(iii) There is a cardinal function \( f : \alpha \rightarrow \beta \) from an ordinal \( \alpha < \beta \), whence again \( \lambda, \delta \in V_\xi \) and since \( \lambda \) is singular in \( V_\xi \), so no cardinal in \( V_\xi \) is inaccessible. \( \Box \)

**The reflection scheme**

**Definition.** Suppose \( C \) is a class and \( \phi(x_1, \ldots, x_n) \) is a formula all of whose parameters belong to \( C \).

We say that \( \phi(x) \) is **absolute** for \( C \) if for all \( a_1, \ldots, a_n \) in \( C \),

\[
\phi(a_1, \ldots, a_n) \iff \phi^C(a_1, \ldots, a_n),
\]

i.e., it and only if

\[
\forall x_1, \ldots, x_n \ (C(x_1) \land \cdots \land C(x_n) \rightarrow \\
( \phi(x_1, \ldots, x_n) \iff \phi^C(x_1, \ldots, x_n) )).
\]

Since the relativization \( \phi^C \) at a quantifier-free formula \( \phi \) is itself, any
A formula \( \phi(x_1, \ldots, x_n) \) is said to be in

\textit{prenex-form} if

\[ \phi = Q_1 y_1 Q_2 y_2 \cdots Q_m y_m \psi, \]

where

\( Q_i \) are quantifiers and \( \psi \) is quantifier-free.

The class of formulas absolute for \( C \)

is closed under logical equivalence and Boolean combinations. That is, if \( \models \psi(x) \leftrightarrow \phi(x) \),

then \( \psi \) is absolute for \( C \) if and only if \( \phi \) is absolute for \( C \). This follows from:

\[ \models (\psi \rightarrow \phi) \quad \Rightarrow \quad \models (\psi^C \rightarrow \phi^C), \]

which can easily be proved by induction on

propositional or

model-theoretic considerations.

Since every formula is logically equivalent to

one in prenex-form, when dealing with

absoluteness, it suffices to consider formulas

in prenex-form.
Lemma. Suppose \( \phi(x_1, \ldots, x_n) \) is a formula w/o parameters in prenex-form and that 

\((X_n)_{n \in \mathbb{N}}\) is an increasing sequence of sets. 

If \( \phi \) and all its sub-formulas are absolute for every \( X_n \), then \( \phi \) and all of its sub-formulas are absolute for \( X = \bigcup X_n \).

Proof. The result is proved by induction on the length of the prenex of \( \phi \).

If \( \phi \) is quantifier-free, then \( \phi \) is absolute for any class or set, so the result is trivial.

Suppose now that the result is true for

\( \psi(y, x_1, \ldots, x_n) \) and let \( \phi(x) = \exists y \psi(y, x) \).

Then for any \( a_1, \ldots, a_n \in X \), choose \( k < \omega \) such that \( a_1, \ldots, a_n \in X_k \).

Now, if \( \phi(\bar{x}) \) holds, then since \( \phi \) is absolute for \( X_k \), also \( \phi^{|X_k}(\bar{x}) \) holds, so also some \( b \in X_k \), \( \psi^{|X_k}(b, \bar{x}) \) holds.

As \( \psi \) is absolute for \( X_k \), we get that
\( \forall (b, z) \), and as \( \forall \) is absolute for \( x \) we get that \( \forall^x (b, z) \).

Thus, finally, \( \exists b \in x \forall^x (b, z) \), i.e., \( \forall^x (z) \).

Conversely, if \( \forall^x (z) \), then for some \( b \in x \), \( \forall^x (b, z) \). Since \( \forall \) is absolute for \( x \), also \( \forall (b, z) \) and so \( \exists y \forall (y, z) \), i.e., \( \forall^x (z) \).

Universal quantification is proved similarly. \( \square \)

**Theorem** (The reflection scheme) (ZF).

Suppose \( \forall^x \phi \) is a formula w/o parameters.

Then for every \( \alpha \) there is a limit ordinal \( \beta > \alpha \) such that \( \phi \) is absolute for \( V_\beta \).

**Proof**. Wlog we can suppose that \( \phi \) is in prenex form. We show by induction on the length of the quantifier prefix of \( \phi \) that

\( \forall x \exists y > \alpha \text{ limit } (\forall \text{ subformulas of } \phi \text{ is absolute for } V_\beta) \).
The base case where \( \phi \) is quantifier-free is trivial, since \( \phi \) is absolute in \( V_{2+\omega} \).

Now, suppose that the induction hypothesis holds for \( \psi(y, z) \) and let \( \phi(z) = \exists y \psi(y, z) \).

Then, by the induction hypothesis, for any \( z \), there is a \( \beta > z \) limit \( \delta \). \( \psi \) and all its subformulas are absolute in \( V_\beta \). Fix \( \delta \).

We define a class function \( F(z) = z \) by

\[ z = F(z) \] is the set of all \( y \) of minimal rank such that \( \psi(y, z) \).

Thus \( z \) belongs to the domain of \( F \) if and only if \( \exists y (\psi(y, z) \land y \in F(z)) \).

We now define a strictly increasing sequence of ordinals \( (\beta_n)_{n<\omega} \) as follows:

\[ \beta_0 = z \]
\[ \beta_{2n+1} = \text{smallest ordinal} > \beta_{2n} \text{ such that} \]

\[ F(\bar{c}) \in V_{\beta_{2n+1}} \text{ for every tuple} \]

\[ \bar{c} = (c_1, \ldots, c_k) \text{ in the domain of } F \]

\[ \text{with } c_i \in V_{\beta_{2n}}. \]

\[ \beta_{2n+2} = \text{smallest ordinal} > \beta_{2n+1} \text{ and } \]

\[ \text{all its subformulas are absolute in } V_{\beta_{2n+2}}. \]

Now set \( \beta = \sup_{n<\omega} \beta_n \), which is a limit ordinal.

\[ > \kappa. \text{ Also, since } V_\beta = \bigcup_{n<\omega} V_{\beta_{2n+2}}, \text{ the previous lemma implies that } \Psi \text{ and all its subformulas are absolute for } V_\beta. \]

To finish the proof at the inductive step, it thus suffices to prove that also \( \phi \) is absolute for \( V_\beta \).

We fix \( c_1, \ldots, c_k \in V_\beta \), say \( c_1, \ldots, c_k \in V_{\beta_{2n+1}} \).

First let \( \psi_{V_\beta}(\bar{c}) \), then there is \( b \in V_\beta \)

such that \( \psi_{V_\beta}(b, \bar{c}) \). Since \( \Psi \) is absolute for \( V_\beta \), also \( \psi(b, \bar{c}) \), whence

\[ \exists y \psi(y, \bar{c}) \], i.e., \( \phi(\bar{c}) \). \]
Conversely, if \( \phi(\bar{x}) \), then there is some \( b \) of minimal rank such that \( \Psi(b, \bar{x}) \), whence \( b \in F(\bar{x}) \).

It follows that \( F(\bar{x}) \in V^{\beta_{2m+2}} \)
and so also \( b \in F(\bar{x}) \subseteq V^{\beta_{2m+2}} \subseteq V^\beta \).

Thus, as \( \Psi \) is absolute to \( V^\beta \),
we have \( \Psi^{V^\beta}(b, \bar{x}) \) and hence
\( \exists y \in V^\beta \Psi^{V^\beta}(y, \bar{x}) \), i.e., \( \phi^{V^\beta}(\bar{x}) \).

The case of universal quantifiers is similar.
Alternatively, by using \( \forall \in \mathcal{P} \), one can reduce it to the existential quantifiers.

Corollary: For any true sentence \( \sigma \) w/o parameters,
there are arbitrarily large limit ordinals \( \beta \leq \sigma^{V^\beta} \) holds.

Using the preceding arguments, one can prove a more general statement.
Theorem (ZF) Suppose $W : \text{Ord} \to \mathcal{U}$ is a class function such that

$\forall x < \beta \Rightarrow W_x \subseteq W_{\beta}$ (increasing)

If limit $\Rightarrow W_\alpha = \bigcup_{\beta < \alpha} W_\beta$.

Let $W$ be the class $\bigcup_{\beta \in \text{ordinal}} W_\beta$.

Then for any formula $\phi(\overline{x})$ with parameters any any ordinal $\chi$, there is a limit ordinal $\beta > \chi$ such that

$\forall x_1, \ldots, x_n \in W_\beta \quad (\phi^{W_\beta}(\overline{x}) \iff \phi^W(\overline{x}))$.

Formalising logic in $\mathcal{U}$

Our universe of sets $\mathcal{U}$ should be a place for all mathematics to be done. That is, all groups, manifolds, function spaces etc.

Can be constructed as elements of $\mathcal{U}$ and all reasoning about these objects should ultimately track back to $\mathcal{U}$.
to an underlying reasoning based on ZFC. Thus, in many ways, the set theoretical language is our machine language, while concepts such as fibre bundles, C*-maps, solution spaces of partial differential equations are special kinds of sets defined by more or less involved definitions upon definitions.

As all other mathematical topics, logic also admits a formalisation in $U$, in such a way that formulas, proofs and models simply are objects within $U$. We shall give a cursory treatment of this.

\textbf{Definition.} Let $v, z, \Sigma, e, \varepsilon$ be distinct sets in $U$, e.g., 0, 1, 2, 3, 4, and let $V$ be a disjoint countable set.
\[ \varphi = \{ n < \omega \mid n \geq 5 \} \], called the set of \textit{\( \omega \)-variables}.

By induction on \( n < \omega \), define a function \( n \rightarrow \mathcal{F}_n \) with domain \( \omega \) by:

\[
\mathcal{F}_0 = \{ (\varphi, x, y) \mid x, y \in \varphi \},
\]

\[
\mathcal{F}_n + 1 = \mathcal{F}_n \cup \{ (\varphi, f), (\varphi, f, g), (\varphi, x, f) \mid f, g \in \mathcal{F}_n, x \in \varphi \}.
\]

Finally, \( \mathcal{F} = \bigcup_{n<\omega} \mathcal{F}_n \). Elements of \( \mathcal{F}_0 \) are called \textit{atomical} \( \omega \)-formulas, while others elements of \( \mathcal{F} \) are simply \( \omega \)-formulas.

For \( t \in \mathcal{F} \), \( \text{length}(t) = \text{minimal } n < \omega \) such that \( t \in \mathcal{F}_n \).

\textbf{Lemma (Unique readability)}

For any \( \omega \)-formula \( t \in \mathcal{F} \), exactly one of the following holds:
(i) \( f \) is an atomic \( \mathcal{U} \)-formula

(ii) \( f = (\neg, g) \) for some unique \( g \in \mathcal{F} \)

(iii) \( f = (\vee, g, h) \) \( g, h \in \mathcal{F} \)

(iv) \( f = (\exists x, g) \) \( g \in \mathcal{F}, x \in \theta \)

Moreover, in each of these cases \( l(g), l(h) \leq l(f) \).

\[ \text{Notation: For simplicity of notation, we shall write} \]

\[ (x \in y), (x \ni y), (\neg f), (f \vee g), \exists x (f) \]

for the \( \mathcal{U} \)-formulas

\[ (\exists x, y), (x \ni y), (\neg f), (f \vee g), (\exists x, f) \]

Similarly, the \( \mathcal{U} \)-formulas

\[ ((\neg f) \vee g), (\neg ((\neg f) \vee (\neg g))), (\neg \exists x (\neg f)) \]

are written

\[ (f \rightarrow g), (f \vee g), \Pi_x (f) \]
By induction on \( k(f) \), we define for any \( f \in \mathcal{F} \) the set \( \text{var}(f) \) of free variables in \( f \) by

- if \( f \) is \( (x \in y) \) or \( (x \equiv y) \), then \( \text{var}(f) = \{x, y\} \)
- \( \text{var}(\neg f) = \text{var}(f) \), \( \text{var}(f \lor g) = \text{var}(f) \cup \text{var}(g) \)
- \( \text{var}(\exists x(f)) = \text{var}(f) \setminus \{x\} \).

Also, \( f \in \mathcal{F} \) is said to be a \( U \)-sentence if \( \text{var}(f) = \emptyset \).

Note. For any formula \( \phi(x_1, \ldots, x_n) \) of set theory, there is a corresponding \( U \)-formula \( f \) which we will denote by \( \Gamma^f \). Thus, while \( f \) is an object of our metalanguage, \( \Gamma^f \) is a set belonging to our universe \( U \).

So, for example, it makes sense to quantify over \( U \)-formulas in the language of set theory, which is not the case for true formulas of the metalanguage.
Remark also that if our universe \( U \) contains non-standard natural numbers, then there may be non-standard \( U \)-formulas, i.e., \( U \)-formulas not at
the form \( \phi^U \) for some formula \( \phi \)

at the language of set theory.

---

Model Theory for \( U \)-formulas

By induction on the length of \( f \in F \), we
define for every non-empty set \( X \)
a set \( \text{Val}(f, X) \) by:

(i) \( \text{Val}(\, (x \in y),X \, ) = \{ \delta \in X \mid \delta(x) \in \delta(y) \} \)

(ii) \( \text{Val}(\, (x \neq y),X \, ) = \{ \delta \in X \mid \delta(x) \neq \delta(y) \} \)

(iii) \( \text{Val}(\, f, X \, ) = X^{\text{var}(f)} \setminus \text{Val}(f, X) \)

(iv) \( \text{Val}(\, f \lor g, X \, ) = \{ \delta \in X^{\text{var}(f \lor g)} \mid \exists \delta \in \text{Val}(f, X) \lor \delta \in \text{Val}(g, X) \} \)

(v) \( \text{Val}(\, \Sigma f, X \, ) = \{ \delta \in X^{\text{var}(\Sigma f)} \mid \exists \delta \in \text{Val}(f, X) \}

\delta^{\text{var}(f)} \in X^2 = \delta \} \)
Note For any formula \( \phi(x_1, \ldots, x_n) \) of our metalanguage, we have (modulo changing variables)

\[
\text{Val}(\neg \phi, X) = \{ S : \exists x_1, \ldots, x_n . S \Rightarrow x \mid \phi^X(x_1, \ldots, x_n) \text{ holds} \}
\]

so we can use \( \text{Val}(\neg \phi, X) \) as the set

\[
\{ (a_1, \ldots, a_n) \in X^n \mid \phi^X(a_1, \ldots, a_n) \text{ holds} \}
\]

Suppose \( \theta \) is a \( U \)-formula with some variables among \( x_1, \ldots, x_n \), written \( \theta(x_1, \ldots, x_n) \).

Assume also that \( \gamma \) is a function from a subset of \( \text{var}(\theta) \) into a set \( X \).

Then we say that \((\theta, \gamma)\) is a \( U \)-formula

with parameters in \( X \).

For simplicity of notation, if \( \theta(x_1, \ldots, x_n, y_1, \ldots, y_k) \)

is given with \( x_1, \ldots, x_n \in \text{var}(\theta) \) and

\[ \gamma : \{ x_1, \ldots, x_n \} \rightarrow X \text{ with } \gamma(x_1) = a_1, \ldots, \gamma(x_n) = a_n \]

we write \( \theta(a_1, \ldots, a_n, y_1, \ldots, y_k) \) or just

\( \theta(a, \overline{y}) \) is \((\theta, \gamma)\).

In this case, \( \text{var}(\theta(a, \overline{y})) = \text{var}(\theta) \setminus \{x_1, \ldots, x_n\} \).
A \textit{U-lemma} \( l \) (possibly with parameters) is said to be a \textit{U-sentence} if \( \text{var}(l) = \emptyset \). Also, \( \text{Val}(l, \emptyset, X) = \{ \delta \in X^{\text{var}(l, X)} \mid \delta \cup \emptyset \in \text{Val}(l, X) \} \).

If \( l \) is a U-sentence whose parameters belong to a set \( X \), then \( \text{Val}(l, X) \) is a subset of \( X^\emptyset = \{ \emptyset \} \). If \( \text{Val}(l, X) = \{ \emptyset \} = 1 \), we say that \( l \) is true in \( X \), written \( X \models l \), and if \( \text{Val}(l, X) = \emptyset = 0 \), \( l \) is false in \( X \).

\textbf{Theorem (Łoewenheim–Skolem) (AC)}

Suppose \( P \subseteq X \) is a set. Then there is a subset \( Y \subseteq X \) containing \( P \), \( |Y| \leq |P| + \aleph_0 \), such that for any U-sentence \( l \) with parameters in \( Y \),

\[ X \models l \iff Y \models l. \]

\textbf{Proof} Fix a choice function \( \tau : \mathcal{P}(X) \setminus \{ \emptyset \} \to X \), i.e., \( \tau(A) \in A \) for \( A \subseteq X \), \( A \neq \emptyset \).
We define inductively an increasing sequence 
$(P_n)_{n<\omega}$ of subsets of $X$ as follows:

- $P_0 = \emptyset$

- Given $P_n$, let $\alpha_n = \{ g(\bar{a}, x) \mid g(\bar{a}, x) \in \Sigma \text{ a } \Sigma \text{-formula with parameters } \bar{a} \text{ in } X \text{ and } X \models \Sigma x g(\bar{a}, x) \}$.

For any $g(\bar{a}, x) \in \alpha_n$, set

$$b_{g(\bar{a}, x)} = \bigcap \{ b \in X \mid X \models g(\bar{a}, b) \}$$

and set $P_{n+1} = P_n \cup \{ b_{g(\bar{a}, x)} \mid g(\bar{a}, x) \in \alpha_n \}$

Since there are only countably many $\Sigma$-formulas, $|P_{n+1}| \leq |P_n| + \aleph_0$, so by induction $|P_n| \leq |P_1| + \aleph_0$.

Set $Y = \bigcup_{n<\omega} P_n$. We show by induction on the length of a formula that if $\phi$ is a $\Sigma$-sentence with parameters in $Y$, then

$$X \models \phi \iff Y \models \phi$$

This is trivial if $\phi$ is atomic and the induction steps for $\land$ and $\lor$ are easy.
So suppose instead that
\[ f = f(\bar{a}) = \Sigma x \cdot g(\bar{a}, x) \]
where the induction hypothesis holds for \( g \).

If \( Y = f(\bar{a}) \), then turn to \( \alpha \) if \( Y \models g(\bar{a}, b) \), whence also \( X = g(\bar{a}, b) \) and so \( X = f(\bar{a}) \).

Conversely, if \( X = f(\bar{a}) \), find \( n \) large enough such that the parameters
\[ \bar{a} = (a_1, \ldots, a_n) \]
all belong to \( P_n \).

It follows that \( g(\bar{a}, x) \in \mathcal{O}_n \) and so \( b_g(\bar{a}, x) \in P_{n+1} \subseteq Y \). Since
\[ X = g(\bar{a}, b_g(\bar{a}, x)) \]
we have by the induction hypothesis that \( Y = g(\bar{a}, b_g(\bar{a}, x)) \) and so \( Y = f(\bar{a}) \).

\[ \square \]

Relativisation: As for formulas of set theory, when given a \( N \)-formula \( f(\bar{a}, x) \) with parameters in a set \( X \), we can define the relativised formula
\[ f_X(\bar{a}, x) \] by induction on the length.
of \( f \). We then have the following easy fact:

**Theorem** Suppose \( X \subseteq Y \) are sets with \( x \in Y \) and \( f(x, x) \) is a \( U \)-formula with parameters in \( X \). Then

\[
\text{Val}(f(x, x), X) = \text{Val}(f(x, x), Y) \cap x^{\text{val}(f(x, x))}
\]

**Ordinal definability**

To simplify notation, let \( f(x, x) \) is a \( U \)-formula with one free variable \( x \) and parameters in a set \( X \), \( \text{Val}(f(x, x), X) \) is a subset of \( X^{x \times y} \), which we canonically can identify with a subset of \( X \).

**Definition (ZF)** Let OD be the class of ordinal definable sets given by

\[
\text{OD}(a) \iff \text{there are ordinals } x_1, \ldots, x_k, k \leq \omega, \\
\text{and } \beta \text{ and a } U \text{-formula } f(x_1, x_2, \ldots, x_k) \text{ with one free variable } \xi, \text{ with } x_1, \ldots, x_k < \beta, \ a \in V_\beta, \text{ and } \\
\text{Val}(f(x_1, x_2, \ldots, x_k), V_\beta) = a \text{.}
\]
Note that every ordinal is ordinal definable by using itself as a parameter.

**Proposition.** Suppose $\Phi(x, x_1, \ldots, x_n)$ is a formula of the language of set theory with ordinal parameters $x_1, \ldots, x_n$ and suppose that $a$ is the unique set satisfying $\Phi(x, x_1, \ldots, x_n)$. Then $a$ is ordinal definable, i.e., belongs to $\text{OD}$.

**Proof.** By the reflection scheme, we can find an ordinal $\beta$ satisfying

1. $x_1, \ldots, x_n < \beta$
2. $a \in V_\beta$
3. $\Phi(x, x_1, \ldots, x_n)$ is absolute for $V_\beta$.

In particular, $a$ is the unique element of $V_\beta$ satisfying $\Phi^V_\beta(x, x_1, \ldots, x_n)$ and hence $\text{Val} (\Phi(x, x_1, \ldots, x_n), V_\beta) = a$. So $a$ is ordinal definable. \(\square\)
Proposition. There is a formula \( \Psi(x, y) \) of the language of set theory such that for any set \( a \),

\[
\text{OD}(a) \iff \exists \alpha \text{ ordinal } \forall x \left( \Psi(x, \alpha) \iff x = a \right).
\]

Thus, this formula \( \Psi \) provides a uniform characterization of ordinal definability.

Proof.

Let \( \mathcal{S} \) be the class of all ordinal valued functions \( s : n \to \text{Ord} \), with \( \text{dom}(s) = n \) a finite ordinal. That is, \( \mathcal{S} \) is the class of finite sequences of ordinals.

For \( s, t \in \mathcal{S} \), we put

\[
s < t \iff \sup(s) < \sup(t) \quad \text{or} \quad \begin{array}{l}
\sup(s) = \sup(t) \quad \text{and} \quad \text{dom}(s) < \text{dom}(t) \\
\text{or} \quad \sup(s) = \sup(t), \text{dom}(s) = \text{dom}(t) \quad \text{and} \quad s < \text{lex} t.
\end{array}
\]

Then one can check that \( < \) defines a well-ordering of \( \mathcal{S} \) where proper initial segments are sets. It follows that there is an order preserving class function \( \mathcal{F} : \text{Ord} \to \mathcal{S} \).
Also, in ZF we can construct a bijective

\( \kappa : \omega \to V_\omega \). So, in particular, since

any \( \mathcal{U} \)-formula is an object in \( V_\omega \),

\( \kappa \) maps onto the set of \( \mathcal{U} \)-formulas

\( \mathcal{F} \).

We can now define \( \Psi(x, y) \) as follows:

\[ \Psi(x, y) \iff y \text{ is an ordinal and } \]

\[ \exists (y) = (\eta, \beta, x_1, \ldots, x_n) \text{ is a finite sequence of ordinals } \leq \beta, \]

\[ x_1, \ldots, x_n < \beta, \eta \leq \omega, x \in V_\beta \text{ and } \]

\[ \kappa(\eta) = t(z, y_1, \ldots, y_n) \text{ is a } \mathcal{U} \text{-formula with } \]

\[ \text{Val}(t(z, x_1, \ldots, x_n), V_\beta) = \{x\}. \]

Thus clearly

\[ \text{OD}(a) \iff \exists \alpha \text{ ordinal } \forall x (\Psi(x, \alpha) \iff x = a) \]

\[ \iff \exists \alpha \text{ ordinal } \Psi(a, \alpha). \]

Definition. The class \( \text{HOD} \) of hereditarily ordinal definable sets is given by

\[ \text{HOD}(a) \iff \text{OD}(a) \land \text{cl}(a) \text{ is a subset of OD}, \]
Lemma: \( \text{HOD}(a) \leftrightarrow \text{OD}(a) \land \forall x \in a \; \text{HOD}(x) \).

This follows easily from some fact that
\[
\varepsilon(a) = a \cup \bigcup_{x \in a} \varepsilon(x).
\]

Theorem: Suppose \( U \) is a model of ZF. Then \( \text{HOD} \) is a model of ZFC.

Proof: Note first that \( \text{HOD} \) is a transitive class, i.e., \( x \in b \land \text{HOD}(b) \Rightarrow \text{HOD}(a) \), containing all ordinals.

Extensibility follows from transitivity of \( \text{HOD} \) plus the fact that \( \text{HOD} \) is a transitive class.

Unique: Note that if \( \text{HOD}(a) \) and \( b = \bigcup_{x \in a} x \), then \( b \) is a subset of \( \text{HOD} \). So to see that \( b \) belongs to \( \text{HOD} \), we only need to see that \( b \) is ordinal definable.

So pick \( x \in b. \psi(a,x) \) holds. Then \( b \) is the unique object \( x \) satisfying
\[
\phi(x,a) : \forall y \left( y \in x \iff \exists z \left( z \in a \land y \in z \right) \right)
\]
whence the lemma
\[
\exists v \left( \psi(v,a) \land \phi(x,v) \right)
\]
defines \( b \). Thus \( \text{OD}(b) \).
Power set: Suppose $\text{HOD}(a)$ and let

$b = \mathcal{P}(a) \cap \text{HOD}$ be the set of all hereditarily ordinal definable subsets of $a$. Then $b$ is a subset of $\text{HOD}$ and can be seen to be ordinal definable by methods as above. Thus $b = \mathcal{P}^{\text{HOD}}(a)$.

Replacement

Suppose $\sigma(x, y, a_1, \ldots, a_n)$ is a formula

with parameters $a_1, \ldots, a_n$ definable in $\text{HOD}$ that defines a class function in $\text{HOD}$, i.e.,

$\forall x \ (\forall \text{HOD}(x) \rightarrow \exists^1 y \ (\forall \text{HOD}(y) \land \sigma^{\text{HOD}}(x, y, a_1, \ldots, a_n))).$

Suppose $X$ is in $\text{HOD}$ and $Y$ is the set of images in $\text{HOD}$ of elements of $X$ by this class function. Clearly, $Y$ is a subset of $\text{HOD}$, so we need only show that $Y$ is in $\text{OD}$.

So the ordinals $\beta, \xi_1, \ldots, \xi_n$ satisfy $\forall X, \beta \ (\forall (a_1, \ldots, a_n) \ (\forall (a, x) \ (\forall (a_1, x_1) \ (\forall (a, x_1) \ (\forall (a_n, x_n) \ (\forall (a_1, x_1, \ldots, a_n) \ (\forall z (z \subseteq x \leftrightarrow \exists u (u \in x \land \forall \text{HOD}(z) \land \exists^1 u \ (\forall \text{HOD}(u) \land \sigma^{\text{HOD}}(u, z, a_1, \ldots, a_n)))))))))$
where $\psi$ is defined by

$$
\exists \alpha \exists \beta \cdots \exists \gamma_n \ (\psi(\alpha, \beta) \land \psi(\gamma_1, \xi_1) \land \cdots \land \psi(\gamma_n, \xi_n) \land \phi(x, v, \gamma_n, \cdots, \gamma_1))
$$

Indeed, since $\text{HOD}(\omega)$, $\text{HOD}$ satisfies the axiom of infinity.

**Foundation** If $\text{HOD}(\alpha)$ and $\alpha \neq \emptyset$, let $b \in a$

have minimal rank. Then $b \cap a = \emptyset$ and $\text{HOD}(b)$.

**Choice** Note first that we have a well-ordering of $\text{HOD}$ by

$$a \prec b \iff \exists \alpha (\psi(a, x) \land \forall \beta (\psi(b, \beta) \rightarrow x \prec \beta))$$

That is, we well-order elements of $\text{HOD}$ according to the minimal ordinal defining them via $\psi$.

So if $X \neq \emptyset$ belongs to $\text{HOD}$,

$$R = \{ (a, b) \in X^2 \mid a \prec b \}$$

is a well-ordering of $X$, and thus a subset of $\text{HOD}$, that is clearly ordinal definable.

So $R$ belongs to $\text{HOD}$ and hence $X$ can be well-ordered in $\text{HOD}$.

$\square$
The principle of choice

Recall that we have a formula without parameters in the language of set theory, \( \psi(x, y) \), such that for any set \( a \):

\[
\text{OD}(a) \iff \exists \gamma \text{ ordinal } \forall x (\psi(x, \gamma) \iff x = a)
\]

\[
\iff \exists \gamma \text{ ordinal } \psi(a, \gamma).
\]

Using \( \psi \) we can define a well-ordering of the class \( \text{OD} \) by:

\[
a < b \iff \text{OD}(a) \land \text{OD}(b) \land \exists x (\psi(a, x) \land \forall \beta (\psi(b, \beta) \implies x < \beta)).
\]

**Definition** The principle of choice is the statement that there is a first order formula \( \phi(x, y) \) without parameters defining a well-ordering of \( \mathcal{U} \).

**Note** Since this is really a disjunction over all formulas of set theory, the principle of choice is not even an axiom scheme and even less a first order axiom. However, on the basis of ZF, we shall see that it is first order.
Proposition: \( U \) satisfies the principle of choice if and only if \( \forall x \, \text{OD}(x) \) holds.

Proof: Note that in \( U \), \( \varphi \) defines a well-ordering of \( \text{OD} \). Thus, if \( \forall x \, \text{OD}(x) \), i.e., \( U = \text{OD} \), then \( \varphi \) is a well-ordering of \( U \).

Conversely, suppose \( \varphi(x,y) \) is a formula without parameters defining a well-ordering of \( U \).

Then there is a unique class function
\[
F : \text{Ord} \to U \suchthat x < y \iff \varphi(F(x), F(y)).
\]

It follows that for every \( x \), \( F(x) \) is ordinal definable with parameter \( x \), and hence \( \forall x \, \text{OD}(x) \).

Thus, principle of choice \( \iff \) \( V = \text{OD} \iff V = \text{HOD} \).

Constructibility (d'après K. Gödel)

Definition (ZF): Suppose \( A \) is a set and \( X \subseteq A \) is a subset. We say that \( X \) is definable with parameters in \( A \) if
\[
\exists \varphi(x, y_1, \ldots, y_n) \exists a_1, \ldots, a_n \in A \suchthat X = \text{Val}(\varphi(x, a_1, \ldots, a_n) ; A).
\]

Again, this is a first-order property in \( X \) and \( A \), so we can deduce the set.
\[ \mathcal{D}(A) = \{ x \in A \mid x \text{ is definable with parameters in } A^2 \} \]

**Example.** Suppose \( \phi(x, y_1, \ldots, y_n) \) is a formula in the language of set theory and \( a_1, \ldots, a_n \in A \) are
\[
X = \{ x \in A \mid A \models \phi(x, a_1, \ldots, a_n) \}.
\]

Thus,
\[
X = \text{Val}(\phi(x, a_1, \ldots, a_n), A) \in \mathcal{D}(A).
\]

**Remark (ZFC).** Suppose \( |A| > \aleph_0 \). Then \( |\mathcal{D}(A)| = |A| \).

To see this, note that the set of \( U \)-formulas with parameters in \( A \) has size \( |A| \) itself, and thus also \( |\mathcal{D}(A)| = |A| \).

In particular, we conclude \( \mathcal{F} \), \( |\mathcal{D}(A)| < |\mathcal{P}(A)| \).

Also, \( A \in \mathcal{D}(A) \) for any \( A \). But if \( A \subseteq B \) is not definable, then \( A \notin \mathcal{D}(B) \) and so \( \mathcal{D}(A) \neq \mathcal{D}(B) \). Nevertheless, we have

**Theorem.** If \( A \subseteq B \), then \( \mathcal{D}(A) \subseteq \mathcal{D}(B) \).

**Proof.** Suppose \( x \in \mathcal{D}(A) \) and that \( \phi(x, a_1, \ldots, a_n) \) is a \( U \)-formula with parameters in \( A \).
\[
X = \text{Val}(\phi(x, a_1, \ldots, a_n), A), \quad \text{Then else}
\]
\[
X = \text{Val}(\phi^A(x, a_1, \ldots, a_n), B) \cap A.
\]
By transfinite induction, we now define a hierarchy of sets by

\[ L_\alpha = \bigcup_{\beta < \alpha} D(L_\beta), \]

so \( L_0 = \emptyset \) and \( L_\beta \subseteq L_\alpha \) for \( \beta < \alpha \). Also,

for \( \beta < \alpha \), \( L_\beta \subseteq D(L_\alpha) \subseteq L_\alpha \) and hence,

by the preceding theorem, \( D(L_\beta) \subseteq D(L_\alpha) \). It follows that the hierarchy can alternatively be described by

\[ L_0 = \emptyset, \quad L_{\alpha+1} = D(L_\alpha) \]

and for \( \gamma \) a limit,

\[ L_\gamma = \bigcup_{\beta < \gamma} L_\beta. \]

Moreover, since \( D(A) \subseteq P(A) \) for every set \( A \), we see by induction on \( \alpha \) that \( L_\alpha \subseteq V_\alpha \).

Let \( L \) be the class of constructible sets defined by \( L = \bigcup_{\alpha \in \text{ord}} L_\alpha \), so \( L \subseteq V \).

**Definition.** The **axiom of constructibility** is the statement \( V = L \), which assuming \( \text{ZF} \) is just \( V = L \), i.e.,

\[ \forall x \exists \alpha x \in L_\alpha. \]
Lemma. \( L \) is a transitive class, that is, if \( A \) is constructible, then so is every element of \( A \). Also, if \( A \in L_x \), then \( P \subseteq L_\beta \) for some \( \beta < \alpha \).

Proof. Note that if \( A \in L_{\xi + 1} = P(\alpha) \), then \( A \in L_{\xi + 1} \).

By the proof we see that any \( L_x \) is a transitive set.

For any constructible set \( X \), we let \( \text{ord} (X) = \text{min} (\alpha \mid X \in L_\alpha) \).

Theorem. \( \text{Ord} \subseteq L \) and \( \text{Ord} \cap L_x = \alpha \), \( \forall x \).

So any ordinal \( \alpha \) is constructible with \( \text{ord} \alpha \).

Proof. By induction on \( \alpha \), we prove \( \text{Ord} \cap L_\alpha = \alpha \).

To support this holds for all \( \alpha \) less than some ordinal \( \beta \).

If \( \beta \) is limit, then

\[
\text{Ord} \cap L_\beta = \bigcup \text{Ord} \cap L_x = \bigcup (x = \sup x = \beta, \ x < \beta) = \beta
\]

On the other hand, if \( \beta = \alpha + 1 \) for some \( \alpha \), then, by assumption, \( \text{Ord} \cap L_\alpha = \alpha \) and so \( \alpha \subseteq L_\alpha \). On the other hand, since \( \alpha \subseteq L_\alpha \), \( \forall \gamma \subseteq L_\alpha \) for any \( \gamma \geq \beta = \alpha \cup \{ \alpha \} \).

Thus, do see that \( \text{Ord} \cap L_\beta = \beta \).
we need only show that $x$ is definable within $L_x$, i.e., $x \in D(L_x)$. For this, consider the formula $\phi(x)$:

\[ \forall u \forall v \left( (u \in x \land v \in x) \rightarrow (u = v \lor u \in u \lor u \in v) \right) \]

\[ \forall u \forall v \left( u \in v \leftrightarrow v \in x \right) \]

which given $\mathcal{P}$ states that $x$ is an ordinal.

Also, in any transitive class $C$ containing a set $x$, the formulas $\phi(x)$ and $\phi^C(x)$ are equivalent, so, in particular,

\[ x = \text{Val} \left( \left\{ \phi(x), \forall x \right\} \right) \in D(L_x) = L \]

**$\Sigma_1$ formulas and absoluteness**

We shall now consider a subclass of formulas without parameters of the language of set theory obtained by restricting quantification.

**Definition.** The $\Sigma_1$ formulas is the smallest class of first order formulas of the language of set theory obtained by restricting quantification.

1. any quantifiers free formula is $\Sigma_1$;
2. if $\phi$ and $\psi$ are $\Sigma_1$, then so are $(\phi \land \psi)$ and $(\phi \lor \psi)$.  

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(iii) If \( \phi \in \Sigma_1 \), then so is \( \exists x \phi \).

(iv) If \( \phi \in \Sigma_1 \), then so is \( \forall x \exists y \phi \),

i.e., \( \forall x (x \in y \rightarrow \phi) \).

A class, class function or class relation is said to be \( \Sigma_1 \) if it is defined in the universe \( M \) by some \( \Sigma_1 \)-formula. So a class \( A \) is \( \Sigma_1 \) if it has \( \Sigma_1 \) graph.

**Remark** The subclass of \( \Delta_0 \)-formulas is obtained by replacing (iii) by the following two conditions and otherwise changing \( \Sigma_1 \) to \( \Delta_0 \).

(a) If \( \phi \in \Delta_0 \), then so is \( \exists x \epsilon y \phi \),
(b) If \( \phi \in \Delta_0 \), then so is \( \neg \phi \).

**Theorem** Suppose \( \phi(x_1, \ldots, x_n) \) is \( \Sigma_1 \) and \( M \subseteq N \) are classes with \( M \) transitive. Then for any \( a_1, \ldots, a_k \) in \( M \):

\[
\phi^M(a_1, \ldots, a_k) \Rightarrow \phi^N(a_1, \ldots, a_k). \tag{*}
\]

**Remark** Before proving this, we should acknowledge that most formulas are not \( \Sigma_1 \), but only equivalent to \( \Sigma_1 \)-formulas in some background theory. However, suppose \( T \) is a theory and \( \phi(x), \theta(x) \) are formulas with \( \phi \in \Sigma_1 \) such that \( T \vdash (\phi(x) \rightarrow \theta(x)) \).
Assume also that $M, N$ satisfy $T$. Then (iv) implies that also
\[ \forall^n(x, \bar{a}) \Rightarrow \forall^n(x, \bar{a}) \]
for any $\bar{a}$ in $M$.

Proof of Theorem:

We prove the result by induction on the construction of $\phi$ by the principles (ii) - (iv).

(i) and (ii) are trivial, so suppose that
\[ \forall^n(x, x_1, \ldots, x_k) \]
satisfies the induction hypothesis
and that $a_1, \ldots, a_k$ belong to $M$.

Consider first case (iii):

If \( (\exists x \forall^n(x, a_1, \ldots, a_k)) \)
holds, then there is some $b$ in $M \subseteq N$ s.t. \( \forall^n(b, a_1, \ldots, a_k) \),
whence, by the induction hypothesis, also
\[ \forall^n(b, a_1, \ldots, a_k) \]
and so \( (\exists x \forall^n(x, a_1, \ldots, a_k)) \).

For case (iv),

Suppose $b$ is in $M$ and that
\[ (\forall x \in b \forall^n(x, \bar{a})) \]
holds, i.e., \( \forall x (M(x) \land x \in b \Rightarrow \forall^n(x, \bar{a})) \).

By the induction hypothesis for $\forall^n$, we have for any $x$ in $M$,
\[ \forall^n(x, \bar{a}) \Rightarrow \forall^n(x, \bar{a}) \].
\[ \forall x \left( \forall \bar{v} \exists x \neq b \rightarrow \forall \bar{v}^\prime \left( x, \bar{v} \right) \right) \]

Proof: as \( M \) is countable, \( b \in M \subseteq \mathbb{N} \), so

\[ \forall x \left( \forall \bar{v} \exists x \neq b \rightarrow \forall \bar{v}^\prime \left( x, \bar{v} \right) \right) \]

that is

\[ \left( \forall \bar{v} \exists x \left( x, \bar{v} \right) \right)^\mathbb{N} \]

which shows the induction hypothesis for

the formulas \( \exists x \psi \) and \( \forall x \psi \).

Lemma Suppose \( \psi \left( y, \bar{z} \right) \) is a \( \Sigma_1 \)-formula

and \( y = F \left( x \right) \) a \( \Sigma_1 \) class function. Then

\( \psi \left( F \left( x \right), \bar{z} \right) \) defines a \( \Sigma_1 \) class relation.

Proof Just note that \( \psi \left( F \left( x \right), \bar{z} \right) \) is given

by

\[ \psi \left( F \left( x \right), \bar{z} \right) \iff \exists y \left( F \left( x \right) = y \land \psi \left( y, \bar{z} \right) \right) \]

Lemma Suppose \( a \) is a set defined by a

\( \Sigma_1 \) formula \( \phi \left( x \right) \), i.e.,

\[ x = a \iff \phi \left( x \right) \]

and that \( \psi \left( y, \bar{z} \right) \) is \( \Sigma_1 \). Then also

\[ \psi \left( a, \bar{z} \right) \text{ is } \Sigma_1 \]

...
Proof. \[ \forall (x, z) \Leftrightarrow \exists x \left( \phi(x) \land \psi(x, z) \right). \]

**Fact (ZF):** Ord is a \( \Delta_0 \)-class and \( \omega \) is definable by a \( \Sigma_1 \)-formula.

**Proof:**

\[ \text{Note that} \]

\[ \text{Ord}(x) \leftrightarrow \forall y \exists z (y \in z \land \forall y \exists z (y \in z \land z = y)) \]

\[ \land \forall y \forall z (y \in z \land z \in x) \]

For the definition of \( \omega \), we check that the following formulas are \( \Sigma_1 \).

\[ x = \emptyset \quad \forall y \in x \quad y \neq x \]

\[ y = x \cup \{x\} \quad \forall z (z \in x \lor z = x) \]

\[ \forall z (z \in x \land \forall y (y \in z \land z \in y)) \land \forall y \exists z (y = \emptyset \lor \exists z (y = z \cup z)) \]

**Prop (ZF):** Suppose \( H(.) \) is a \( \Sigma_1 \)-class function of one variable defined on all functions with ordinal domain. Then the unique class function \( F \) defined on all ordinals and satisfying \( F(\alpha) = H(F(\alpha)) \) is \( \Sigma_1 \) too.
Proof. Again we successively verify that certain objects, classes and class relatives are $\Xi$.

$Z = \{ x, y \} \quad x \in Z \land y \in Z \land \forall u \in Z \ (x = u \lor y = u)$

$Z = \{ x, y \} \quad Z = \{ x, y \} \cup \{ x, y \}$

$\forall y \in x \quad \forall z \in y \quad \forall u \in z \quad \forall v \in u \quad (u \in z \land v \in u) \lor \forall v \in u \quad (u \in z \land v \in u)$

$Z = \{ x, y \} \quad \forall u \in z \quad \forall v \in u \quad (u \in z \land v \in u) \lor \forall v \in u \quad (u \in z \land v \in u)$

$Z = \{ x, y \} \quad \forall u \in z \quad \forall v \in u \quad (u \in z \land v \in u) \lor \forall v \in u \quad (u \in z \land v \in u)$

$Z \subset X \times Y \quad \exists a \exists b \quad (a \in X \land b \in Y \land a = (x, y))$

$Z \subset X \times Y \quad \exists a \exists b \quad (a \in X \land b \in Y \land a = (x, y))$

$f$ is a function from $x \to y : f \in X \times Y \land$

$\forall z \in X \exists v \ (v \in Y \land (z, v) \in f) \land$

$\forall z \in X \exists v \ (v \in Y \land (z, v) \in f) \land$

$\forall z \in X \exists v \ (v \in Y \land (z, v) \in f) \land$

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$\forall z \in X \exists v \ (v \in Y \land (z, v) \in f) \land$

$\forall z \in X \exists v \ (v \in Y \land (z, v) \in f) \land$
\( f \) is a function \( \exists a \in b \) \( f : a \rightarrow b \)

\( f \) is a function \( \forall x = y \) \( f \) is a function \( \exists (x, y) \in f \).

So finally we can write \( F(x) = y \) by

\[ \text{Ord}(x) \uplus \exists f \ (x \text{ is a function with domain } x \land \forall p \in x (f(p) = H(f(p))) \land y = H(f)) \] \( \square \)

**Fact** The following are \( \Sigma_1 \)

1. \( f \) is an injection from \( x \) to \( y \) : \( f : x \rightarrow y \land \\
\forall a \in x \forall b \in y \forall c \in y \ (a, c) \not\in f \lor (b, c) \not\in f \lor a = b \)

2. \( f \) is a surjection from \( x \) to \( y \) : \( f : x \rightarrow y \land \\
\forall b \in y \exists a \ (a \in x \land f(a) = b) \).

\[ h = f \circ g \ \exists x \in y \exists z \ (g : x \rightarrow y \land f : y \rightarrow z \land \\
h : x \rightarrow z \land \forall a \in x \exists b \in y \ (f(a) = b \land g(b) = c \land \neg h(a) = c)) \]

**Proposition** \( B = D(f) \) is \( \Sigma_1 \) in the variables \( x, y \)

The proof proceeds by successively verifying that the following classes and properties are \( \Sigma_1 \).
\[ z = \overline{\overline{F_0}} \]  
\( z \) is the set of atomic formulas.

For this, note that "\( x = 0 \)" and "\( x = y \lor z \)" 
are \( \Sigma_1 \) and so also "\( x = 1 \)", "\( x = 2 \)", "\( x = 3 \)" 
and "\( x = 4 \)" are \( \Sigma_1 \). Since \( 7, 0, 1, 2, 3, 4 \) 
are resp. \( 0, 1, 2, 3, 4 \) and \( \mathcal{V} = \mathcal{V} \setminus \{ 0, 1, 2, 3, 4 \} \), 
it is also \( \Sigma_1 \), we have that 
\[ \overline{F_0} = (\exists x \forall y \forall z x \lor y) \lor (\exists x \forall y x \lor z) \]
\[ \in \Sigma_1. \]

\[ k < \omega \]  
\[ z = \overline{\overline{F_k}} \]  
in the variables \( k \) and \( z \):

\[ \exists f \left[ \left( f \text{ is a set with } \text{dom} (f) = \omega \right) \land \left( f(0) = \overline{F_0} \right) \right] \land \]
\[ \forall \omega \left( f(\omega + 1) = f(\omega) \lor (\exists \eta < \omega f(\omega)) \lor (\exists \eta < \omega f(\omega) \land f(\omega)) \lor \right. \]
\[ \left. (\exists \eta < \omega x \lor y \land f(\omega)) \right) \land \left( k < \omega \land f(k) \right) \]

\( f \) is a \( \Sigma \)-formula:  
\[ \exists k \left( f \in \overline{\overline{F_k}} \right) \]

\[ z = \overline{\overline{F}} \]  
due to the \( \Sigma \)-formula.

\[ f \in \overline{\overline{F}} \land a = \text{var} (f) \]  
in \( \Sigma_1 \) in the variables \( f \) and \( a \).

We now show that the class relation

\[ Y = \times \text{var} (f) \land f \in \overline{\overline{F}} \]

in the three variables \( x, y, f \) is \( \Sigma_1 \).

Note however, that the class relation

\[ Y = \times a \]

is not \( \Sigma_1 \). To see this, observe
that if \( Z \) is a transitive set, then for \( y, x, a \in Z \),
\[
(\forall x \in Z) \iff y = \{ z \in Z \mid z : a \to x \}.
\]

Since we can construct transitive sets \( Z \) with elements \( y, x, a \) such that
\[
y = \{ z \in Z \mid z : a \to x \} \iff \{ z : z : a \to x \},
\]
being the set of limitness from \( a \to x \) is not preserved from \( Z \) to \( \mathbb{N} \) and hence cannot be \( \mathbb{Z}_1 \).

\[k \in \omega \implies y = x^k \in \mathbb{Z}_1 \text{ in } k, y, x: \]

\[k \in \omega \implies \exists f \left( \text{\( f \) is a set with domain } = \omega \right) \text{ and } f(\omega) = \emptyset \quad \text{and} \quad \forall n \in \omega \left( (\forall z \in f(n+1) \quad z : n+1 \to x) \land (\forall a \in f(n) \quad x \in \text{the set of limitness from } a \text{ to } x) \right) \land \left( y = f(k) \right).\]

\[f \in \mathbb{F} \land y = x^{\varphi(f)} \]

\[f \in \mathbb{F} \land \exists p : \exists k : \omega \left( (p : \varphi(f) \to k \text{ is a bijection}) \land (\forall g \in x^k \quad \varphi(p \circ g) \in y) \land (\forall h \in y \exists g \in x^k \quad \varphi(p \circ g) = h) \right).\]

\( f \) is a \( U \)-sentence with parameters in \( x \)

\[\exists g \in \mathbb{F} \exists \delta : \varphi(g) \to x \quad f = (g, \delta).\]
\[ z = \frac{\mathcal{F}}{X} \quad \text{where \( \mathcal{F} \) denotes the set of \( U \)-sentences with parameters in \( X \):} \\
\forall f \in \mathcal{F} \quad (f \text{ is a } U \text{-sentence with parameters in } X) \\
\forall f \in \mathcal{F} \quad \forall \delta \in X \text{ var}(f) \quad (f, \delta) \in z \\
\text{\( f \) is a } U \text{-formula with parameters in } X \text{ and } x \text{ as a single free variable} \\
\]

\[ z = \frac{\mathcal{F}}{X} \quad \text{where \( \mathcal{F} \) denotes the set of } U \text{-formulas with parameters in } X \text{ and a single free variable } x \]

\[ f \in \mathcal{F} \quad \text{and } t = \text{Val}(f, X) \quad \text{is } z \text{ in variables } f, X, t \]

and where \( t \) can take the values 0, 1.

To see this, note that \( t = \text{Val}(f, X) \) if and only if there is a function satisfying Tarski's recursive definition of truth eventually ending up with \( t \).

\[ f \in \mathcal{F} \quad \text{and } y = \text{Val}(f, X) \]

\[ \exists \delta \in \text{D}(X) : \exists x \in \text{D}(X) : \exists f \in \mathcal{F} \quad y = \text{Val}(f, X) \]

\[ z = \text{D}(X) : \forall y \in z \quad (y \in \text{D}(X) \quad \text{and } \forall x \in \mathcal{F} \quad (\text{Val}(f, X) \in z)) \]

\[ \Box \]
**Corollary (ZF)** The class function \( L : \text{Ord} \to V \) is \( \varepsilon \).

**Proof** \( L \) is a class function defined by transfinite recursion from the \( \varepsilon \), class function \( D(.) \). \( \square \)

**Theorem (ZF)** \( L \) satisfies \( \text{ZF} + \forall V = L \).

**Proof** Recall that \( L \) is a transitive class. Assume \( L \) satisfies \( \text{ZF} \). Then, as the class function

\[
\chi \mapsto L\chi
\]

is well-defined in any model of \( \text{ZF} \), we have that

\[
\left[ \chi \mapsto L\chi \right]_L
\]

is a class function defined on all ordinals \( \chi \) in \( L \). Moreover, since \( y = L\chi \) is \( \varepsilon \) in due variables \( y, \chi \), we have for all \( y, \chi \) in \( L \):

\[
\left[ y = L\chi \right]_L \Rightarrow y = L\chi .
\]

Now, suppose \( \chi \) belongs to \( L \) and find an ordinal \( \alpha \) such \( \chi \in L\alpha \). Then \( \chi \) belongs to \( L \) and since the class of ordinals is \( \Delta_0 \) and \( L \) has \( \varepsilon \), complement, we have

\[
\left[ \text{Ord}(\alpha) \right]_L
\].

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Now, let \( y \) be the set in \( L \) sat.
\[
[y = L^x].
\]
Thus also
\[
y = L^x
\]
and so \( x \in y \), whence also \([x \in y]^L\).
So, finally,
\[
[x \in L^x]^L,
\]
showing that
\[
[V = L]^L.
\]
It now remains to show that \( L \) satisfies \( \text{ZF} \).

**Extensionality** holds in \( L \) since \( L \) is transitive.

**Union** Suppose \( a \) is constructible, say \( a \in L^x \).
Thus \( b = \bigcup_{x \in a} x \in L^x \) since \( L^x \) is transitive and
\[
b = \text{Val}(\Sigma y (y \in a \land y \in x), L^x) \in L^{a+1}.
\]

**Power Set** Suppose \( a \) is in \( L \) and let
\[
b = \{x \in a \mid x \text{ is in } L\}.
\]
Find also \( x \) sufficiently large such that \( b \in L^x \) and then also \( a \in L^x \). But then
\[
b = \text{Val}(\Pi y (y \in a \rightarrow y \in a), L^x) \in L^{a+1}.
\]
Suppose \( \phi(x, y, \bar{a}) \) is a formula with parameters \( a_1, \ldots, a_n \) in \( L \) that defines a class function in \( L \).

Suppose \( b \) is a constructible cut and set

\[
\sigma = \exists y \in L(y) \land \exists x \in c \phi^L(x, y, \bar{a})
\]

Then \( b \in L_x \) for some \( x \) large enough to

\( a_1, \ldots, a_n \in L_x \). By the reflection scheme

we can find \( x_0 > x \) such that

\[
\forall x, y, z \in L_{x_0} \quad \left( \phi^{L_{x_0}}(x, y, z) \iff \phi^L(x, y, z) \right)
\]

and hence, in particular,

\[
\forall x, y \in L_{x_0} \quad \left( \phi^{L_{x_0}}(x, y, \bar{a}) \iff \phi^L(x, y, \bar{a}) \right).
\]

It follows that

\[
b = \text{Val} \left( \exists x \in c \phi(x, y, \bar{a})^L_{x_0} \right) \in L_{x_0 + 1}.
\]

Foundation: If \( x \neq \emptyset \) belongs to \( L \), then

\( \langle a \rangle \) is a set of minimal rank. Then \( b \) belongs

to \( L \) and \( a \cap b = \emptyset \). \( \square \)
Theorem

\[ V = L \implies \text{principle of choice}. \]

In particular, the principle of choice is consistent. Also

\[ V = L \implies V = L = OD = HOD. \]

Proof

List the set of U-formulas as \((f_n)_{n \in \omega} \).

Suppose \( X \) is a set well-ordered by a relation \( \leq \). We then define a well-oriendg \( \leq' \) of \( \mathcal{D}(X) \) as follows:

The ordering \( \leq' \) of \( X \) canonically induces a well-oriendg \( \leq' \) of the set \( X < \omega \) of finite sequences of elements of \( X \).

Now if \( R, B \in \mathcal{D}(X) \) put

\[ R \leq' B \iff \exists n \in \omega \ (f_n(x, \bar{y}) \in R \text{ and } \bar{x} \in X < \omega \text{ with } R = \text{Val}(f_n(x, \bar{y}), X), \] and

\( \exists \bar{y} \in X < \omega \) with \( B = \text{Val}(f_m(x, \bar{y}), X) \) either

(i) \( n < m \), or

(ii) \( n = m \) and \( \bar{x} \preceq' \bar{y} \).

Finally, let for \( R, B \in \mathcal{D}(X) \)

\[ A \leq' B \iff \begin{cases} R, B \in X & R \leq' B \\bar{y} \in X \& R \leq' B \bar{x} \in X \& B \preceq' X \end{cases} . \]
Note that then $\Delta \leq \omega$ is a well-ordering of $X = \Delta^\omega$, and $\leq^*$ is a well-ordering of $\Delta^{\omega + 1} = D(\Delta^\omega)$ in which $\Delta^\omega$ is an initial segment on which the ordering agrees with $\leq$.

Now, by transfinite induction, we define

$\leq_0 = \text{initial ordering on } \Delta^0 = \emptyset$

$\leq_{\alpha + 1} = \leq^*$

$\leq_\beta = \bigcup \leq_\alpha$ for $\alpha < \beta$.

Thus each $\leq_\alpha$ is a well-ordering of $\Delta^\alpha$, for $\alpha < \beta$, $\leq_\alpha = \leq_\beta \restriction \Delta^\alpha$, and $\Delta^\alpha$ is an initial segment of $(\Delta^\beta, \leq_\beta)$.

Finally, let $\leq = \bigcup \leq_\alpha$, which is a class well-ordering of $\Delta$ in which each initial segment is contained in some $\Delta^\alpha$. \qed
Theorem (ZFC) Suppose $F(x)$ is a $\Xi_1$ class function of one variable. Then for any $a$ in the domain of $F$, we have

$$|F(a)| \leq |C_1(a)| + \kappa_0$$

where $C_1(a)$ is the measurable closure of $a$.

Proof Suppose $\phi(x,y)$ is a $\Xi_1$ formula with $\phi(x,y) \iff F(x) = y$.

Suppose $a$ belongs to the domain of $F$ and let $\alpha$ be large enough so that $F(a) \in V_\alpha$ and

$$\forall x, y \in V_\alpha \left( \phi^V(x,y) \iff \phi(x,y) \right)$$

(such an $\alpha$ can be found by reflection applied to $\phi$).

Note that $C_1(\{a\}) = \{a\} \cup C_1(a) \in V_\alpha$. Since $V_\alpha$ is transitive and $\phi$ holds in $C_1(\{a\})$, $|C_1(\{a\})| = |C_1(a)| + 1$.

Let $\delta$ denote the set of $\kappa$-admissible $\alpha$ with parameters in $C_1(\{a\})$. Then $V_\alpha \models \delta$.

By Löwenheim–Skolem applied to $C_1(\{a\}) \subseteq V_\alpha$, there is a subset $C_1(\{a\}) \subseteq X \subseteq V_\alpha$ with $|X| \leq |C_1(a)| + \kappa_0$ so, $X \models \delta$ for any $\delta$.

Since $V_\alpha$ is transitive, it is extendable and

$$V_\alpha \models \forall x \forall y (x = y \iff \forall z (z \in x \iff z \in y))$$
Thus, \( X \) satisfies the axiom of foundation and hence so does \( Y \).

Let us now denote the Mostowski collapse and let \( f : X \to Y \) be the corresponding isomorphism.

That is, \( Y \) is a transitive set and \( f \) is a bijection such that

\[
\forall x, y \in X \quad (x \in y \implies f(x) \in f(y)).
\]

It follows that \( Y \) satisfies every sentence in \( A \) where any parameter \( a \in E(\tau;Z) \) is replaced by \( f(a) \).

However, since \( E(\tau;Z) \) is already transitive, the collapsing map \( f \) is the identity on \( E(\tau;Z) \) and so \( E(\tau;Z) \subseteq Y \) and \( Y = f \) for any \( f : \tau \to X \).

Now, \( \forall y \in X \quad x \in f(y) \text{ and so } Y \models \forall y \phi(x,y) \)

whenever \( \phi \text{ is some } a \in A \), we have \( \forall y \phi(a,y) \).

Since \( f \) is \( \Sigma_1 \) and \( Y \) is transitive, it follows that also \( \phi(a,b) \), whence \( F(a) = b \in Y \).

Again as \( Y \) is transitive, \( F(a) \subseteq Y \) and

\[
|F(a)| \leq |Y| = |X| \leq |E(\tau;Z)| + b_0.
\]

\( \square \)

Remark: Since clearly \( |P(w)| > b_0 = |E(\tau;Z)| + b_0 \),

we see that the class function \( P(\cdot) \) cannot be \( \Sigma_1 \).
Remark: If $a$ is a $\Sigma_1$ definable set, i.e., the statement $x = a$ is $\Sigma_1$ in the variable $x$, then $|a| \leq \omega$. For in this case $a = F(\alpha)$ is a $\Sigma_1$ class function.

Theorem (ZFC)

(i) $|L_\alpha| = |\alpha|$ for any ordinal $\alpha \geq \omega$.

(ii) In any constructible set $a$, $|\text{order}(a)| \leq |\text{Cl}(a)| + \aleph_0$.

Proof

(i) Since $\alpha \subseteq L_\alpha$, we have $|\alpha| \leq |L_\alpha|$. Conversely, note that $\alpha \rightarrow L_\alpha$ is $\Sigma_1$, so, by the previous theorem, $|L_\alpha| \leq |\text{Cl}(\alpha)| + \aleph_0 = |\alpha|$.

(ii) Again note that order $(\cdot)$ is a $\Sigma_1$ class function since

$\text{order}(a) = \alpha \iff a \in L_\alpha \land \forall \beta \exists \gamma. a \in L_\beta$.

So the result follows from the preceding theorem.

Theorem (ZFC) If $\forall \alpha$ then the generalized continuum hypothesis (GCH) holds; i.e., in any infinite cardinal $\kappa$, $2^\kappa = \kappa^+$.  

Proof. Suppose $a \leq \kappa$, then (order $(a) | \leq |\text{cl}(a)| + \kappa$, 
$\leq \kappa$ and hence $a \in \kappa$ for some $\alpha < \kappa^+$.

So $\mathfrak{P}(\kappa) \subseteq \kappa^+$ and so $|\mathfrak{P}(\kappa)| = 2^\kappa \leq 1\cdot \kappa^+ = \kappa^+$. \( \blacksquare \)

**Definition.** A sentence $\phi$ is **arithmetic** if all quantifiers are of the form 
$$\exists x \in V_\omega \quad \forall x \in V_\omega.$$ 

For example, since Peano arithmetic is countable in $V_\omega$, any statement in Peano arithmetic is an arithmetic statement.

**Theorem (ZF).** If an arithmetic statement $\phi$ is provable from $\text{ZFC} + V=L + \text{GCH}$, then $\phi$ is provable from ZF.

Proof. $V_\omega = L_\omega \subseteq L$, $\phi \in \Sigma_2$ and so from ZF we get that $\phi^L \rightarrow \phi$.

Now, suppose $\psi_1, \psi_2, \ldots, \psi_n, \phi$ is a proof of $\phi$ from the axioms of ZFC + $V=L$.

If $\psi_i$ is an axiom, then also $\psi_i^L$ holds (as can be proven only supposing ZF in $V$), so $\psi_1^L, \psi_2^L, \ldots, \psi_n^L, \phi^L \rightarrow \phi$ is a proof of $\phi$ only using axioms of ZF. \( \blacksquare \)
Forcing

Whereas Gödel's construction of $L$ provided us with a model of $\text{ZFC} + \forall \alpha \exists L_{\alpha+1}$, we shall now present P. Cohen's method of forcing giving us a model of $\text{ZFC} + \neg \text{CH}$.

Main idea: If $M$ is a model of $\text{ZFC}$ and $N$ is a countable, transitive set in $M$, then forcing is a method for adjoining a new set $x$ to $M$, assumed to be somehow generic, to obtain a new countable transitive set $M[x]$ still satisfying $\text{ZF}$.

We also have tools for studying this adjoining of $x$ to $M$ and to control certain properties of $M[x]$ in terms of $M$ and $x$.

First, let us see how we can obtain countable transitive set models of $\text{ZFC}$.

Theorem: Suppose $T$ is a theory in the language of set theory extending $\text{ZFC}$ and let $m$ be a new constant symbol. Then if $T$ is consistent, so is the theory $T^*$: $T + T^m + \{m \text{ is a countable, transitive set}\}$.
Proof
Suppose towards a contradiction that $T^*$ is inconsistent. Then there is a finite fragment of $T^*$ that is inconsistent and so there are sentences $\phi_1, \ldots, \phi_n \in T^*$ such that $T + \bigwedge_{i=1}^n \phi_i + \text{"m is a countable transitive set"}$ is inconsistent.

Now, since $T$ is consistent, let $U$ be a model of $T$. By the reflection scheme, find an ordinal $\alpha$ such that $(\bigwedge_{i=1}^n \phi_i) \uparrow \alpha$ holds.

Also, by Löwenheim–Skolem, there is a countable set $X \subseteq V_\alpha$ such that for any $U$-formula $\phi$:

$$X \models \phi \iff V_\alpha \models \phi.$$  

In particular, since $V_\alpha$ satisfies the scheme of extensionality, so does $X$, and as $V_\alpha \models \phi_i$ for all $i$, we have $X \models \phi_i$ for all $i$.

Let $\bar{f} : X \to Y$ be the canonical map from $X$ onto the Mostowski collapse. Then $Y$ is a countable transitive set and $Y \models \phi_i$ for all $i$.  

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We can therefore expand \( U \) to a model of \( \mathcal{T} + \bigwedge_{i \in \omega} \Phi_i \ + \ \text{"w is a countable transitive set"} \) by interpreting \( w \) as \( Y \), contradicting our assumption.

---

**Generic extensions**

In the following, suppose \( M \) is a countable transitive set satisfying \( ZF \) in a universe \( U \) satisfying \( ZFC \).

Assume also that \( (P, \leq) \in M \) is a poset (partially ordered set) in \( M \), \( P \neq \emptyset \).

Elements of \( P \) are called forcing conditions and if \( p \leq q \), we say that \( p \) is stronger than \( q \). Two conditions \( p, q \) are said to be compatible if \( \exists r \in P \ (r \leq p \land r \leq q) \).

Otherwise, \( p \) and \( q \) are incompatible, written \( p \perp q \).

A subset \( D \subseteq P \) is dense if \( \forall p \in P \exists q \in D \ q \leq p \)

and is saturated if \( \forall p \in D \ \forall q \leq p \ q \in D \).
Moreover, \( D \subseteq P \) is predicate if
\[
\forall p \in P \exists q \in D \ p \sqsubseteq q \text{ are compatible.}
\]
For any set \( X \subseteq P \), let
\[
\bar{X} = \{ p \in P \mid \exists q \in X \ p \sqsubseteq q \}
\]
denote the saturation of \( X \). Note that if \( X \) is predicate, then \( \bar{X} \) is dense.

Now, suppose \( G \subseteq P \) is a subset, not necessarily belonging to \( M \), but only to \( U \). We say that \( G \) is \( P \)-generic over \( M \) if
\begin{enumerate}
  \item \( \forall p \in G \ \forall q \in P \ (p \sqsubseteq q \rightarrow q \in G) \)
  \text{(that is, \( G \) is upwards closed)}
  \item \( \forall p \in G \ \forall q \in G \ p \sqsubseteq q \text{ are compatible} \)
  \item \( \forall D \in M \) (if \( D \) is a dense subset of \( P \), then \( D \cap G \neq \emptyset \))
\end{enumerate}

Since \( P \)-genericity are upwards closed, we see that (iii) can be replaced by either
\begin{enumerate}
  \item[(iii)'] \( \forall D \in M \) (if \( D \) is a dense subset of \( P \), then \( D \cap G \neq \emptyset \))
  \item[(iii)''] \( \forall D \in M \) (if \( D \) is a dense and saturated subset of \( P \), then \( D \cap G \neq \emptyset \)).
\end{enumerate}
Lemma: Suppose $G$ is $P$-generic over $M$. Then

$$\forall p \in P \ (p \in G \iff \exists q \in G \ s.t. \ p \vdash q).$$

Proof: Since any two elements of $G$ are compatible if $q \in G$ and $p \vdash q$, then $p \in G$.

Conversely, suppose $p \in G$ and consider the set

$$D = \{ q \in P \mid q \leq p \text{ or } q \vdash p \}.$$

We claim that $D$ is dense. For if $r \in P$ is given, then either $r \vdash p$, in which case $r \in D$, or there is $q \in P$ s.t. $q \leq r$ and $q \in D$, in which case $q \in D$, showing density.

Also, since $M$ satisfies $ZF$, the construction of $D$ can be done inside $M$ and so $D \in M$.

In other words, $D \in M$ is a dense subset of $P$, so, as $G$ is $P$-generic over $M$, we have $G \cap D \neq \emptyset$.

So let $q \in G \cap D$ be any element. Note that if $q \leq p$, then as $G$ is upwards closed else $p \in G$, which is not the case. So instead we must have $p \vdash q$.
Lemma Suppose \( G \) is \( \mathcal{P} \)-generic over \( H \). Then
\[ \forall p \in G \forall q \in G \exists r \in G \ (r \leq p \land r \leq q), \]
That is, any two elements of \( G \) have a common minorant.

Proof Let
\[ D = \{ r \in \mathcal{P} \mid r \perp p \ \text{or} \ (r \leq p \land r \perp q) \} \]
Again, since \( H \) satisfies \( ZF \), the construction of \( D \) can be done in \( H \) and so \( D \subset H \).
Moreover, \( D \) is dense: For given any \( t \in \mathcal{P} \) either \( t \perp p \), and so \( t \in D \), or there is \( s \in \mathcal{P} \) with \( s \subseteq t \land s \neq p \). In the latter case, either \( s \perp q \), whence \( s \in \mathcal{D} \), or there is some \( r \leq s \leq p \land r \perp q \), whence \( r \in D \).

So pick some \( r \in G \cap D \). Since any two elements of \( G \) are compatible, this must mean that \( r \perp \), and so \( p, q \) have a common minorant in \( G \). \( \square \)

By induction, we see that

**Lemma** Any finite subset of \( G \) has a common minorant in \( G \).
Definition: Suppose $D \in P$ and $p \in P$. We say that $D$ is dense below $p$ if
\[ \forall q \in P \ (q \leq p \implies \exists r \in D \ r \leq q) \]

Lemma: Suppose $G$ is $P$-generic over $M$ and assume $D \in M$ is dense below some $p \in G$. Then $G \cap D \neq \emptyset$.

Proof: Note that $E = D \cup \{ q \in P \mid q \leq p \}$ and $E$ is dense in $P$. For if $q \in P$ and $q$ has no minimalant in $D$, then $q$ and $p$ cannot have any common minimalants, whence $q \perp q$ and thus also $q \in D$.

It follows that $G \cap E \neq \emptyset$ and so, as any two elements of $G$ are compatible, also $G \cap D \neq \emptyset$.

Definition: A subset $X \subseteq P$ is an antichain if
\[ \forall p \in X \forall q \in X \ (p \neq q \implies p \perp q) \]

An antichain is said to be maximal if it is not contained in any larger antichain.

Note: An antichain is maximal if and only if it is predense in $P$. 122
In particular, if $X \in M$ is a maximal antichain and $G \subseteq P$ -generic over $M$, then $G \cap X \neq \emptyset$.
(In particular, being a maximal antichain is $\Delta_0$.)

**Lemma** (Assuming $M$ satisfies AC)

Suppose $D \subseteq M$ is a dense subset of $P$. Then there is a maximal antichain $X \subseteq D$, $X \subseteq M$.

**Proof** Work in $M$:

We order the sets of all antichains $X \subseteq D$ by inclusion and note that by Zorn's lemma, this family has a maximal element $X$, which $\text{then}$ is predicable in $(D, \subseteq)$.

So for any $p \in P$, there is $q \in D$, $q \leq p$, and so, by predicability of $X$ in $D$, some $p \in X$ compatible with $q$ and thus also with $p$. So $X$ is predicable in $P$ and hence a maximal antichain.

**Theorem** (Assuming $M$ satisfies AC) Assume $G \subseteq P$.

Then $G$ is $P$-generic over $M$ if and only if

(a) any two elements of $G$ are compatible,
(b) if $X \subseteq M$ is a maximal antichain in $P$, then $G \cap X \neq \emptyset$.
Proof. We have already seen that if \( G \) is \( P \)-generic over \( M \), then (a) and (b) hold.

For the inverse, note that if \( G \) intersects any maximal antichain \( X \subseteq M \), then \( G \) also intersects any dense subset \( D \subseteq M \).

Finally, to see that \( G \) is closed upwards, suppose \( p \in G \), \( q \in P \) and \( p \leq q \). We let
\[
D_q = \{ r \in P \mid r \leq q \} \quad \text{and let} \quad X \subseteq D_q, \quad X \subseteq M,
\]
be a maximal antichain of the poset \((D_q, \leq)\).

Then also \( X \cup \{ q \} \subseteq M \) and \( X \cup \{ q \} \subseteq M \) is a maximal antichain in \( P \) so \( G \cap (X \cup \{ q \}) \neq \emptyset \), and hence \( q \in G \), since otherwise \( G \) would contain two incompatible elements. \( \square \)

The following result tells us that for the purposes of forcing, we can work with \((D, \leq)\) instead of \((P, \leq)\) for any dense subset \( D \subseteq M \).

Then suppose \( D \subseteq M \) is a dense subset of \( D \). Thus if \( G \) is \( P \)-generic over \( M \), also \( G \cap D \) is \( D \)-generic over \( M \). Conversely, for any \( H \subseteq D \) which is \( D \)-generic over \( M \), there is a unique \( G \subseteq P \) which is \( P \)-generic over \( M \) and such that \( H = G \cap D \). In fact, \( G = \{ p \in P \mid \exists q \in H \, q \leq p \} \).