $\phi = Q_1 y_1 Q_2 y_2 \cdots Q_m y_m \psi$, where Q_i are quantifiers and ψ is quantifier-free.

<u>Obs</u> The class of formulas absolute for C is closed under <u>logical</u> equivalence and Boolean combinations. That is, if $\vdash \psi(\bar{x}) \leftrightarrow \phi(\bar{x})$, then ψ is absolute for C if and only if ϕ is absolute for C. This follows from:

$$\vdash (\psi \rightarrow \phi) \quad \Longrightarrow \quad \vdash (\psi^C \rightarrow \phi^C),$$

which can easily be proved by induction on proofs or by model-theoretic considerations.

Since every formula is logically equivalent to one in prenex-form, when dealing with absoluteness it suffices to consider formulas in prenex-form.

Lemma: Suppose $\phi(x_1, \ldots, x_n)$ is a formula w/o parameters in prenex-form and that $(X_n)_{n\in\omega}$ is an increasing sequence of sets.

If ϕ and all its subformulas are absolute for every X_n, then ϕ and all of its subformulas are absolute for $X = \bigcup_{n\in\omega} X_n$.

Proof The result is proved by induction on the length of the prefix of ϕ.

If ϕ is quantifier-free, then ϕ is absolute for any class or set, so the result is trivial.

Suppose now that the result is true for $\psi(y, x_1, \ldots, x_n)$ and let $\phi(\bar{x}) = \exists y\, \psi(y, \bar{x})$.

Then for any $c_1, \ldots, c_n \in X$, choose $k < \omega$ s.t. $c_1, \ldots, c_n \in X_k$.

Now, if $\phi(\bar{c})$ holds, then since ϕ is absolute for X_k, also $\phi^{X_k}(\bar{c})$ holds, so for some $b \in X_k$, $\psi^{X_k}(b, \bar{c})$ holds.

As ψ is absolute for X_k, we get that

$\psi(b,\bar{c})$, and as ψ is absolute for X we get that $\psi^X(b,\bar{c})$.

Thus, finally, $\exists b \in X\ \psi^X(b,\bar{c})$, i.e., $\phi^X(\bar{c})$.

Conversely, if $\phi^X(\bar{c})$, then for some $b \in X$, $\psi^X(b,\bar{c})$. Since ψ is absolute for X, also $\psi(b,\bar{c})$ and so $\exists y\, \psi(y,\bar{c})$, i.e., $\phi(\bar{c})$.

Universal quantification is proved similarly. $\square$

<u>Theorem</u> (The reflection scheme) (ZF).

Suppose $\phi(\bar{x})$ is a formula w/o parameters. Then for every α there is a limit ordinal $\beta > \alpha$ such that ϕ is absolute for V_β.

<u>Proof</u> Wlog we can suppose that ϕ is in prenex-form. We show by induction on the length of the quantifier-prefix of ϕ that:

$\forall \alpha\, \exists \beta > \alpha$ limit (any subformula of ϕ is absolute for V_β).

The base case when ϕ is quantifier-free is trivial, since ϕ is absolute for $V_{\alpha+\omega}$.

Now, suppose that the induction hypothesis holds for $\psi(y, \bar{x})$ and let $\phi(\bar{x}) = \exists y\, \psi(y, \bar{x})$.

Then, by the induction hypothesis, for any α there is $\beta > \alpha$ limit s.t. ψ and all its subformulas are absolute for V_β. Fix α.

We define a class function $F(\bar{x}) = z$

by

" $z = F(\bar{x})$ is the set of all y of minimal rank such that $\psi(y, \bar{x})$ ".

Then $\bar{x}$ belongs to the domain of F if and only if $\exists y \left(\psi(y, \bar{x}) \;\&\; y \in F(\bar{x})\right)$

We now define a strictly increasing sequence of ordinals $(\beta_n)_{n<\omega}$ as follows:

$\beta_0 = \alpha$.

β_{2n+1} = smallest ordinal $> \beta_{2n}$ such that

$F(\bar{c}) \in V_{\beta_{2n+1}}$ for every tuple

$\bar{c} = (c_1, \dots, c_k)$ in the domain of F

with $c_1, \dots, c_k \in V_{\beta_{2n}}$.

β_{2n+2} = smallest ordinal $> \beta_{2n+1}$ s.t. ψ and all its subformulas are absolute for $V_{\beta_{2n+2}}$

Now set $\beta = \sup_{n<\omega} \beta_n$, which is a limit ordinal $> \alpha$. Also, since $V_\beta = \bigcup_{n<\omega} V_{\beta_{2n+2}}$, the previous lemma implies that ψ and all its subformulas are absolute for V_β.

To finish the proof of the induction step, it thus suffices to prove that also ϕ is absolute for V_β.

We fix $c_1, \dots, c_k \in V_\beta$, say $c_1, \dots, c_k \in V_{\beta_{2n+1}}$. First let $\phi^{V_\beta}(\bar{c})$, then there is $b \in V_\beta$ such that $\psi^{V_\beta}(b, \bar{c})$. Since ψ is absolute for V_β, also $\psi(b, \bar{c})$, whence $\exists y\, \psi(y, \bar{c})$, i.e., $\phi(\bar{c})$.

Conversely, if $\phi(\bar{c})$, then there is some b of minimal rank such that $\psi(b, \bar{c})$, whence $b \in F(\bar{c})$.

It follows that $F(\bar{c}) \in V_{\beta_{2n+2}}$, and so also $b \in F(\bar{c}) \subseteq V_{\beta_{2n+2}} \subseteq V_\beta$. Thus, as ψ is absolute for V_β, we have $\psi^{V_\beta}(b, \bar{c})$ and hence $\exists y \in V_\beta \ \psi^{V_\beta}(y, \bar{c})$, i.e., $\phi^{V_\beta}(\bar{c})$.

The case of universal quantifiers is similar. Alternatively, by using $\forall = \neg\exists\neg$, one can reduce it to existential quantifiers. ∎

<u>Corollary</u> For any true sentence σ w/o parameters, there are arbitrarily large limit ordinals β s.t. σ^{V_β} holds.

Using the preceding arguments, one can prove a more general statement.

Theorem (ZF^-) Suppose $W : \mathrm{Ord} \to \mathcal{U}$ is a class function such that

- $\alpha < \beta \Rightarrow W_\alpha \subseteq W_\beta$ (increasing)
- λ limit $\Rightarrow W_\lambda = \bigcup_{\beta<\lambda} W_\beta$.

Let W be the class $\bigcup_{\beta \text{ ordinal}} W_\beta$.

Then for any formula $\phi(\bar{x})$ w/o parameters any any ordinal α, there is a limit ordinal $\beta > \alpha$ such that

$$\forall x_1, \dots, x_n \in W_\beta \left(\phi^{W_\beta}(\bar{x}) \longleftrightarrow \phi^{W}(\bar{x}) \right).$$

Formalising logic in $\mathcal{U}$

Our universe of sets $\mathcal{U}$ should be a place for all mathematics to be done. That is, all groups, manifolds, function spaces etc can be constructed as elements of $\mathcal{U}$ and all reasoning about these objects should ultimately hark back to an underlying reasoning based on ZFC. Thus, in many ways, the set theoretical language is our machine language, while concepts such as fibrebundles, C^∞-maps, solution spaces of partial differential equations are special kinds of sets defined by more or less involved definitions upon definitions.

As all other mathematical topics, logic also admits a formalisation in $\mathcal{U}$, in such a way that formulas, proofs and models simply are objects within $\mathcal{U}$. We shall give a cursory treatment of this.

<u>Definition</u> Let $\vee, \neg, \Sigma, \varepsilon, \approx$ be distinct sets in $\mathcal{U}$, e.g., $0, 1, 2, 3, 4$, and let $\mathcal{V}$ be a disjoint countable set,

say $\mathcal{V} = \{ n < \omega \mid n \geq 5 \}$, called the set of U-variables.

By induction on $n < \omega$, define a function $n \mapsto \mathcal{F}_n$ with domain ω by:

- $\mathcal{F}_0 = \{ (\varepsilon, x, y), (\approx, x, y) \mid x, y \in \mathcal{V} \}$
- $\mathcal{F}_{n+1} = \mathcal{F}_n \cup \{ (\neg, f), (\vee, f, g), (\exists x, f) \mid f, g \in \mathcal{F}_n, x \in \mathcal{V} \}$.

Finally, $\mathcal{F} = \bigcup_{n<\omega} \mathcal{F}_n$. Elements of $\mathcal{F}_0$ are called atomic U-formulas, while the elements of $\mathcal{F}$ are simply U-formulas.

For $f \in \mathcal{F}$, $l(f) = \text{length}(f) =$ minimal $n < \omega$ such that $f \in \mathcal{F}_n$.

Lemma (Unique readability)
For any U-formula $f \in \mathcal{F}$ exactly one of the following holds:

(i) f is an atomic $\mathcal{U}$-formula

(ii) $f = (\neg, g)$ for some unique $g \in \mathcal{F}$

(iii) $f = (\vee, g, h)$ —— " —— $g, h \in \mathcal{F}$

(iv) $f = (\Sigma x, g)$ —— " —— $g \in \mathcal{F}$, $x \in \mathcal{V}$.

Moreover, in each of these case $\ell(g), \ell(h) < \ell(f)$.

<u>Notation</u> For simplicity of notation, we shall write

$(x \varepsilon y)$, $(x \approx y)$, $(\neg f)$, $(f \vee g)$, $\Sigma x\,(f)$

for the $\mathcal{U}$-formulas

(ε, x, y), $(\approx, x, y)$, $(\neg, f)$, $(\vee, f, g)$, $(\Sigma x, f)$.

Similarly, the $\mathcal{U}$-formulas

$((\neg f) \vee g)$, $(\neg((\neg f) \vee (\neg g)))$, $(\neg \Sigma x\,(\neg f))$

are written

$(f \rightarrow g)$, $(f \wedge g)$, $\Pi x\,(f)$.

By induction on $l(f)$, we define for any $f \in \mathcal{F}$ the set $\mathrm{var}(f)$ of free variables in f by

- if f is $(x \varepsilon y)$ or $(x \approx y)$, then
  $\mathrm{var}(f) = \{x, y\}$
- $\mathrm{var}(\neg f) = \mathrm{var}(f)$, $\mathrm{var}(f \vee g) = \mathrm{var}(f) \cup \mathrm{var}(g)$
- $\mathrm{var}(\Sigma x(f)) = \mathrm{var}(f) \setminus \{x\}$.

Also, $f \in \mathcal{F}$ is said to be a $\mathcal{U}$-<u>sentence</u> if $\mathrm{var}(f) = \emptyset$.

<u>Note</u> For any formula $\phi(x_1, \ldots, x_n)$ of set theory there is a corresponding $\mathcal{U}$-formula f which we will denote by $\ulcorner \phi \urcorner$. Thus, while ϕ is an object of our metalanguage, $\ulcorner \phi \urcorner$ is a <u>set</u> belonging to our universe $\mathcal{U}$.

So, for example, it makes sense to quantify over $\mathcal{U}$-formulas in the language of set theory, which is not the case for true formulas of the metalanguage.

Remark also that if our universe $\mathcal{U}$ contains non-standard natural numbers, then there may be non-standard $\mathcal{U}$-formulas, i.e., $\mathcal{U}$-formulas f not of the form $\ulcorner \phi \urcorner$ for some formula ϕ of the language of set theory.

—

<u>Model theory for $\mathcal{U}$-formulas</u>

By induction on the length of $f \in \mathbb{F}$, we define for every non-empty set X, a set $\mathrm{Val}(f, X)$ by:

(i) $\mathrm{Val}((x \in y), X) = \{ \delta \in X^{\{x,y\}} \mid \delta(x) \in \delta(y) \}$

(ii) $\mathrm{Val}((x \approx y), X) = \{ \delta \in X^{\{x,y\}} \mid \delta(x) = \delta(y) \}$

(iii) $\mathrm{Val}((\neg f), X) = X^{\mathrm{var}(f)} \setminus \mathrm{Val}(f, X)$

(iv) $\mathrm{Val}((f \vee g), X) = \{ \delta \in X^{\mathrm{var}(f \vee g)} \mid$
$\delta \restriction_{\mathrm{var}(f)} \in \mathrm{Val}(f, X)$ or $\delta \restriction_{\mathrm{var}(g)} \in \mathrm{Val}(g, X) \}$.

(v) $\mathrm{Val}(\Sigma x (f), X) = \{ \delta \in X^{\mathrm{var}(f) \setminus \{x\}} \mid \exists \tilde{\delta} \in \mathrm{Val}(f, X)$
$\tilde{\delta} \restriction_{\mathrm{var}(f) \setminus \{x\}} = \delta \}$.

<u>Note</u> For any formula $\phi(x_1,\dots,x_n)$ of our meta-language, we have (modulo changing variables)

$$\mathrm{Val}(\ulcorner\phi\urcorner, X) = \{\delta : \{x_1,\dots,x_n\} \to X \mid \phi^X(\delta x_1,\dots,\delta x_n) \text{ holds}\}.$$

So we can see $\mathrm{Val}(\ulcorner\phi\urcorner, X)$ as the set

$$\{(a_1,\dots,a_n) \in X^n \mid \phi^X(a_1,\dots,a_n) \text{ holds}\}.$$

Suppose f is a $\mathcal{U}$-formula with free variables among $x_1,\dots,x_n$, written $f(x_1,\dots,x_n)$.
Assume also that γ is a function from a subset of $\mathrm{var}(f)$ into a set X.
Then we say ~~that~~ (f,γ) is a <u>$\mathcal{U}$-formula with parameters</u> in X.

For simplicity of notation, if $f(x_1,\dots,x_n,y_1,\dots,y_k)$ is given with $x_1,\dots,x_n \in \mathrm{var}(f)$ and $\gamma : \{x_1,\dots,x_n\} \longrightarrow X$ with $\gamma(x_i) = a_i$, we write $f(a_1,\dots,a_n,y_1,\dots,y_k)$ or just $f(\bar{a},\bar{y})$ for (f,γ).

In this case, $\mathrm{var}(f(\bar{a},\bar{y})) = \mathrm{var}(f) \setminus \{x_1,\dots,x_n\}$

A $\mathcal{U}$-formula f (possibly with parameters) is said to be a <u>$\mathcal{U}$-sentence</u> if $var(f) = \emptyset$. Also, $Val((f,\gamma), X) = \{\delta \in X^{var(f,\gamma)} \mid \delta \cup \gamma \in Val(f,X)\}$

If f is a $\mathcal{U}$-sentence whose parameters belong to a set X, then $Val(f,X)$ is a subset of $X^{\emptyset} = \{\emptyset\}$. If $Val(f,X) = \{\emptyset\} = 1$, we say that f is <u>true</u> in X, written $X \models f$, and if $Val(f,X) = \emptyset = 0$, f is <u>false</u> in X.

<u>Theorem</u> (Löwenheim - Skolem) (AC)
Suppose $P \subseteq X$ are sets. Then there is a subset $Y \subseteq X$ containing P, $|Y| \leq |P| + \aleph_0$, such that for any $\mathcal{U}$-sentence f with parameters in Y,

$$X \models f \iff Y \models f.$$

<u>Proof</u> Fix a choice function $\pi : \mathcal{P}(X) \setminus \{\emptyset\} \to X$, i.e., s.t. $\pi(A) \in A$ for $A \subseteq X$, $A \neq \emptyset$.

We define inductively an increasing sequence $(P_n)_{n<\omega}$ of subsets of X as follows:

- $P_0 = P$

- Given P_n, let $\mathfrak{a}_n = \{ g(\bar{a},x) \mid g(\bar{a},x)$ is a $\mathcal{U}$-formula with parameters $\bar{a}$ in X and $X \models \Sigma x\, g(\bar{a},x) \}$.

  For any $g(\bar{a},x) \in \mathfrak{a}_n$, let

  $$b_{g(\bar{a},x)} = \pi(\{ b \in X \mid X \models g(\bar{a},b) \})$$

  and set $P_{n+1} = P_n \cup \{ b_{g(\bar{a},x)} \mid g(\bar{a},x) \in \mathfrak{a}_n \}$

Since there are only countably many $\mathcal{U}$-formulas, $|P_{n+1}| \leq |P_n| + \aleph_0$, so by induction $|P_n| \leq |P| + \aleph_0$.

Set $Y = \bigcup_{n<\omega} P_n$. We show by induction on the length of a formula that if f is a $\mathcal{U}$-sentence with parameters in Y, then

$$X \models f \iff Y \models f$$

This is trivial if f is atomic and the induction steps for $\neg$ and $\vee$ are easy.

So suppose instead that

$$f = f(\bar{a}) = \Sigma x\, g(\bar{a}, x)$$

where the induction hypothesis holds for g.

If $Y \models f(\bar{a})$, then there is $b \in Y$ s.t. $Y \models g(\bar{a}, b)$, whence also $X \models g(\bar{a}, b)$ and so $X \models f(\bar{a})$.

Conversely, if $X \models f(\bar{a})$, find n large enough such that the parameters $\bar{a} = (a_1, \ldots, a_k)$ all belong to P_n. It follows that $g(\bar{a}, x) \in \mathfrak{A}_n$ and so $b_{g(\bar{a},x)} \in P_{n+1} \subseteq Y$. Since $X \models g(\bar{a}, b_{g(\bar{a},x)})$, we have by the induction hypothesis that $Y \models g(\bar{a}, b_{g(\bar{a},x)})$ and so $Y \models f(\bar{a})$. $\square$

Relativisation: As for formulas of set theory, when given a $\mathcal{U}$-formula $f(\bar{a}, \bar{x})$ with parameters in a set X, we can define the relativised formula $f^X(\bar{a}, \bar{x})$ by induction on the length of f. We then have the following easy fact:

<u>Theorem</u> Suppose $X \subseteq Y$ are sets with $X \in Y$ and $f(\bar{a}, \bar{x})$ is a $\mathcal{U}$-formula with parameters in X. Then

$$\mathrm{Val}(f(\bar{a},\bar{x}), X) = \mathrm{Val}(f^X(\bar{a},\bar{x}), Y) \cap X^{\mathrm{var}(f(\bar{a},\bar{x}))}$$

Ordinal definability

To simplify notation, if $f(\bar{a}, x)$ is a $\mathcal{U}$-formula with one free variable x and parameters in a set X, $\mathrm{Val}(f(\bar{a},x), X)$ is a subset of $X^{\{x\}}$, which we canonically can identify with a subset of X.

<u>Definition</u> (ZF) Let OD be the class of <u>ordinal</u> <u>definable</u> <u>sets</u> given by:

$OD(a) \iff$ there are ordinals $\alpha_1, \dots, \alpha_k$, $k<\omega$, and β and a $\mathcal{U}$-formula $f(x, \alpha_1, \dots, \alpha_k)$ with one free variable s.t. $\alpha_1, \dots, \alpha_k < \beta$, $a \in V_\beta$ and $\mathrm{Val}(f(x, \alpha_1, \dots, \alpha_k), V_\beta) = \{a\}$.

Obs. Note that every ordinal is ordinal definable by using itself as a parameter.

Proposition Suppose $\phi(x, \alpha_1, \dots, \alpha_n)$ is a formula of the language of set theory with ordinal parameters $\alpha_1, \dots, \alpha_n$ and suppose that a is the unique set satisfying $\phi(x, \alpha_1, \dots, \alpha_n)$. Then a is ordinal definable, i.e., belongs to OD.

Proof By the reflection scheme, we can find an ordinal β satisfying

- $\alpha_1, \dots, \alpha_n < \beta$
- $a \in V_\beta$
- $\phi(x, y_1, \dots, y_n)$ is absolute for V_β.

In particular, a is the unique element of V_β satisfying $\phi^{V_\beta}(x, \alpha_1, \dots, \alpha_n)$ and hence
$\mathrm{Val}(\ulcorner \phi(x, \alpha_1, \dots, \alpha_n) \urcorner, V_\beta) = \{a\}$. So
a is ordinal definable. $\square$

Proposition There is a formula $\Psi(x,y)$ of the language of set theory such that for any sets a,

$$OD(a) \iff \exists \gamma \text{ ordinal } \forall x \, (\Psi(x,\gamma) \leftrightarrow x = a).$$
$$\iff \exists \gamma \text{ ordinal } \Psi(a,\gamma).$$

Thus, this formula Ψ provides a uniform characterisation of ordinal definability.

Proof

Let $\mathcal{S}$ be the class of all ordinal valued functions $s : n \to \mathrm{Ord}$, with $\mathrm{dom}(s) = n$ a finite ordinal. That is, $\mathcal{S}$ is the class of finite sequences of ordinals.

For s, t in $\mathcal{S}$, we put

$$s < t \iff \sup(s) < \sup(t) \quad \text{or}$$
$$\sup(s) = \sup(t) \;\&\; \mathrm{dom}(s) < \mathrm{dom}(t)$$
$$\text{or}$$
$$\sup(s) = \sup(t),\ \mathrm{dom}(s) = \mathrm{dom}(t) \;\&$$
$$s <_{lex} t.$$

Then one can check that $<$ defines a well-ordering of $\mathcal{S}$ whose proper initial segments are sets. It follows that there is an order preserving class function

$$J : \mathrm{Ord} \to \mathcal{S}$$

Also, in ZF we can construct a bijection $K : \omega \to V_\omega$. So, in particular, since any $\mathcal{U}$-formula is an object in V_ω, K maps onto the set of $\mathcal{U}$-formulas $\mathfrak{F}$.

We can now define $\Psi(x,y)$ as follows:

$\Psi(x,y) \iff y$ is an ordinal & $J(y) = (n, \beta, \alpha_1, \dots, \alpha_k)$ is a finite sequence of ordinals s.t. $\alpha_1, \dots, \alpha_k < \beta$, $n < \omega$, $x \in V_\beta$ and $K(n) = f(z, y_1, \dots, y_k)$ is a $\mathcal{U}$-formula with

$$\mathrm{Val}\left(f(z, \alpha_1, \dots, \alpha_k), V_\beta\right) = \{x\}.$$

Then clearly

$$\begin{aligned} OD(a) &\iff \exists \gamma \text{ ordinal } \forall x \left(\Psi(x,\gamma) \leftrightarrow x = a\right) \\ &\iff \exists \gamma \text{ ordinal } \Psi(a,\gamma). \end{aligned}$$

$\square$

<u>Definition</u> The class HOD of <u>hereditarily ordinal definable sets</u> is given by

$HOD(a) \iff OD(a)$ & $cl(a)$ is a subset of OD.

Lemma $HOD(a) \iff OD(a)$ & $\forall x \in a\ HOD(x)$.

This follows easily from the fact that

$$cl(a) = a \cup \bigcup_{x \in a} cl(x).$$

Theorem Suppose $\mathcal{U}$ is a model of ZF. Then HOD is a model of ZFC.

Proof Note first that HOD is a transitive class, i.e., $a \in b$ & $HOD(b) \Rightarrow HOD(a)$, containing all ordinals.

Extensionality follows from transitivity of HOD plus extensionality in $\mathcal{U}$.

Union: Note that if $HOD(a)$ and $b = \bigcup_{x \in a} x$, then b is a subset of HOD. So to see that b belongs to HOD, we only need to see that b is ordinal definable.

So pick α s.t. $\Psi(a, \alpha)$ holds. Then b is the unique object x satisfying

$\phi(x, a) : \forall y\ (y \in x \leftrightarrow \exists z\ (z \in a$ & $y \in z))$

whence the formula

$$\exists v\ (\Psi(v, \alpha)\ \&\ \phi(x, v))$$

defines b. Thus $OD(b)$.

Powersets: Suppose HOD(a) and let

$b = \mathcal{P}(a) \cap HOD$ be the set of all hereditarily ordinal definable subsets of a. Then b is a subset of HOD and can be seen to be ordinal definable by methods as above. Then $b = \mathcal{P}^{HOD}(a)$.

Replacement

Suppose $\sigma(x, y, a_1, \ldots, a_k)$ is a formula with parameters $a_1, \ldots, a_k$ from HOD that defines a class function in HOD, i.e.

$$\forall x \left(HOD(x) \to \exists^{\leq 1} y \left(HOD(y) \;\&\; \sigma^{HOD}(x, y, \bar{a})\right)\right).$$

Suppose X is in HOD and Y is the set of images in HOD of elements of X by this class function. Clearly, Y is a subset of HOD, so we need only show that Y is in OD.

So fix ordinals $\beta, \alpha_1, \ldots, \alpha_k$ s.t. $\Psi(X, \beta)$, $\Psi(a_1, \alpha_1), \ldots, \Psi(a_k, \alpha_k)$. Then Y is the unique object satisfying

$$\phi(x, X, a_1, \ldots, a_k): \forall z \left(z \in x \leftrightarrow \exists u \left(u \in X \;\&\; HOD(z) \;\&\; \sigma^{HOD}(u, z, \bar{a})\right)\right)$$

where Υ is defined by

$$\exists v \, \exists y_1 \cdots \exists y_k \, \big(\Psi(v,\beta) \,\&\, \Psi(y_1,\alpha_1) \,\&\, \cdots \,\&\, \Psi(y_k,\alpha_k) \,\&\, \phi(x, v, y_1, \ldots, y_k) \big).$$

Infinity Since HOD(ω), HOD satisfies the axiom of infinity.

Foundation If HOD(a) and $a \neq \emptyset$, let $b \in a$ have minimal rank. Then $b \cap a = \emptyset$ and HOD(b).

Choice Note first that we have a well-ordering of HOD by

$$a < b \iff \exists \alpha \, \big(\Psi(a,\alpha) \,\&\, \forall \beta \, (\Psi(b,\beta) \rightarrow \alpha < \beta) \big)$$

That is, we well-order elements of HOD according to the minimal ordinal defining them via Ψ.

So if $X \neq \emptyset$ belongs to HOD,

$$R = \{ (a,b) \in X^2 \mid a < b \}$$

is a well-ordering of X, and thus a subset of HOD, that is clearly ordinal definable.
So R belongs to HOD and hence X can be well-ordered in HOD. $\square$

The principle of choice

Recall that we have a formula without parameters in the language of set theory, $\Psi(x,y)$, such that for any set a

$$OD(a) \iff \exists \gamma \text{ ordinal } \forall x \left(\Psi(x,\gamma) \leftrightarrow x = a\right)$$
$$\iff \exists \gamma \text{ ordinal } \Psi(a,\gamma).$$

Using Ψ we can define a well-ordering of the class OD by

$$a \prec b \iff OD(a) \,\&\, OD(b) \,\&$$
$$\exists \alpha \left(\Psi(a,\alpha) \,\&\, \forall \beta \left(\Psi(b,\beta) \to \alpha < \beta\right)\right).$$

<u>Definition</u> The principle of choice is the statement that there is a first order formula $\Phi(x,y)$ without parameters defining a well-ordering of $\mathcal{U}$.

<u>Note</u> Since this is really a disjunction over all formulas of set theory, the principle of choice is not even an axiom scheme and even less a first order axiom. However, on the basis of ZF, we shall see that it is first order.

Proposition $\mathcal{U}$ satisfies the principle of choice if and only if "$\forall x\, OD(x)$" holds.

Proof Note that in $\mathcal{U}$, $<$ defines a well-ordering of OD. Thus, if "$\forall x\, OD(x)$", i.e., $\mathcal{U} = OD$, then $<$ is a well-ordering of $\mathcal{U}$.

Conversely, suppose $\phi(x,y)$ is a formula without parameters defining a well-ordering of $\mathcal{U}$. Then there is a unique class function

$F : Ord \to \mathcal{U}$ s.t. $\alpha < \beta \iff \phi(F(\alpha), F(\beta))$.

It follows that for every α, $F(\alpha)$ is ordinal definable with parameter α, and hence $\forall x\, OD(x)$. $\square$

Thus: principle of choice $\iff V = OD \iff V = HOD$.

Constructibility d'après K. Gödel

Definition (ZF) Suppose A is a set and $X \subseteq A$ is a subset. We say that X is *definable with parameters in* A if

$\exists k<\omega\ \exists f(x, y_1, \dots, y_k)$ a $\mathcal{U}$-formula $\exists a_1, \dots, a_k \in A$

$X = Val(f(x, a_1, \dots, a_k), A)$.

Again this is a first order property in X and A, so we can define the set

$\mathcal{D}(A) = \{X \subseteq A \mid X$ is definable with parameters in $A\}$.

Example Suppose $\phi(x, y_1, \ldots, y_n)$ is a formula in the language of set theory and $a_1, \ldots, a_n \in A$ are s.t.

$$X = \{x \in A \mid A \models \phi(x, a_1, \ldots, a_n)\}.$$

Then
$$X = \mathrm{Val}(\ulcorner \phi(x, a_1, \ldots, a_n) \urcorner, A) \in \mathcal{D}(A).$$

Remark (ZFC) Suppose $|A| \geq \aleph_0$. Then $|\mathcal{D}(A)| = |A|$.

To see this, note that the set of $\mathcal{U}$-formulas with parameters in A has size $|A|$ itself, and thus also $|\mathcal{D}(A)| = |A|$.

In particular, for infinite A, $|\mathcal{D}(A)| < |\mathcal{P}(A)|$.

Also, $A \in \mathcal{D}(A)$ for any A. But if $A \subseteq B$ is not definable, then $A \notin \mathcal{D}(B)$ and so $\mathcal{D}(A) \not\subseteq \mathcal{D}(B)$. Nevertheless, we have

Theorem If $A \subseteq B$ & $A \in B$, then $\mathcal{D}(A) \subseteq \mathcal{D}(B)$.

Proof Suppose $X \in \mathcal{D}(A)$ and that $f(x, a_1, \ldots, a_n)$ is a $\mathcal{U}$-formula with parameters in A s.t.
$X = \mathrm{Val}(f(x, a_1, \ldots, a_n), A)$. Then also
$X = \mathrm{Val}(f^A(x, a_1, \ldots, a_k), B) \cap A$

$= \mathrm{Val}\left(f^A(x, a_2, \ldots, a_n) \ \& \ x \in A, B \right)$, so $x \in \mathcal{D}(B)$. ∎

By transfinite induction, we now define a hierarchy of sets by

$$L_\alpha = \bigcup_{\beta < \alpha} \mathcal{D}(L_\beta) .$$

So $L_0 = \emptyset$ and $L_\beta \subseteq L_\alpha$ for $\beta < \alpha$. Also, for $\beta < \alpha$, $L_\beta \in \mathcal{D}(L_\beta) \subseteq L_\alpha$ and hence, by the preceding theorem, $\mathcal{D}(L_\beta) \subseteq \mathcal{D}(L_\alpha)$.

It follows that the hierarchy can alternatively be described by

$$L_0 = \emptyset \quad , \quad L_{\alpha+1} = \mathcal{D}(L_\alpha)$$

and for λ limit,

$$L_\lambda = \bigcup_{\xi < \lambda} L_\xi \quad ,$$

Moreover, since $\mathcal{D}(A) \subseteq \mathcal{P}(A)$ for every set A, we see by induction on α that $L_\alpha \subseteq V_\alpha$.

Let L be the class of <u>constructible</u> <u>sets</u> defined by $L = \bigcup_{\alpha \text{ ord}} L_\alpha$. So $L \subseteq V$.

<u>Definition</u> The <u>axiom</u> of <u>constructibility</u> is the statement $\mathcal{U} = L$, which assuming ZF is just $V = L$, i.e.,

$$\forall x \, \exists \alpha \ \ x \in L_\alpha \quad .$$

Lemma L is a transitive class, that is, if A is constructible, then so is every element of A. Also, if $A \in L_\alpha$, then $A \subseteq L_\beta$ for some $\beta < \alpha$.

Proof Note that if $A \in L_{\xi+1} = \mathcal{D}(L_\xi)$, then $A \subseteq L_\xi$. $\square$

By the proof we see that any L_α is a transitive set.

For any constructible set X, we let $\operatorname{order}(X) = \min(\alpha \mid X \in L_\alpha)$.

Theorem $Ord \subseteq L$ and $Ord \cap L_\alpha = \alpha$, $\forall \alpha$. So any ordinal α is constructible with order $\alpha+1$.

Proof By induction on α, we prove $Ord \cap L_\alpha = \alpha$. So suppose this holds for all α less than some ordinal β.

If β is limit, then

$$Ord \cap L_\beta = \bigcup_{\alpha<\beta} Ord \cap L_\alpha = \bigcup_{\alpha<\beta} \alpha = \sup_{\alpha<\beta} \alpha = \beta.$$

On the other hand, if $\beta = \alpha+1$ for some α, then, by assumption, $Ord \cap L_\alpha = \alpha$ and so $\alpha \subseteq L_\alpha$. On the other hand, since $\alpha \notin L_\alpha$, $\eta \notin L_\alpha$ for any $\eta \geq \beta = \alpha \cup \{\alpha\}$.

 Thus, to see that $Ord \cap L_\beta = \beta$;

we need only show that α is definable subset of L_α, i.e., $\alpha \in \mathcal{D}(L_\alpha)$.

For this, consider the formula $\phi(x)$:

$$\forall u \forall v \left((u \in x \ \& \ v \in x) \rightarrow (u \in v \vee v \in u \vee u = v) \right)$$

$$\& \ \forall u \forall v \left(u \in v \in x \rightarrow u \in x \right),$$

which given ZF states that x is an ordinal.

Also, for any transitive class C containing a set x, the formulas $\phi(x)$ and $\phi^C(x)$ are equivalent, so, in particular,

$$\alpha = \mathrm{Val}\left(\ulcorner \phi(x) \urcorner, L_\alpha \right) \in \mathcal{D}(L_\alpha) = L_\beta. \ \blacksquare$$

Σ_1 formulas and absoluteness

We shall now consider a subclass of formulas without parameters of the language of set theory obtained by restricting quantification.

<u>Definition</u> The Σ_1-<u>formulas</u> is the smallest class of first order formulas of the language $\{\in\}$ s.t.

(i) any quantifier-free formula is Σ_1,

(ii) if ϕ, ϑ are Σ_1, then so are $(\phi \wedge \vartheta)$, $(\phi \vee \vartheta)$,

(iii) if ϕ is Σ_1, then so is $\exists x\, \phi$,

(iv) if ϕ is Σ_1, then so is $\forall x \in y\, \phi$,

i.e., $\forall x\, (x \in y \rightarrow \phi)$.

A class, class function or class relation is said to be Σ_1 if it is defined in the universe $\mathcal{U}$ by some Σ_1-formula. So a class fct is Σ_1 if it has Σ_1 graph.

<u>Remark</u> The subclass of Δ_0-formulas is obtained by replacing (iii) by the following two conditions and otherwise changing "Σ_1" to "Δ_0".

(a) if ϕ is Δ_0, then so is $\exists x \in y\, \phi$,

(b) if ϕ is Δ_0, then so is $\neg \phi$.

<u>Theorem</u> Suppose $\phi(x_1, \ldots, x_k)$ is Σ_1 and $M \subseteq N$ are classes with M transitive. Then for any $a_1, \ldots, a_k$ in M:

$$\phi^M(a_1, \ldots, a_k) \implies \phi^N(a_1, \ldots, a_k). \quad (*)$$

<u>Remark</u>
Before proving this, we should acknowledge that most formulas are not Σ_1, but only equivalent to Σ_1-formulas in some background theory. However, suppose T is a theory and $\phi(\bar{x}), \vartheta(\bar{x})$ are formulas with ϕ Σ_1 such that $T \models (\phi(\bar{x}) \rightarrow \vartheta(\bar{x}))$.

Assume also that M, N satisfy T. Then (*) implies that also

$$\vartheta^M(\bar{a}) \Rightarrow \vartheta^N(\bar{a})$$

for any $\bar{a}$ in M.

<u>Proof of theorem</u>:

We prove the result by induction on the construction of ϕ by the principles (i) – (iv). (i) and (ii) are trivial, so suppose that $\psi(x, x_1, \dots, x_k)$ satisfies the induction hypothesis and that $a_1, \dots, a_k$ belong to M.

Consider first case (iii):

If $(\exists x \psi(x, a_1, \dots, a_k))^M$ holds, then there is some b in $M \subseteq N$ s.t. $\psi^M(b, a_1, \dots, a_k)$, whence, by the induction hypothesis, also $\psi^N(b, a_1, \dots, a_k)$ and so $(\exists x \psi(x, a_1, \dots, a_k))^N$.

For case (iv).

Suppose b is in M and that $(\forall x \in b\, \psi(x, \bar{a}))^M$, i.e., $\forall x (M(x) \& x \in b \rightarrow \psi^M(x, \bar{a}))$. By the induction hypothesis for ψ, we have for any x in M,

$$\psi^M(x, \bar{a}) \Rightarrow \psi^N(x, \bar{a}).$$

So $\quad \forall x\, (M(x) \;\&\; x \in b \rightarrow \psi^N(x, \bar{a}))$.

But as M is transitive, $b \subseteq M \subseteq N$, so

$$\forall x\, (N(x) \;\&\; x \in b \rightarrow \psi^N(x, \bar{a}))$$

that is

$$(\forall x \in b\ \psi(x, \bar{a}))^N,$$

which shows the induction hypothesis for the formulas $\exists x \psi$ and $\forall x \in y\, \psi$. $\blacksquare$

<u>Lemma</u> Suppose $\psi(y, \bar{z})$ is a Σ_1-formula and $y = F(\bar{x})$ a Σ_1 class function. Then $\psi(F(\bar{x}), \bar{z})$ defines a Σ_1 class relation.

<u>Proof</u> Just note that $\psi(F(\bar{x}), \bar{z})$ is given by

$$\psi(F(\bar{x}), \bar{z}) \Leftrightarrow \exists y\, (F(\bar{x}) = y \;\&\; \psi(y, \bar{z})). \blacksquare$$

<u>Lemma</u> Suppose a is a set defined by a Σ_1 formula $\phi(x)$, i.e.,

$$x = a \iff \phi(x)$$

and that $\psi(y, \bar{z})$ is Σ_1. Then also $\psi(a, \bar{z})$ is Σ_1.

Proof $\psi(a,\bar z) \iff \exists x\ (\phi(x)\ \&\ \psi(x,\bar z))$. $\boxtimes$

Fact (ZF) Ord is a Δ_0-class and ω is defined by a Σ_1-formula.

Pf Note ~~that~~

$\mathrm{Ord}(x) \iff \forall y\in x\ \forall z\in x\ (y\in z \vee z\in y \vee z=y)$

$\&\ \forall y\in x\ \forall z\in y\ z\in x$.

For the definition of ω, we check that the following formulas are Σ_1

$\underline{x=\emptyset}$: $\forall y\in x\ y\neq y$

$\underline{y=x\cup\{x\}}$: $\forall z\in y\ (z\in x \vee z=x)\ \&$

$\forall z\in x\ z\in y\ \&\ x\in y$.

$\underline{x=\omega}$: $\emptyset\in x\ \&\ \forall y\in x\ y\cup\{y\}\in x\ \&$

$\forall y\in x\ (y=\emptyset \vee \exists z\ y=z\cup\{z\})$. $\square$

Prop (ZF) Suppose $H(\cdot)$ is a Σ_1 class function of one variable defined on all functions with ordinal domain. Then the unique class function F defined on all ordinals and satisfying

$$F(\alpha) = H(F\restriction\alpha)$$

is Σ_1 too.

Proof Again we successively verify that certain objects, classes and class relations are Σ_1:

$\underline{z = \{x, y\}}$ $\quad x \in z \;\&\; y \in z \;\&\; \forall u \in z\, (x = u \vee y = u)$.

$\underline{z = (x, y)}$ $\quad z = \{\{x\}, \{x, y\}\}$.

$\underline{y \subseteq x}$ $\quad \forall z \in y\; z \in x$

$\underline{z = x \cup y}$ $\quad \forall u \in z\, (u \in x \vee u \in y) \;\&\; \forall u \in x\; u \in z \;\&\; \forall u \in y\; u \in z$

$\underline{z = x \cap y}$ $\quad \forall u \in z\, (u \in x \;\&\; u \in y) \;\&\; \forall u \in x\, (u \in y \to u \in z)$

$\underline{z = x \setminus y}$ $\quad \forall u \in z\, (u \in x \;\&\; u \notin y) \;\&\; \forall u \in x\, (u \notin y \to u \in z)$

$\underline{z \subseteq x \times y}$ $\quad \forall u \in z\, \exists a \exists b\, (a \in x \;\&\; b \in y \;\&\; u = (a, b))$

$\underline{z \supseteq x \times y}$ $\quad \forall a \in x\, \forall b \in y\, ((a, b) \in z)$.

<u>f is a function from x to y</u> : $f \subseteq x \times y$ &

$\forall z \in x\, \exists v\, (v \in y \;\&\; (z, v) \in f)$ &

$\forall z \in x\, \forall v \in y\, \forall u \in y\, ((z, v) \notin f \vee (z, u) \notin f \vee v = u)$

[Here $(z, v) \notin f \iff \exists a\, (a \notin f \;\&\; a = (z, v))$

is Σ_1 in the variables z, v, f.]

<u>f is a function with domain x</u> $\quad \exists y\; f : x \to y$

$\underline{g = f \restriction x}$ $\quad \exists a \exists b\, (x \subseteq a \;\&\; f : a \to b \;\&\; g : x \to b \;\&\; g \subseteq f)$.

<u>f is a function</u> $\exists a \exists b \; f : a \to b$

<u>f is a function & $f(x) = y$</u> f is a function & $(x, y) \in f$.

So finally we can write $F(\alpha) = y$ by

$Ord(\alpha)$ & $\exists f$ (f is a function with domain α & $\forall \beta \in \alpha \, (f(\beta) = H(f \restriction_\beta))$ & $y = H(f)$). $\square$

<u>Fact</u> The following are Σ_1

2) <u>f is an injection from x to y</u>: $f : x \to y$ & $\forall a \in x \, \forall b \in x \, \forall c \in y \; ((a, c) \notin f \vee (b, c) \notin f \vee a = b)$

<u>f is a surjection from x to y</u> $f : x \to y$ & $\forall b \in y \, \exists a \, (a \in x \; \& \; f(a) = b)$.

<u>$h = f \circ g$</u> $\exists x \exists y \exists z \; (g : x \to y$ & $f : y \to z$ & $h : x \to z$ & $\forall a \in x \, \exists b \, \exists c \, (f(a) = b$ & $g(b) = c$ & $h(a) = c))$.

<u>Proposition</u> $B = \mathcal{D}(A)$ is Σ_1 in the variables A, B

The proof proceeds by successively verifying that the following classes and properties are Σ_1.

$z = \mathcal{F}_0$, i.e., z is the set of atomic formulas.

For this, note that "$x = 0$" and "$x = y \cup \{y\}$" are Σ_1 and so also "$x = 1$", "$x = 2$", "$x = 3$" and "$x = 4$" are Σ_1. Since $\neg, \vee, \Sigma, \varepsilon, \approx$ are resp. $0, 1, 2, 3, 4$ and $\mathcal{V} = \omega \setminus \{0, 1, 2, 3, 4\}$ is also Σ_1, we have that

$$\mathcal{F}_0 = (\{\varepsilon\} \times \mathcal{V} \times \mathcal{V}) \cup (\{\approx\} \times \mathcal{V} \times \mathcal{V})$$

is Σ_1 too.

$k < \omega$ & $z = \mathcal{F}_k$ is Σ_1 in the variables k and z:

$$\exists f \Big[(f \text{ is a fct with } \mathrm{dom}(f) = \omega) \,\&\, (f(0) = \mathcal{F}_0) \,\&\, \forall u \in \omega \big(f(u+1) = f(u) \cup (\{\neg\} \times f(u)) \cup (\{\vee\} \times f(u) \times f(u)) \cup (\{\Sigma\} \times \mathcal{V} \times f(u)) \big) \,\&\, (k < \omega) \,\&\, (z = f(k)) \Big]$$

f is a $\mathcal{U}$-formula: $\exists k \, (f \in \mathcal{F}_k)$

$z = \mathcal{F}$ the set of $\mathcal{U}$-formula.

$f \in \mathcal{F}$ & $a = \mathrm{var}(f)$ is Σ_1 in the variables f and a.

We now show that the class relation

$$\text{"} Y = X^{\mathrm{var}(f)} \ \& \ f \in \mathcal{F} \text{"}$$

in the three variables X, Y, f is Σ_1.

Note however, that the class relation

$$\text{"} Y = X^a \text{"}$$

 is not Σ_1. To see this, observe

that if Z is a transitive set, then for $Y, X, a \in Z$,

$$\left(Y = X^a\right)^Z \Leftrightarrow Y = \{ g \in Z \mid g : a \to X \}.$$

Since we can construct transitive sets Z with elements Y, X, a s.t.

$$Y = \{ g \in Z \mid g : a \to X \} \subsetneq \{ g \mid g : a \to X \},$$

being the set of functions from a to X is not preserved from Z to $\mathcal{U}$ and hence cannot be Σ_1.

<u>$k < \omega$ & $Y = X^k$</u> is Σ_1 in k, Y, X:

$$k \in \omega \ \& \ \exists f \Big[(f \text{ is a fct. with domain} = \omega) \ \& \ (f(0) = \{\emptyset\}) \ \& \ \forall n \in \omega \Big((\forall g \in f(n+1) \ g : n+1 \to X) \ \& \ (\forall h \in f(n) \ \forall x \in X \ h \cup \{(n,x)\} \in f(n+1)) \Big) \ \& \ (Y = f(k)) \Big].$$

<u>$f \in \mathcal{F}$ & $Y = X^{var(f)}$</u>

$$f \in \mathcal{F} \ \& \ \exists p \exists k < \omega \Big[(p : var(f) \to k \text{ is a bijection}) \ \& \ (\forall g \in X^k \ g \circ p \in Y) \ \& \ (\forall h \in Y \ \exists g \in X^k \ g \circ p = h) \Big].$$

<u>f is a $\mathcal{U}$-sentence with parameters in X</u>

$$\exists g \in \mathcal{F} \ \exists \delta : var(g) \to X \ \ f = (g, \delta).$$

$\underline{z = \mathcal{F}_X^\circ}$ where this latter is the set of $\mathcal{U}$-sentences with parameters in X :

$\forall f \in z$ (f is a $\mathcal{U}$-sentence with parameters in X)

& $\forall f \in \mathcal{F}$ $\forall \delta \in X^{var(f)}$ $(f, \delta) \in z$.

<u>f is a $\mathcal{U}$-formula with parameters in X and x as a single free variable</u>

$\underline{z = \mathcal{F}_X^x}$ where the latter denotes the set of $\mathcal{U}$-formulas with parameters in X and a single free variable x.

$\underline{f \in \mathcal{F}_X^\circ \;\&\; t = Val(f, X)}$ is Σ_1 in variables f, X, t and where t can take the values $0, 1$.

To see this, note that $t = Val(f, X)$ if and only if there is a function satisfying Tarski's recursive definition of truth eventually ending up with t.

$\underline{f \in \mathcal{F}_X^x \;\&\; y = Val(f, X)}$

$\underline{y \in \mathcal{D}(X)}$: $\exists x \in \mathcal{V}\ \exists f \in \mathcal{F}_X^x\ \ y = Val(f, X)$

$\underline{z = \mathcal{D}(X)}$: $\forall y \in z\ (y \in \mathcal{D}(X))$ &

$\forall x \in \mathcal{V}\ \forall f \in \mathcal{F}_X^x\ (Val(f, X) \in z)$. $\square$

<u>Corollary</u> (ZF) The class function $L : \mathrm{Ord} \to \mathcal{U}$ is Σ_1.

<u>Proof</u> L is a class function defined by transfinite recursion from the Σ_1 class function $\mathcal{D}(\cdot)$. $\square$

<u>Theorem</u> (ZF) L satisfies ZF + "$V=L$".

<u>Proof</u> Recall that L is a transitive class. Assume that L satisfies ZF. Then, as the class function

$$\alpha \longmapsto L_\alpha$$

is well-defined in any model of ZF, we have that

$$\left[\alpha \longmapsto L_\alpha\right]^L$$

is a class function defined on all ordinals α in L. Moreover, since $y = L_\alpha$ is Σ_1 in the variables y, α, we have for all y, α in L:

$$\left[y = L_\alpha\right]^L \implies y = L_\alpha .$$

Now, suppose x belongs to L and find an ordinal α s.t. $x \in L_\alpha$. Then α belongs to L and since the class of ordinals is Δ_0 and thus has Σ_1 complement, we have

$$\left[\mathrm{Ord}(\alpha)\right]^L .$$

Now, let y be the set in L s.t.

$$[y = L_\alpha].$$

Then also

$$y = L_\alpha$$

and so $x \in y$, whence also $[x \in y]^L$.

So, finally, $[x \in L_\alpha]^L$, showing that

$$[\forall x \; \exists \alpha \; x \in L_\alpha]^L$$

i.e., $[V = L]^L$.

It now remains to show that L satisfies ZF

__Extensionality__ holds in L since L is transitive.

__Union__ Suppose a is constructible, say $a \in L_\alpha$.

Then $b = \bigcup_{x \in a} x \subseteq L_\alpha$ since L_α is transitive

and

$$b = \mathrm{Val}(\Sigma_y (y \in a \;\&\; x \in y), L_\alpha) \in L_{\alpha+1}.$$

__Power set__ Suppose a is in L and let

$b = \{x \subseteq a \mid x \text{ is in } L\}$. Find also

α sufficiently large such that $b \subseteq L_\alpha$

and thus also $a \in L_\alpha$. But then

$$b = \mathrm{Val}(\Pi_y (y \in x \to y \in a), L_\alpha) \in L_{\alpha+1}.$$

Infinity ω belongs to L.

Replacement Suppose $\phi(x, y, \bar{a})$ is a formula with parameters $a_1, \dots, a_k$ in L that defines a class function in L.

Suppose c is a constructible set and set

$$b = \{ y \mid L(y) \ \& \ \exists x \in c \ \phi^L(x, y, \bar{a}) \}$$

Then $b \subseteq L_\alpha$ for some α large enough s.t. $a_1, \dots, a_k, c \in L_\alpha$. By the reflection scheme we can find $\beta > \alpha$ such that

$$\forall x, y, \bar{z} \in L_\beta \ \left(\phi^{L_\beta}(x, y, \bar{z}) \leftrightarrow \phi^L(x, y, \bar{z}) \right)$$

and so, in particular,

$$\forall x, y \in L_\beta \ \left(\phi^{L_\beta}(x, y, \bar{a}) \leftrightarrow \phi^L(x, y, \bar{a}) \right).$$

It follows that

$$b = \mathrm{Val}\left(\ulcorner \exists x \in c \ \phi(x, y, \bar{a}) \urcorner, L_\beta \right) \in L_{\beta+1}.$$

Foundation If $a \neq \emptyset$ belongs to L, pick $b \in a$ of minimal rank. Then b belongs to L and $a \cap b = \emptyset$. $\square$

Theorem

$V = L \Rightarrow$ principle of choice.

In particular, the principle of choice is consistent. Also

$$V = L \Rightarrow V = L = OD = HOD.$$

Proof List the set of $\mathcal{U}$-formulas as $(f_n)_{n<\omega}$. Suppose X is a set well-ordered by a relation $\preccurlyeq$. We then define a well-ordering $\preccurlyeq'$ of $\mathcal{D}(X)$ as follows:

The ordering $\preccurlyeq$ of X canonically induces a well-ordering $\preccurlyeq_1$ of the set $X^{<\omega}$ of finite sequences of elements of X.

Now if $A, B \in \mathcal{D}(X)$ put

$A \preccurlyeq_2 B \iff$ there is $f_n(x, \bar{y}) \in \mathcal{F}$ and $\bar{a} \in X^{<\omega}$ with $A = Val(f_n(x, \bar{a}), X)$ and for any $f_m(x, \bar{z}) \in \mathcal{F}$ and $\bar{b} \in X^{<\omega}$ with $B = Val(f_m(x, \bar{b}), X)$ either

(i) $n < m$, or

(ii) $n = m$ and $\bar{a} \preccurlyeq_1 \bar{b}$.

Finally, let for $A, B \in \mathcal{D}(X)$

$$A \preccurlyeq' B \iff \begin{cases} A, B \in X \;\&\; A \preccurlyeq_0 B \\ A, B \notin X \;\&\; A \preccurlyeq_2 B \\ A \in X \;\&\; B \notin X. \end{cases}$$

Note that then if $\preceq$ is a well-ordering of $X = L_\alpha$, then $\preceq'$ is a well-ordering of $L_{\alpha+1} = \mathcal{D}(L_\alpha)$ in which L_α is an initial segment on which the ordering agrees with $\preceq$.

Now, by transfinite induction, we define

$\preceq_0$ = trivial ordering on $L_0 = \emptyset$

$\preceq_{\alpha+1} = \preceq'_\alpha$

$\preceq_\lambda = \bigcup_{\alpha<\lambda} \preceq_\alpha$ for λ limit.

Then each $\preceq_\alpha$ is a well-ordering of L_α, for $\alpha < \beta$, $\preceq_\alpha = \preceq_\beta \upharpoonright L_\alpha$ and L_α is an initial segment of $(L_\beta, \preceq_\beta)$.

Finally, let $\preceq = \bigcup_{\alpha \text{ ord}} \preceq_\alpha$, which is a class well-ordering of L in which each initial segment is contained in some L_α.

$\square$

The generalised continuum hypothesis in L

<u>Theorem</u> (ZFC) Suppose $F(\cdot)$ is a Σ_1 class function of one variable. Then for any a in the domain of F, we have

$$|F(a)| \le |Cl(a)| + \aleph_0$$

where $Cl(a)$ is the transitive closure of x.

<u>Proof</u> Suppose $\phi(x,y)$ is a Σ_1 formula with $\phi(x,y) \Leftrightarrow F(x)=y$. Suppose a belongs to the domain of F and let α be large enough s.t. $a, F(a) \in V_\alpha$ and

$$\forall x, y \in V_\alpha \quad \left(\phi^{V_\alpha}(x,y) \longleftrightarrow \phi(x,y) \right)$$

(such an α can be found by reflection applied to ϕ).

Note that $Cl(\{a\}) = \{a\} \cup Cl(a) \subseteq V_\alpha$ since V_α is transitive and that $|Cl(\{a\})| = |Cl(a)| + 1$.

Let $\mathcal{A}$ denote the set of $\mathcal{U}$-sentences f with parameters in $Cl(\{a\})$ s.t. $V_\alpha \models f$.

By Löwenheim-Skolem applied to $Cl(\{a\}) \subseteq V_\alpha$, there is a subset $Cl(\{a\}) \subseteq X \subseteq V_\alpha$ with $|X| \le |Cl(a)| + \aleph_0$ s.t. $X \models f$ for any $f \in \mathcal{A}$.

Since V_α is transitive, it is extensional and so $V_\alpha \models \ulcorner \forall x \forall y \left(x = y \longleftrightarrow \forall z \left(z \in x \longleftrightarrow z \in y\right)\right) \urcorner$

Thus, V_α satisfies the axiom of extensionality and hence so does X.

Let now Y denote the Mostowski collapse and let $j : X \to Y$ be the corresponding isomorphism. That is, Y is a transitive set and j is a bijection such that

$$\forall x, y \in X \; (x \in y \leftrightarrow j(x) \in j(y)).$$

It follows that Y satisfies every sentence in $\mathcal{A}$ when any parameter $x \in Cl(\{a\})$ is replaced by $j(x)$. However, since $Cl(\{a\})$ is already transitive the collapsing map j is the identity on $Cl(\{a\})$ and so $Cl(\{a\}) \subseteq Y$ and $Y \models f$ for any $f \in \mathcal{A}$.

Now, $\ulcorner \exists y \, \phi(a,y) \urcorner \in \mathcal{A}$ and so $Y \models \ulcorner \exists y \, \phi(a,y) \urcorner$, whence for some $b \in Y$, we have $\phi^Y(a,b)$.

Since ϕ is Σ_1 and Y is transitive, it follows that also $\phi(a,b)$, whence $F(a) = b \in Y$.

Again as Y is transitive, $F(a) \subseteq Y$ and so $|F(a)| \leq |Y| = |X| \leq |Cl(a)| + \aleph_0$. $\square$

<u>Remark</u> Since clearly $|\mathcal{P}(\omega)| > \aleph_0 = |Cl(\omega)| + \aleph_0$, we see that the class function $\mathcal{P}(\cdot)$ cannot be Σ_1.

Remark If a is a Σ_1 definable set, i.e., the statement $x = a$ is Σ_1 in the variable x, then $|a| \leq \aleph_0$. For in this case $a = F(\emptyset)$ for a Σ_1 class function.

Theorem (ZFC)

(i) $|L_\alpha| = |\alpha|$ for any ordinal $\alpha \geq \omega$.

(ii) for any constructible set a,

$$|\mathrm{order}(a)| \leq |\mathrm{cl}(a)| + \aleph_0 .$$

Proof

(i) Since $\alpha \subseteq L_\alpha$, we have $|\alpha| \leq |L_\alpha|$.

Conversely, note that $\alpha \longmapsto L_\alpha$ is Σ_1, so, by the previous theorem, $|L_\alpha| \leq |\mathrm{cl}(\alpha)| + \aleph_0 = |\alpha|$.

(ii) Again note that $\mathrm{order}(\cdot)$ is a Σ_1 class function since

$$\mathrm{order}(a) = \alpha \iff a \in L_\alpha \;\&\; \forall \beta \in \alpha \;\; a \notin L_\beta .$$

So the result follows from the preceding theorem. $\square$

Theorem (ZFC) If $V = L$ then the generalized continuum hypothesis (GCH) holds, i.e., for any infinite cardinal κ, $2^\kappa = \kappa^+$.

Proof Suppose $a \subseteq \kappa$, then $|\text{order}(a)| \leq |Cl(a)| + \aleph_0$
$\leq \kappa$ and so $a \in L_\alpha$ for some $\alpha < \kappa^+$.
So $\mathcal{P}(\kappa) \subseteq L_{\kappa^+}$ and so $|\mathcal{P}(\kappa)| = 2^\kappa \leq |L_{\kappa^+}| = \kappa^+$. ▨

Definition A sentence is arithmetical if all quantifiers are of the form
$$\exists x \in V_\omega \quad \text{or} \quad \forall x \in V_\omega .$$

For example, since Peano arithmetic is definable in V_ω, any statement in Peano arithmetic is an arithmetical statement.

Theorem (ZF) If an arithmetical statement ϕ is provable from ZFC + V=L + GCH, then ϕ is provable from ZF.

Proof $V_\omega = L_\omega \subseteq L$, ϕ is Σ_1 and so from ZF we get that $\phi^L \Rightarrow \phi$.
Now, suppose $\psi_1, \psi_2, \ldots, \psi_n, \phi$ is a proof of ϕ from the axioms of ZFC + V=L.
If ψ_i is an axiom, then also ψ_i^L holds (as can be proven only supposing ZF in $\mathcal{U}$), so $\psi_1^L, \psi_2^L, \ldots, \psi_n^L, \phi^L, \phi$ is a proof of ϕ only using axioms of ZF. ▨

Forcing

Whereas Gödel's construction of L provided us with a model of $ZFC + V=L + GCH$, we shall now present P. Cohen's method of forcing giving us a model of $ZFC + \neg CH$.

Main idea: If $\mathcal{U}$ is a model of ZFC and M is a countable, transitive set in $\mathcal{U}$, then forcing is a method for adjoining a new set x to M, assumed to be somehow generic, to obtain a new countable transitive set $M[x]$ still satisfying ZF.

We also have tools for studying this adjoining of x to M and to control the properties of $M[x]$ in terms of M and x.

First, let us see how we can obtain countable transitive set models of ZFC.

<u>Theorem</u> Suppose T is a theory in the language of set theory extending ZFC and let m be a new constant symbol. Then if T is consistent, so is the theory T^*:

$T + T^m +$ "m is a countable, transitive set".

Proof Suppose towards a contradiction that T^* is inconsistent. Then there is a finite fragment of T^* that is inconsistent and so there are sentences $\phi_1, \ldots, \phi_n \in T$ s.t.

$$T + \bigwedge_{i=1}^{n} \phi_i^m + \text{"}m \text{ is a countable transitive set"}$$

is inconsistent.

Now, since T is consistent, let $\mathcal{U}$ be a model of T. By the reflection scheme, find an ordinal α such that $\left(\bigwedge_{i=1}^{n} \phi_i\right)^{V_\alpha}$ holds.

Also, by Löwenheim-Skolem, there is a countable set $X \subseteq V_\alpha$ such that for any $\mathcal{U}$-formula f:

$$X \models f \iff V_\alpha \models f.$$

In particular, since V_α satisfies the axiom of extensionality, so does X, and as $V_\alpha \models \ulcorner \phi_i \urcorner$ for all i, we have $X \models \ulcorner \phi_i \urcorner$ for all i.

Let $j : X \to Y$ be the canonical map from X onto its Mostowski collapse. Then Y is a countable transitive set and $Y \models \ulcorner \phi_i \urcorner$ for all i.

We can therefore expand $\mathcal{U}$ to a model of $T + \bigwedge_{i=1}^{n} \varphi_i^m$ + "m is a countable transitive set" by interpreting m as Y, contradicting our assumption. $\square$

Generic extensions

In the following, suppose M is a countable transitive set satisfying ZF in a universe $\mathcal{U}$ satisfying ZFC.

Assume also that $(\mathbb{P}, \leq) \in M$ is a poset (partially ordered set) in M, $\mathbb{P} \neq \emptyset$. Elements of $\mathbb{P}$ are called <u>forcing conditions</u> and if $p \leq q$, we say that p is <u>stronger</u> than q. Two conditions p, q are said to be <u>compatible</u> if $\exists r \in \mathbb{P}\ (r \leq p \ \&\ r \leq q)$. Otherwise, p and q are <u>incompatible</u> written $p \perp q$.

A subset $D \subseteq \mathbb{P}$ is <u>dense</u> if

$$\forall p \in \mathbb{P}\ \exists q \in D \quad q \leq p$$

and is <u>saturated</u> if

$$\forall p \in D\ \forall q \leq p \quad q \in D.$$

Moreover, $D \subseteq \mathbb{P}$ is *predense* if

$$\forall p \in \mathbb{P} \; \exists q \in D \quad p \;\&\; q \text{ are compatible.}$$

For any set $X \subseteq \mathbb{P}$, let

$$\tilde{X} = \{ p \in \mathbb{P} \mid \exists q \in X \;\; p \leq q \}$$

denote the *saturation* of X. Note that if X is predense, then $\tilde{X}$ is dense.

Now, suppose $G \subseteq \mathbb{P}$ is a subset, not necessarily belonging to M, but only to $\mathcal{U}$. We say that G is *$\mathbb{P}$-generic over M* if

(i) $\forall p \in G \; \forall q \in \mathbb{P} \; (p \leq q \rightarrow q \in G)$

(that is, G is upwards closed)

(ii) $\forall p \in G \; \forall q \in G \quad p \;\&\; q$ are compatible

(iii) $\forall D \in M$ (if D is a dense subset of $\mathbb{P}$ then $D \cap G \neq \emptyset$)

Since $\mathbb{P}$-generics are upwards closed, we see that (iii) can be replaced by either

(iii)' $\forall D \in M$ (if D is a predense subset of $\mathbb{P}$ then $D \cap G \neq \emptyset$)

or

(iii)'' $\forall D \in M$ (if D is a dense and saturated subset of $\mathbb{P}$, then $D \cap G \neq \emptyset$).

Lemma Suppose G is $\mathbb{P}$-generic over M. Then

$$\forall p \in \mathbb{P} \ (p \notin G \leftrightarrow \exists q \in G \quad p \perp q).$$

Pf Since any two elements of G are compatible if $q \in G$ and $p \perp q$, then $p \notin G$.

Conversely, suppose $p \notin G$ and consider the set

$$D = \{ q \in \mathbb{P} \mid q \leq p \ \text{ or } \ q \perp p \}.$$

We claim that D is dense. For if $r \in \mathbb{P}$ is given, then either $r \perp p$, in which case $r \in D$, or there is $q \in \mathbb{P}$ s.t. $q \leq r$ & $q \leq p$, in which case $q \in D$, showing density.

Also, since M satisfies ZF, the construction of D can be done inside M and so $D \in M$.

In other words, $D \in M$ is a dense subset of $\mathbb{P}$. So, as G is $\mathbb{P}$-generic over M, we have $G \cap D \neq \emptyset$.

So let $q \in G \cap D$ be any element. Note that if $q \leq p$, then as G is upwards closed also $p \in G$, which is not the case. So instead we must have $p \perp q$. ☒

Lemma Suppose G is $\mathbb{P}$-generic over M. Then

$$\forall p \in G \; \forall q \in G \; \exists r \in G \; (r \le p \;\&\; r \le q).$$

That is, any two elements of G have a common minorant.

Proof Let

$$D = \{ r \in \mathbb{P} \mid r \perp p \text{ or } (r \le p \;\&\; r \perp q) \text{ or } (r \le p \;\&\; r \le q) \}.$$

Again, since M satisfies ZF, the construction of D can be done in M and so $D \in M$. Moreover, D is dense: For given any $t \in \mathbb{P}$ either $t \perp p$, and so $t \in D$, or there is $s \in \mathbb{P}$ with $s \le t$ & $s \le p$. In the latter case, either $s \perp q$, whence $s \in D$, or there is some $r \le s \le p$ & $r \le q$, whence $r \in D$.

So pick some $r \in G \cap D$. Since any two elements of G are compatible, this must mean that $r \le p$ & $r \le q$, and so p, q have a common minorant in G. $\square$

By induction, we see that

Lemma Any finite subset of G has a common minorant in G.

Definition Suppose $D \subseteq \mathbb{P}$ and $p \in \mathbb{P}$. We say that D is <u>dense below</u> p if

$$\forall q \in \mathbb{P} \ (q \leq p \rightarrow \exists r \in D \ r \leq q).$$

Lemma Suppose G is $\mathbb{P}$-generic over M and assume $D \in M$ is dense below some $p \in G$. Then $G \cap D \neq \emptyset$.

Proof Note that $E = D \cup \{q \in \mathbb{P} \mid q \perp p\} \in M$ and E is dense in $\mathbb{P}$. For if $q \in \mathbb{P}$ and q has no minorant in D, then q and p cannot have any common minorant, whence $p \perp q$ and thus also $q \in E$.

It follows that $G \cap E \neq \emptyset$ and so, as any two elements of G are compatible, also $G \cap D \neq \emptyset$. $\square$

Definition A subset $X \subseteq \mathbb{P}$ is an <u>antichain</u> if $\forall p \in X \ \forall q \in X \ (p \neq q \rightarrow p \perp q)$.

An antichain is said to be <u>maximal</u> if it is not contained in any larger antichain.

<u>Note</u> An antichain is maximal if and only if it is predense in $\mathbb{P}$.

In particular, if $X \in M$ is a maximal antichain and G is $\mathbb{P}$-generic over M, then $G \cap X \neq \emptyset$. (In particular, being a maximal antichain is Δ_0).

<u>Lemma</u> (Assuming M satisfies AC)

Suppose $D \in M$ is a dense subset of $\mathbb{P}$. Then there is a maximal antichain $X \subseteq D$, $X \in M$.

<u>Proof</u> Work in M:

We order the set of all antichains $X \subseteq D$ by inclusion and note that by Zorn's lemma, this family has a maximal element X, which then is predense in $(D, \leq)$. So for any $p \in \mathbb{P}$, there is $q \in D$, $q \leq p$, and so, by predensity of X in D, some $r \in X$ compatible with q and thus also with p. So X is predense in $\mathbb{P}$ and hence a maximal antichain. $\square$

<u>Theorem</u> (Assuming M satisfies AC) Assume $G \subseteq \mathbb{P}$. Then G is $\mathbb{P}$-generic over M if and only if

(a) any two elements of G are compatible,

(b) if $X \in M$ is a maximal antichain in $\mathbb{P}$, then $G \cap X \neq \emptyset$.

Proof We have already seen that if G is $\mathbb{P}$-generic over M, then (a) and (b) hold.
For the converse, note that if G intersects any maximal antichain $X \in M$, then G also intersects any dense subset $D \in M$.
Finally, to see that G is closed upwards, suppose $p \in G$, $q \in \mathbb{P}$ and $p \leq q$. We let $D_q = \{ r \in \mathbb{P} \mid r \perp q \}$ and let $X \subseteq D_q$, $X \in M$, be a maximal antichain of the poset $(D_q, \leq)$.
Then also $X \cup \{q\} \in M$ and is a maximal antichain in $\mathbb{P}$. So $G \cap (X \cup \{q\}) \neq \emptyset$, and hence $q \in G$, since otherwise G would contain two incompatible elements. $\square$

The following result tells us that for the purposes of forcing, we can work with $(D, \leq)$ instead of $(\mathbb{P}, \leq)$ for any dense subset $D \in M$.

Thm Suppose $D \in M$ is a dense subset of $\mathbb{P}$. Then if G is $\mathbb{P}$-generic over M, also $G \cap D$ is D-generic over M. Conversely, for any $H \subseteq D$ which is D-generic over M, there is a unique $G \subseteq \mathbb{P}$ which is $\mathbb{P}$-generic over M and such that $H = G \cap D$. In fact, $G = \{ p \in \mathbb{P} \mid \exists q \in H \; q \leq p \}$.