Proof Suppose G is $\mathbb{P}$-generic over M. Then $G \cap D$ is D-generic over M. For if $E \subseteq D$, $E \in M$, is dense in D, i.e., $\forall p \in D \, \exists q \in E \; q \leq p$, then E is also dense in $\mathbb{P}$, whence

$G \cap E = (G \cap D) \cap E \neq \emptyset$.

Also, if $p, q \in G \cap D$, then p, q are compatible in $\mathbb{P}$, whence $r \leq p$, $r \leq q$ for some $r \in \mathbb{P}$. Since D is dense in $\mathbb{P}$, there is $s \in D$, $s \leq r$, whence $s \leq p$, $s \leq q$ and so p, q are compatible in G.

Finally, $G \cap D$ is clearly closed upwards in D.

Conversely, suppose $H \subseteq D$ is D-generic over M and let $G = \{p \in \mathbb{P} \mid \exists q \in H \; q \leq p\}$ be the upwards closure of H in $\mathbb{P}$. Clearly, since any $\mathbb{P}$-generic over M is upwards closed, so G is contained in any $F \supseteq H$ that is $\mathbb{P}$-generic over M. Also, if $F \supseteq H$ is $\mathbb{P}$-generic over M, $F \cap D = H$, and $p \in F$, let $X = \{q \in D \mid q \leq p\}$. Then obviously $X \in M$ is dense below p in $\mathbb{P}$ and so $F \cap X \neq \emptyset$. It follows that there is $q \in F \cap D = H$, $q \leq p$, whence $p \in G$. So $F = G$.

So if G is actually $\mathbb{P}$-generic over M, then G is unique with the property that $G \cap D = H$. For this, we note that G is clearly upwards closed. And if $p, q \in G$, there are $r, s \in H$ s.t. $r \leq p$, $s \leq q$. Since r, s are compatible, so are p, q. Finally, if $E \subseteq \mathbb{P}$, $E \in M$, is a saturated dense set, then $E \cap D \in M$ is dense in D and so $H \cap E \cap D \neq \emptyset$, whereby also $G \cap E \supseteq G \cap D \cap E = H \cap D \cap E \neq \emptyset$. $\square$

<u>Theorem</u> (Rasiowa - Sikorski) (ZFC) Suppose M is a countable transitive set satisfying ZF and $(\mathbb{P}, \leq) \in M$ is a poset. Then for any $p \in \mathbb{P}$, there exists $G \subseteq \mathbb{P}$ which is $\mathbb{P}$-generic over M and such that $p \in G$.

<u>Proof</u> Since M is countable, so is $\mathcal{P}(\mathbb{P}) \cap M$ and we can therefore find a sequence $(D_n)_{n<\omega}$ enumerating all dense subsets $D \subseteq \mathbb{P}$, $D \in M$. Let also $\pi : \mathcal{P}(\mathbb{P}) \setminus \{\emptyset\} \to \mathbb{P}$ be a choice function on $\mathbb{P}$.

By induction on $n<\omega$, we define a decreasing sequence $(p_n)_{n<\omega}$ of elements of $\mathbb{P}$ as follows:

$p_0 = p$.

$p_{n+1} = \pi(\{q \in D_n \mid q \leq p_n\})$ (note that since D_n is dense in $\mathbb{P}$, $\{q \in D_n \mid q \leq p_n\} \neq \emptyset$).

Then $p_0 \geq p_1 \geq \dots$ and $p_{n+1} \in D_n$ for any $n < \omega$. Letting $G = \{q \in \mathbb{P} \mid \exists n < \omega \ p_n \leq q\}$, we see that G is upwards closed, any two elements of G are compatible and G intersects any dense $D \subseteq \mathbb{P}$, $D \in M$. Moreover $p \in G$. $\square$

Mostowski collapse of a well-founded relation

<u>Definition</u> Suppose A is a set and $R \subseteq A \times A$ is a binary relation. We say that R is <u>well-founded</u> if there is no sequence $(a_n)_{n<\omega}$ such that $a_{n+1} R a_n$ for all $n < \omega$. Equivalently, R is well-founded if for any non-empty $X \subseteq A$ there is $a \in X$ s.t.

$\forall b \in X \ \neg bRa$.

Suppose R is a binary relation on a set A. A function Φ with domain A is said to be an <u>R-contraction</u> if for any $a \in A$

$$\Phi(a) = \{\Phi(b) \mid b \in A \ \& \ bRa\}.$$

Theorem (ZF$^-$) If R is a well-founded relation on a set A, then there exists a unique R-contraction Φ.

Proof Uniqueness: Suppose towards a contradiction that Φ and Ψ are distinct R-contractions and let $X = \{a \in A \mid \Phi(a) \neq \Psi(a)\} \neq \emptyset$. Since R is well-founded, there is $a \in X$ st. $\forall b \in A\ (bRa \rightarrow b \notin X)$. But then

$$\Phi(a) = \{\Phi(b) \mid b \in A \ \&\ bRa\}$$

$$= \{\Psi(b) \mid b \in A \ \&\ bRa\} = \Psi(b)$$

contradicting $a \in X$.

Existence: Note that if $X \subseteq A$ is downwards R-closed and Φ is an R-contraction on A, then $\Phi \restriction_X$ is an R-contraction on X.

Also, if $X, Y \subseteq A$ are both downwards closed, then so is $X \cap Y$. Therefore, if $X, Y \subseteq A$ and Φ, Ψ are R-contractions on X and Y respectively, then $\Phi \restriction_{(X \cap Y)}$, $\Psi \restriction_{(X \cap Y)}$ are R-contractions on $X \cap Y$ and must therefore agree.

Therefore, if Φ is the union of all

 R-contractions defined on downwards

closed subsets of A, we see that Φ is an R-contraction whose domain is a downwards closed subset of A. We claim that $A = \mathrm{dom}(\Phi)$. For if not, let $a \in A \setminus \mathrm{dom}(\Phi)$ be s.t. $\forall b \in A\ (bRa \to b \in \mathrm{dom}(\Phi))$, which exists since R is well-founded. But then we can define Ψ on $\mathrm{dom}(\Phi) \cup \{a\} \supsetneq \mathrm{dom}(\Phi)$ by $\Psi(b) = \Phi(b)$ for $b \in \mathrm{dom}(\Phi)$ and $\Psi(a) = \{\Phi(b) \mid b \in A \;\&\; bRa\}$. Then Ψ is an R-contraction defined on a strictly larger downwards closed subset of A, which is absurd. $\square$

Construction of generic extensions

Suppose M is a countable transitive set model of ZF and $(\mathbb{P}, \leq) \in M$ is a poset. Suppose also that G is $\mathbb{P}$-generic over M.

We define the following relation R on M using G: For $x, y \in M$

$$xRy \iff \exists p \in G\ [(x,p) \in y].$$

Since $xRy \Rightarrow \mathrm{rk}(x) < \mathrm{rk}(y)$, we see that R is well-founded on M.

By the preceding theorem, there is a unique R-contraction, Φ with domain M and we shall denote the image $\Phi[M]$ by $M[G]$. $M[G]$ is called a generic extension of M. Our goal is now to show that $M[G]$ itself is a countable, transitive set model of ZF and eventually study the relation between $M, (\mathbb{P}, \leq), G$ and $M[G]$.

Lemma $M \subseteq M[G]$.

Proof Working in M, we define a class function denoted $y = \hat{x}$ by induction on the rank of x:

$$\hat{x} = \{ (\hat{z}, p) \mid z \in x \ \& \ p \in \mathbb{P} \} \in M.$$

Let $\hat{M}$ denote the set $\hat{M} = \{ \hat{x} \mid x \in M \}$, which is a class in M.

We claim that $\hat{M} \subseteq M$ is downwards R-closed. For if $\hat{x} \in \hat{M}$ and $u R \hat{x}$, then there is some $p \in G$ such that $(u, p) \in \hat{x}$, i.e., $u = \hat{z}$ for some $z \in x \subseteq M$. Thus $u = \hat{z} \in \hat{M}$ too.

Also, the map $x \in M \mapsto \hat{x} \in \hat{M}$ is injective. For if not let $x \in M$ have minimal rank such that there is $y \in M$, $y \neq x$, $\hat{y} = \hat{x}$.

Then for any $z, u \in M$ if $z \in x$, we have

$$z = u \iff \hat{z} = \hat{u}.$$

On the other hand, if $z \in x \setminus y$ and $p \in \mathbb{P}$, then $(\hat{z}, p) \in \hat{x} = \hat{y}$, and so $(\hat{z}, p) = (\hat{u}, p)$ for some $u \in y$, whence $z = u \in y$, which is absurd.

Similarly, if $u \in y \setminus x$ and $p \in \mathbb{P}$, then $(\hat{u}, p) \in \hat{y} = \hat{x}$, and so $(\hat{u}, p) = (\hat{z}, p)$ for some $z \in x$, whence $u = z \in x$, which is also absurd.

We therefore have that for any $x, y \in M$

$$x \in y \iff \hat{x} R \hat{y}. \tag{$*$}$$

Here the implication from left to right is direct from the definition of $\hat{y}$, and conversely, if $\hat{x} R \hat{y}$, then $(\hat{x}, p) \in \hat{y}$ for some $p \in G$, whence $\hat{x} = \hat{z}$ for some $z \in M$ s.t. $z \in y$. As $\hat{\ }$ is injective, also $x = z \in y$.

Thus, the inverse map $\hat{x} \mapsto x$ is an isomorphism of $(\hat{M}, R)$ with $(M, \in)$.

Also, by $(*)$

$$x = \{y \in M \mid \hat{y} R \hat{x}\},$$

which shows that $\hat{x} \mapsto x$ is the unique R-contraction defined on $\hat{M}$. It follows that for any $x \in M$, $\Phi(\hat{x}) = x \in M[G]$, so $M \subseteq M[G]$. $\square$

We shall reuse the construction of $\hat{x}$ above, so let us spell it out:

<u>Construction</u> For every $x \in M$,

$$\hat{x} = \{(\hat{y}, p) \mid y \in x \ \& \ p \in \mathbb{P}\}$$

is said to be a <u>$\mathbb{P}$-name</u> for x. Then for any $x, y \in M$,

$$x \in y \iff \hat{x} R \hat{y}$$

and

$$\Phi(\hat{x}) = x.$$

<u>Lemma</u> $G \in M[G]$.

<u>Proof</u> Set $\Gamma = \{(\hat{p}, p) \mid p \in \mathbb{P}\} \in M$. We claim

that $\Phi(\Gamma) = G$, whence $G \in M[G] = \Phi[M]$. To see this, suppose first $p \in G$. Then $(\hat{p}, p) \in \Gamma$ and so $\hat{p} R \Gamma$, whence $p = \Phi(\hat{p}) \in \Phi(\Gamma)$. So $G \subseteq \Phi(\Gamma)$. Conversely, suppose $y \in \Phi(\Gamma)$. Then there is $z \in M$ s.t. $\Phi(z) = y$ and $p \in G$ s.t. $(z, p) \in \Gamma$. It follows that $z = \hat{p}$, whence $y = \Phi(z) = \Phi(\hat{p}) = p \in G$. So $\Phi(\Gamma) = G$ and $G \in M[G]$. $\square$

<u>Remark</u> In general, $G \notin M$ and so no object of M is interdefinable with G. On the other hand, R and Φ are defined using G.

The $\mathbb{P}$-names $\hat{x}$ for $x \in M$ are defined within M without using G, but will be mapped onto $x \in M$ for any choice of $G \subseteq \mathbb{P}$ that is $\mathbb{P}$-generic over M.

<u>Lemma</u> For any $x \in M$, $rk(\Phi(x)) \le rk(x)$

<u>Proof</u> Recall that $x R y \Rightarrow rk(x) < rk(y)$.

Suppose that $rk(\Phi(x)) > rk(x)$ for some $x \in M$ ~~and choose such an x with minimal rank.~~

Then

$$rk(\Phi(x)) = rk(\{\Phi(y) \mid y R x\}) = \left(\sup_{y R x} rk(\Phi(y))\right) + 1$$

$$\leq \left(\sup_{yRx} \mathrm{rk}(y)\right) + 1 \qquad \text{(by the induction hyp.)}$$

$$\leq \mathrm{rk}(x) \quad , \text{ contradicting our assumption on } x. \ \blacksquare$$

<u>Lemma</u> $M \cap \mathrm{Ord} = M[G] \cap \mathrm{Ord}$.

<u>Proof</u> The inclusion from left to right follows from $M \subseteq M[G]$. Conversely, if $\alpha \in M[G]$ is an ordinal, then $\alpha = \Phi(x)$ for some $x \in M$, whence $\alpha + 1 = \mathrm{rk}(\alpha) \leq \mathrm{rk}(x) \in M$. Since M is transitive, also $\alpha \in M$. $\square$

<u>Lemma</u> $M[G]$ satisfies the axioms of extensionality, foundation and infinity.

<u>Proof</u> $M[G]$ is a transitive set and $\omega \in M[G]$. $\square$

<u>Lemma</u> $M[G]$ satisfies the union axiom.

<u>Proof</u> Since $M[G]$ is the image of M by the R-contraction Φ, we need to show that for any $a \in M$ there is some $b \in M$ such that

$$\Phi(b) = \{x \mid \exists y \in \Phi(a)\ x \in y\} = \bigcup \Phi(a).$$

So given $a \in M$, let

$$b = \{(y,r) \in Cl(a) \times \mathbb{P} \mid \exists p, q \geq r \, \exists x ((y,p) \in x \ \& \ (x,q) \in a)\}$$

which belongs to M.

Assume $c \in \Phi(b)$. Then $c = \Phi(y)$ for some $y \in M$, $r \in G$ such that $(y,r) \in b$. Thus for some $p, q \geq r$ and $x \in M$,

$$(y,p) \in x \quad \text{and} \quad (x,q) \in a \ .$$

Since G is upwards closed, $p, q \in G$, so

$$y R x \quad \text{and} \quad x R a$$

whence

$$c = \Phi(y) \in \Phi(x) \in \Phi(a)$$

and so $c \in \bigcup \Phi(a)$. I.e., $\Phi(b) \subseteq \bigcup \Phi(a)$.

Conversely, assume $c \in \bigcup \Phi(a)$ and let $d \in \Phi(a)$ be such that $c \in d$. Then there is $x \in M$ and $q \in G$ such that $(x,q) \in a$ and $\Phi(x) = d$, and $p \in G$, $y \in M$ such that $(y,p) \in x$ and $\Phi(y) = c$. Pick $r \in G$ s.t. $r \leq p$, $r \leq q$. Then $(y,r) \in b$ and so $c = \Phi(y) \in \Phi(b)$, showing $\bigcup \Phi(a) \subseteq \Phi(b)$. $\square$

Definition of Forcing (Strong Forcing)

Suppose M is a countable transitive set model of ZF and $(\mathbb{P}, \leq) \in M$ a poset in M.

We shall write formulas in the language of set theory with the single parameter $(\mathbb{P}, \leq)$ and three free variables p, x, y, where p ranges over $\mathbb{P}$ and x, y over M. These will be denoted

$$p \Vdash \bar{x} \in \bar{y}, \quad p \Vdash \bar{x} \notin \bar{y}, \quad p \Vdash \bar{x} \neq \bar{y}, \quad p \Vdash \bar{x} = \bar{y}$$

and we will ensure that

(1) $p \Vdash \bar{x} \in \bar{y} \iff \exists r \geq p \; \exists z \left((z, r) \in y \;\&\; p \Vdash \bar{z} = \bar{x} \right)$

(2) $p \Vdash \bar{x} \neq \bar{y} \iff \exists r \geq p \; \exists z$

$$\left[\left((z,r) \in y \;\&\; p \Vdash \bar{z} \notin \bar{x} \right) \text{ or } \left((z,r) \in x \;\&\; p \Vdash \bar{z} \notin \bar{y} \right) \right]$$

(3) $p \Vdash \bar{x} \notin \bar{y} \iff \forall q \leq p \; (q \nVdash \bar{x} \in \bar{y})$

(4) $p \Vdash \bar{x} = \bar{y} \iff \forall q \leq p \; (q \nVdash \bar{x} \neq \bar{y})$

The symbol $\Vdash$ is called the forcing relation and, e.g., "$p \Vdash \bar{x} \in \bar{y}$" should be read "$p$ forces $\bar{x} \in \bar{y}$".

Since the statements (1), (2), (3), (4) are defined by induction on the rank of x, y it is intuitively clear that this is well-defined. But to see that they are class relations, we define a well-founded ordering $\prec$ of $M \times M$ by

$$(x_0, y_0) \prec (x_1, y_1) \iff \max(rk(x_0), rk(y_0)) < \max(rk(x_1), rk(y_1))$$
or
$$\big(\text{——— " ———} = \text{——— " ———}$$
and
$$\min(rk(x_0), rk(y_0)) < \min(rk(x_1), rk(y_1)) \big).$$

Let $\rho : M \times M \longrightarrow \mathrm{Ord} \cap M$ be the corresponding rank function (which is a class function in M). Then, by induction on the stratification ρ, there is a unique class function $F : M \times M \longrightarrow \mathcal{P}(\mathbb{P})^4$ defined in M s.t.

$$F(x,y) = \{ (D_1, D_2, D_3, D_4) \in \mathcal{P}(\mathbb{P})^4 \mid p \in D_1 \iff p \Vdash \bar{x} \in \bar{y}, \text{ etc} \}.$$

So using F, we see that (1), (2), (3), (4) are class relations in M.

Having defined (1), (2), (3), (4), we extend the forcing relation to arbitrary formulas with parameters by induction on their construction over <u>literals</u> "$x \in y$", "$x \notin y$", "$x = y$", "$x \neq y$"

If $\phi(a_1, \dots, a_n)$, $\psi(a_1, \dots, a_n)$, $\sigma(y, a_1, \dots, a_n)$ are formulas of the language of set theory with parameters $a_1, \dots, a_n \in M$, let

(5) $p \Vdash (\phi(\bar{a}_1, \dots, \bar{a}_n) \vee \psi(\bar{a}_1, \dots, \bar{a}_n))$

$\iff p \Vdash \phi(\bar{a}_1, \dots, \bar{a}_n)$ or $p \Vdash \psi(\bar{a}_1, \dots, \bar{a}_n)$

(6) $p \Vdash \exists y\, \sigma(y, \bar{a}_1, \dots, \bar{a}_n)$

$\iff \exists b \quad p \Vdash \sigma(\bar{b}, \bar{a}_1, \dots, \bar{a}_n)$

(7) $p \Vdash \neg \phi(\bar{a}_1, \dots, \bar{a}_n)$

$\iff \forall q \leq p \quad q \not\Vdash \phi(\bar{a}_1, \dots, \bar{a}_n)$.

¡ <u>Warning</u>! Contrary to classical logic, for the forcing relation, we consider formulas as being built up from literals and <u>not</u> from atomic formulas.

Eg., $p \Vdash \neg(\bar{x} \notin \bar{y}) \iff \forall q \leq p \;\; q \not\Vdash \bar{x} \notin \bar{y}$

$\iff \forall q \leq p\, \exists r \leq q \;\; r \Vdash \bar{x} \in \bar{y}$

which is weaker than $p \Vdash \bar{x} \in \bar{y}$.

<u>Lemma</u> Suppose $p \le q$. Then

$$q \Vdash \phi(\bar{a}_1, \dots, \bar{a}_n) \implies p \Vdash \phi(\bar{a}_1, \dots, \bar{a}_n).$$

<u>Proof</u> Suppose first that ϕ is one of the formulas $a \in b$, $a \neq b$, $a \notin b$ or $a = b$. Then the result is proved by induction on the stratification $\mathcal{S}$.

Now for any other formula ϕ, the result is proved by induction on the construction of ϕ using (5), (6), (7). $\square$

Formulas involving $\forall$ and $\&$ are introduced by defining

$$\forall x \, \phi \iff \neg \exists x \neg \phi$$

$$\phi \,\&\, \psi \iff \neg(\neg\phi \vee \neg\psi).$$

Thus

(8) $p \Vdash \forall y \, \sigma(y, \bar{a}_1, \dots, \bar{a}_n)$

$\iff p \Vdash \neg \exists y \neg \sigma(y, \bar{a}_1, \dots, \bar{a}_n)$

$\iff \forall q \le p \; \forall b \;\; q \not\Vdash \neg \sigma(\bar{b}, \bar{a}_1, \dots, \bar{a}_n)$

$\iff \forall b \; \forall q \le p \; \exists r \le q \;\; r \Vdash \sigma(\bar{b}, \bar{a}_1, \dots, \bar{a}_n).$

(9) $p \Vdash (\phi(\bar{a}_1, \ldots, \bar{a}_n) \,\&\, \psi(\bar{a}_1, \ldots, \bar{a}_n))$

$$\Longleftrightarrow \forall q \leq p \Big(\exists r \leq q \ \ r \Vdash \phi(\bar{a}_1, \ldots, \bar{a}_n) \quad \text{and}$$
$$\exists s \leq q \ \ s \Vdash \psi(\bar{a}_1, \ldots, \bar{a}_n) \Big).$$

<u>Lemma</u> For all $p \in \mathbb{P}$ and $a, b \in M$ with $(a, p) \in b$,

$$p \Vdash \bar{a} = \bar{a} \quad \text{and} \quad p \Vdash \bar{a} \in \bar{b} .$$

<u>Proof</u> We show this by induction on $rk(a)$. So suppose $q \Vdash \bar{c} = \bar{c}$ for all $q \in \mathbb{P}$ and $c \in M$ with $rk(c) < rk(a)$. Then

$$p \Vdash \bar{a} = \bar{a} \iff \neg \exists q \leq p \quad q \Vdash \bar{a} \neq \bar{a}$$
$$\iff \neg \exists q \leq p \ \exists r \geq q \ \exists c$$
$$\big((c, r) \in a \ \& \ q \Vdash \bar{c} \notin \bar{a} \big).$$

Now, suppose towards a contradiction that $q \leq p$, $r \geq q$ and $(c, r) \in a$ with $q \Vdash \bar{c} \notin \bar{a}$. Then $rk(c) < rk(a)$ and so $q \Vdash \bar{c} = \bar{c}$, whence by (1), $q \Vdash \bar{c} \in \bar{a}$. By (3), this contradicts $q \Vdash \bar{c} \notin \bar{a}$. Therefore $p \Vdash \bar{a} = \bar{a}$.

To see that $p \Vdash \bar{a} \in \bar{b}$, we apply (1) directly. $\square$

The truth lemma

Lemma For any formula $\phi(a_1, \ldots, a_n)$ with parameters $a_1, \ldots, a_n \in M$, there is $p \in G$ s.t.

$$p \Vdash \phi(\bar{a}_1, \ldots, \bar{a}_n) \quad \text{or} \quad p \Vdash \neg \phi(\bar{a}_1, \ldots, \bar{a}_n).$$

Proof
Note that by (7), the set

$$D = \{ p \in \mathbb{P} \mid p \Vdash \phi(\bar{a}_1, \ldots, \bar{a}_n) \text{ or } p \Vdash \neg \phi(\bar{a}_1, \ldots, \bar{a}_n) \}$$

is dense and also belongs to M. So $G \cap D \neq \emptyset$. $\square$

Lemma If $a, b \in M$, we have

$$\Phi(a) \in \Phi(b) \iff \exists p \in G \quad p \Vdash \bar{a} \in \bar{b}$$

$$\Phi(a) \neq \Phi(b) \iff \exists p \in G \quad p \Vdash \bar{a} \neq \bar{b}.$$

Proof The proof is by induction on $\rho(a, b)$. So assume the result holds for all $c, d \in M$ with $\rho(c, d) < \rho(a, b)$.

Suppose $\Phi(a) \in \Phi(b)$ and pick $c \in M$, $q \in G$ s.t. $\Phi(a) = \Phi(c)$, $(c, q) \in b$. Choose also, by the previous lemma, some $p \leq q$, $p \in G$, such that either $p \Vdash \bar{c} \neq \bar{a}$ or $p \Vdash \neg(\bar{c} \neq \bar{a})$, i.e., $p \Vdash \bar{c} = \bar{a}$

Since $\rho(c,a) < \rho(a,b)$, we have

$$p \Vdash \bar{c} \neq \bar{a} \rightarrow \Phi(c) \neq \Phi(a)$$

and we must therefore have $p \Vdash \bar{c} = \bar{a}$.
By (1) it follows that $p \Vdash \bar{a} \in \bar{b}$.

Conversely, if $p \Vdash \bar{a} \in \bar{b}$ for some $p \in G$, then by (1), there is $r \geq p$ and $c \in M$ such that $(c,r) \in b$ and $p \Vdash \bar{c} = \bar{a}$.
Again $\rho(c,a) < \rho(a,b)$, so if $\Phi(c) \neq \Phi(a)$, then by the induction hypothesis, there is $q \leq p$, $q \in G$, s.t. $q \Vdash \bar{c} \neq \bar{a}$, contradicting $p \Vdash \bar{c} = \bar{a}$. Thus, $\Phi(a) = \Phi(c) \in \Phi(b)$.

The proof for the second equivalence is similar. $\square$

<u>Truth Lemma</u> Suppose $\psi(x_1, \dots, x_n)$ is a formula without parameters and $a_1, \dots, a_n \in M$. Then

$$\psi^{M[G]}(\Phi(a_1), \dots, \Phi(a_n))$$
$$\Updownarrow$$
$$\exists p \in G \quad p \Vdash \psi(\bar{a}_1, \dots, \bar{a}_n).$$

<u>Proof</u> While the previous lemma is proved by induction on ρ, this is proved by induction on the construction of ψ over literals "$x \in y$", "$x \notin y$", "$x = y$", "$x \neq y$".

Now if ψ is either $x_1 \in x_2$ or $x_1 \neq x_2$, then result is proved by the previous lemma.

Suppose ψ is $x_1 \notin x_2$. Then for $a_1, a_2 \in M$

$$\exists p \in G \quad p \Vdash \bar{a}_1 \notin \bar{a}_2 \quad \Longleftrightarrow$$

$$\exists p \in G \quad \forall q \leq p \quad q \not\Vdash \bar{a}_1 \in \bar{a}_2$$

so if $\Phi(a_1) \in \Phi(a_2)$, there is $r \in G$ s.t. $r \Vdash \bar{a}_1 \in \bar{a}_2$, and so for any $p \in G$ there is $q \leq r$, $q \leq p$ s.t. $q \Vdash \bar{a}_1 \in \bar{a}_2$, contradicting $\exists p \in G \quad p \Vdash \bar{a}_1 \notin \bar{a}_2$.

Thus,

$$\exists p \in G \; p \Vdash \bar{a}_1 \notin \bar{a}_2 \quad \Rightarrow \quad \Phi(a_1) \notin \Phi(a_2).$$

$$\Rightarrow \forall p \in G \quad p \not\Vdash \bar{a}_1 \in \bar{a}_2 \rightarrow \exists p \in G \; p \Vdash \bar{a}_1 \notin \bar{a}_2.$$

Similarly for $x_1 = x_2$ and $\psi = \neg\phi$.

Suppose $\psi(x_1, -, x_n) = \exists y \; \sigma(y, x_1, -, x_n)$ and the lemma holds for σ. Then for any $a_1, -, a_n \in M$,

$$\psi^{M[G]}(\Phi(a_1), -, \Phi(a_n))$$

$$\Longleftrightarrow \exists b \in M \quad \sigma^{M[G]}(\Phi(b), \Phi(a_1), -, \Phi(a_n))$$

$$\Longleftrightarrow \exists b \in M \; \exists p \in G \quad p \Vdash \sigma(\bar{b}, \bar{a}_1, -, \bar{a}_n)$$

$$\Longleftrightarrow \exists p \in G \quad p \Vdash \exists y \; \sigma(y, \bar{a}_1, -, \bar{a}_n)$$

$\Leftrightarrow \exists p \in G \quad p \Vdash \psi(\bar{a}_1, -, \bar{a}_n)$.

The case of $\vee$ is easy. $\square$

<u>Notation</u> Instead of writing

$$p \Vdash \psi(\bar{\hat{a}}_1, \bar{\hat{a}}_2, --, \bar{\hat{a}}_n, \bar{b}_1, -, \bar{b}_k)$$

for $a_1, -, a_n, b_1, -, b_k \in M$, we simplify notation and write

$$p \Vdash \psi(a_1, -, a_n, \bar{b}_1, -, \bar{b}_k).$$

So "unbarred" a's refer to themselves in the generic extension $M[G]$, while "barred" b's refer to $\Phi(b)$ in $M[G]$. So the truth lemma reads:

- For any formula $\psi(x_1, -, x_n, y_1, -, y_k)$ and any $a_1, -, a_n, b_1, -, b_k \in M$,

$$\psi^{M[G]}(a_1, -, a_n, \Phi(b_1), -, \Phi(b_k))$$

$$\Updownarrow$$

$$\exists p \in G \quad p \Vdash \psi(a_1, -, a_n, \bar{b}_1, -, \bar{b}_k).$$

Theorem $M[G]$ satisfies ZF.

Proof We need only check the power set axiom and the replacement scheme.

For the power set axiom, suppose $\Phi(a) \in M[G]$ for some $a \in M$. We let

$$a' = \{(x,p) \in Cl(a) \times \mathbb{P} \mid \exists q \geq p \quad (x,q) \in a\} \in M$$

and $\quad b = \mathcal{P}^M(a') \times \mathbb{P} \quad \in M$.

We claim that $\Phi(b) = \mathcal{P}^{M[G]}(\Phi(a))$.

To see this, suppose $\Phi(u) \in \Phi(b)$ and assume wlog that there is $r \in G$ s.t. $(u,r) \in b$.

Then $u \subseteq a'$ and so for any $(x,p) \in u$, there is $q \geq p$ with $(x,p) \in a$. Since G is upwards closed, this means that if $\Phi(x) \in \Phi(u)$, also $\Phi(x) \in \Phi(a)$, whence $\Phi(u) \subseteq \Phi(a)$ and $\Phi(b) \subseteq \mathcal{P}^{M[G]}(\Phi(a))$.

Conversely, if $\Phi(u) \subseteq \Phi(a)$, we need to find $v \subseteq a'$ such that $\Phi(v) = \Phi(u)$. It follows then that $(v,r) \in b$ for any $r \in G$, and so $\Phi(u) = \Phi(v) \in \Phi(b)$.

Set

$$v = \{(x,p) \in a' \mid p \Vdash \bar{x} \in \bar{u}\}.$$

Then for $\Phi(v) = \Phi(u)$, suppose that $\Phi(x) \in \Phi(v)$ for some $(x,p) \in a'$, $p \Vdash \bar{x} \in \bar{u}$ and $p \in G$. Then by the truth lemma also $\Phi(x) \in \Phi(u)$, so $\Phi(v) \subseteq \Phi(u)$.

Conversely, if $\Phi(x) \in \Phi(u) \subseteq \Phi(a)$ for some $(x,q) \in a$ with $q \in G$, there is by the truth lemma some $r \in G$ s.t. $r \Vdash \bar{x} \in \bar{u}$. Now let $p \in G$, $p \le r, q$, then $p \Vdash \bar{x} \in \bar{u}$ and so $(x,p) \in v$ and thus also $\Phi(x) \in \Phi(v)$. So $\Phi(u) \subseteq \Phi(v)$.

For replacement: Suppose $\Phi(a) \in M[G]$ and $\psi(x, y, \Phi(a_1), \ldots, \Phi(a_n))$ is a formula that in $M[G]$ defines a class function of the variable x.

Let

$$B = \{\Phi(y) \in M[G] \mid \exists \Phi(x) \in \Phi(a) \ \psi^{M[G]}(\Phi(x), \Phi(y), \Phi(a_1), \ldots, \Phi(a_n))\}.$$

We need to find $b \in M$ s.t. $\Phi(b) = B$.

Now, for all $x \in M$ and $p \in \mathbb{P}$,

$F(x,p) = \{ y \in M \mid y$ has minimal rank s.t.

$p \Vdash \psi(\bar{x}, \bar{y}, \bar{a}_1, \dots, \bar{a}_n) \}$.

So F is a class function in M and thus by applying replacement in M, we see that the following is a set in M:

$b = \{ (y,p) \in M \times \mathbb{P} \mid \exists x \ \exists q \geq p \ \ (x,q) \in a \ \& \ y \in F(x,p) \}$.

Now, if $\Phi(y) \in \Phi(b)$ for some $p \in G$ s.t. $(y,p) \in b$, we find x and $q \geq p$ s.t. $(x,q) \in a$ and $y \in F(x,p)$, i.e.,

$p \Vdash \psi(\bar{x}, \bar{y}, \bar{a}_1, -, \bar{a}_n)$.

By the truth lemma, $\psi^{M[G]}(\Phi(x), \Phi(y), \Phi(a_1), -, \Phi(a_n))$ and so $\Phi(y) \in B$.

Conversely, if $\Phi(y) \in B$ find $\Phi(x) \in \Phi(a)$ such that

$\psi^M(\Phi(x), \Phi(y), \Phi(a_1), -, \Phi(a_n))$

and assume wlog that $(x,q) \in a$ for some $q \in G$. By the truth lemma, there is $p \in G$ s.t. $p \Vdash \psi(\bar{x}, \bar{y}, \bar{a}_1, -, \bar{a}_n)$ and we can assume that $p \leq q$. Pick $y_0 \in F(x,p) \neq \emptyset$. Then $p \Vdash \psi(\bar{x}, \bar{y}_0, \bar{a}_1, -, \bar{a}_n)$ and so, by the truth lemma,

$$\psi^{M[G]}(\Phi(x), \Phi(y_0), \Phi(a_1), -, \Phi(a_n)).$$

Since ψ defined a class function in $M[G]$, this implies that $\Phi(y) = \Phi(y_0)$. As also $(y_0, p) \in b$, we have $\Phi(y) = \Phi(y_0) \in \Phi(b)$. Therefore, $B \subseteq \Phi(b)$ and so $B = \Phi(b)$. $\square$

<u>Theorem</u> M and $M[G]$ contain the same ordinals.

<u>Proof</u> Recall that as $M, M[G]$ satisfy ZF and are transitive, the notion of ordinal is absolute for both. So the result follows from $M \cap \mathrm{Ord} = M[G] \cap \mathrm{Ord}$. $\square$

<u>Theorem</u> $M[G]$ is the smallest transitive set model of ZF such that $M \subseteq M[G]$ and $G \in M[G]$.

<u>Theorem</u> If M satisfies AC, then so does $M[G]$.

<u>Proof</u> Suppose $\Phi(a) \in M[G]$. It suffices to find in $M[G]$ a surjection from an ordinal onto a superset of $\Phi(a)$. So AC in M, let $f : \alpha \to Cl(a)$ be a surjection from