# Polish group actions and model theory.

**Theorem** (Dixmier) Suppose $G$ is a Polish group and $H \leq G$ a closed subgroup. Then there is a Borel set $S \subseteq G$ which is a transversal for the left $H$-coset equivalence relation on $G$.

**Proof** Let $d$ be a compatible complete metric on $G$ and let $(U_s)$ be a Souslin scheme of open sets of $G$ s.t.

(i) $\overline{U_s} \subseteq U_t$ for $t \subseteq s$, $U_s \neq \emptyset$

(ii) $U_s = \bigcup_{i=0}^{\infty} U_{si}$, (iii) diam $U_s < \frac{1}{|s|+1}$, $s \neq \emptyset$.

Let $\phi(\alpha) = \bigcap_{n=0}^{\infty} U_{\alpha|n}$ for $\alpha \in \mathbb{N}^{\mathbb{N}}$. Then $\phi : \mathbb{N}^{\mathbb{N}} \to G$ is a continuous and open surjection.

Thus, for every $s \in \mathbb{N}^{<\mathbb{N}}$, the set

$$D_s = \{ g \in G \mid gH \cap \phi(N_s) \neq \emptyset \}$$

$$= \{ g \in G \mid g \in \phi(N_s) \cdot H \}$$

is open, since $\phi(N_s)$ and thus $\phi(N_s)H$ is open.

Note also that $D_s$ is saturated wrt $H$, so

$$g \in G \longmapsto T_g = \{ s \in \mathbb{N}^{<\mathbb{N}} \mid g \in D_s \}$$

is an $H$-invariant assignment of pruned trees on $\mathbb{N}$, $T : G \to 2^{\mathbb{N}^{<\mathbb{N}}}$.

Let $b(T_g)$ be the left-most branch of $T_g$, i.e., $\beta = b(T_g) \iff \forall s \in \mathbb{N}^{<\mathbb{N}} \big(s <_{lex} \beta|_{|s|} \rightarrow s \notin T_g\big)$. Then $g \mapsto b(T_g)$ has Borel graph and hence is Borel.

We can now set

$$g \in S \iff \phi(b(T_g)) = g . \qquad \square$$

<u>Lemma</u> Let $G$ be a Polish group acting continuously on a Polish space $X$. Then every orbit is Borel.

<u>Pf</u> Fix $x \in X$ and note that the stabiliser $G_x = \{g \in G \mid gx = x\}$ is closed. So let $S$ be a Borel transversal for the left cosets of $G_x$ in $G$. Then $g \in G \longmapsto gx$ restricts to a Borel bijection from $S$ onto $[x]_G$. So $[x]_G$ is the injective Borel image of a Borel set and hence Borel. $\square$

Theorem (Montgomery - Novikov)

[Category quantifiers preserve measurability]

Suppose $X, Y$ are Polish spaces and $A \subseteq X \times Y$ is Borel. Then for any open $\emptyset \neq U \subseteq Y$, the set

$$\{x \in X \mid A_x \text{ is meagre in } U\}$$

is Borel in $X$. Same for non-meagre, comeagre.

Pf Suppose first that $A = B \times C$, where $B$ and $C$ are Borel. Then $A_x = \begin{cases} C & \text{for } x \in B \\ \emptyset & \text{for } x \notin B \end{cases}$.

So $\{x \in X \mid A_x \text{ is meagre in } U\}$

$$= \begin{cases} X & \text{if } C \text{ is meagre in } U \\ \sim B & \text{if not,} \end{cases}$$

which is Borel.

Now, if the result holds for $A$, then if $\{V_n\}$ is a basis

$$\{x \in X \mid (\sim A)_x \text{ is meagre in } U\}$$

$$= \bigcap_{\emptyset \neq V_n \subseteq U} \sim \{x \in X \mid A_x \text{ is meagre in } V_n\}$$

which is Borel.

Again if the result holds for $A_n$, then

$$\{x \in X \mid (\bigcup_n A_n)_x \text{ is meagre in } U\}$$
$$= \bigcap_n \{x \in X \mid (A_n)_x \text{ is meagre in } U\}.$$

By induction it holds for all Borel $A \subseteq X \times Y$. $\square$

Note that we can reformulate the above result as:

if $X, Y$ are Polish, $A \subseteq X \times Y$ is Borel and $U \subseteq Y$ is open, then the sets

$$x \in B \iff \forall^* y \in Y \quad (x,y) \in A$$
$$x \in B \iff \exists^* y \in Y \quad (x,y) \in A$$

are Borel too.

<u>Vaught transforms</u>

Suppose $G$ is a Polish group acting in a Borel fashion on a standard Borel space $X$.

Define for $A \subseteq X$,

$$A^* = \{x \in X \mid \forall^* g \in G \quad gx \in A\}$$
$$A^\Delta = \{x \in X \mid \exists^* g \in G \quad gx \in A\}.$$

Similarly, for non-empty open $U \subseteq G$,

$$A^{*U} = \{x \in X \mid \forall^* g \in U \ gx \in A\}$$

$$A^{\Delta U} = \{x \in X \mid \exists^* g \in U \ gx \in A\}.$$

By the preceding theorem, if $A$ is Borel, then so are $A^*, A^\Delta, A^{*U}, A^{\Delta U}$.

<u>Proposition</u>: The Vaught transforms $A^*$ and $A^\Delta$ are $G$-invariant and

$$(A) \subseteq A^* \subseteq A^\Delta \subseteq [A],$$

where $(A)$ is the largest $G$-invariant set contained in $A$ and $[A]$ is the $G$-saturation of $A$.

<u>Pf</u> Note that if $x \in A^*$, then for any $h \in G$ $D = \{g \in G \mid gx \in A\}$ and also $Dh^{-1}$ are comeagre. So $\forall^* g \in G$ $g = fh^{-1}$, where $fx \in A$, i.e.,

$\forall^* g \in G \ ghx \in A$. So $hx \in A^*$.

Similarly for $A^\Delta$. $\square$

## The logic action

Suppose $L$ is a countable relational language, i.e., $L = (R_i)_{i \in I}$, where $I$ is ctbl and each $R_i$ is an $n_i$-ary relation symbol.

Then any $L$-structure $\mathcal{M}$ with universe $\mathbb{N}$ can be seen as simply the sequence $(R_i^{\mathcal{M}})_{i \in I}$, where $R_i^{\mathcal{M}} \subseteq \mathbb{N}^{n_i}$.

So we can equate $\mathcal{M}$ with the element

$$x = (x_i)_{i \in I} \in \prod_{i \in I} 2^{\mathbb{N}^{n_i}}, \quad x_i = \chi_{R_i^{\mathcal{M}}} \in 2^{\mathbb{N}^{n_i}}.$$

We denote $\mathcal{M}$ by $\mathcal{M}_x$.

In this way, $X_L = \prod_{i \in I} 2^{\mathbb{N}^{n_i}}$ becomes the space of all $L$-structures with universe $\mathbb{N}$ (We call it the space of denumerable $L$-structures).

Note that if $\mathcal{M}_x \sim x = (x_i)$, $\mathcal{M}_y \sim y = (y_i)$ then $\mathcal{M}_x$ and $\mathcal{M}_y$ are <u>isomorphic</u>, $\mathcal{M}_x \cong \mathcal{M}_y$ iff there is a bijection $g : \mathbb{N} \to \mathbb{N}$ s.t.

$$\forall i \ \forall (a_1, \dots, a_{n_i}) \in \mathbb{N}^{n_i} \quad \mathcal{M}_x \models R_i(a_1, \dots, a_{n_i})$$
$$\iff \mathcal{M}_y \models R_i(ga_1, \dots, ga_{n_i})$$

i.e., $\forall i \, \forall a_1, \dots, a_{n_i} \quad x_i(g^{-1}(a_1), \dots, g^{-1}(a_{n_i})) = 1$

$$\leftrightarrow y_i(a_1, \dots, a_{n_i}) = 1.$$

Now, let $S_\infty$ be the group of all permutations of $\mathbb{N}$ and let $S_\infty$ act on $X_L$ by

$$(g \cdot x)_i(a_1, \dots, a_{n_i}) = x_i(g^{-1}(a_1), \dots, g^{-1}(a_{n_i})).$$

Then we see that

$$\mathcal{M}_x \cong \mathcal{M}_y \iff \exists g \in S_\infty \quad gx = y.$$

Now, equip $S_\infty$ with the topology having subbasis

$$\{g \in S_\infty \mid g(n) = m\}, \quad n \in \mathbb{N}.$$

This is the topology induced from $\mathbb{N}^{\mathbb{N}}$, where we identify $g \in S_\infty$ with its graph:

$$x_n = m \iff g(n) = m.$$

So $S_\infty$ is a $G_\delta$ subset of $\mathbb{N}^{\mathbb{N}}$ and hence is Polish. Moreover, the group operation is continuous, so $S_\infty$ is a Polish group.

Also, the action $S_\infty \curvearrowright X_L$ is continuous.

Theorem (Scott)

The isomorphism class $\{x \in X_L \mid \mathcal{M}_x \cong \mathcal{M}_y\}$ of any model $\mathcal{M}_y$ is Borel.

We now extend the notion of $L$-formula as follows:

Defn Let $L_{\omega_1\omega}$ be the smallest set of expressions st.

(i) any first order $L$-formula $\phi$ belongs to $L_{\omega_1\omega}$.

(ii) if $(\phi_n)_{n=1}^{\infty}$ is a sequence in $L_{\omega_1\omega}$ all of whose free variables are among $v_0, \ldots, v_{k-1}$, then $\bigwedge_n \phi_n$ and $\bigvee_n \phi_n$ belong to $L_{\omega_1\omega}$

(iii) if $\phi$ is in $L_{\omega_1\omega}$, then so are $\exists v\, \phi$, $\forall v\, \phi$ for all variables $v$.

(v) if $\phi$ is in $L_{\omega_1\omega}$, then so is $\neg\phi$.

Note: By (ii), any $L_{\omega_1\omega}$ only has finitely many free variables.

We have the obvious extension of the semantics of first order logic.

<u>Proposition</u> Let $\phi(v_0, \ldots, v_{k-1})$ be an $L_{\omega_1\omega}$ formula. Then the sets

$$A_{\phi,k} = \{(x,s) \in X_L \times \mathbb{N}^k \mid \mathcal{M}_x \models \phi(s_0, s_1, \ldots, s_{k-1})\}$$

is Borel in $X_L \times \mathbb{N}^n$, where $\mathbb{N}^n$ is discrete.

<u>Pf</u> This is pretty obvious by induction on the construction of $\phi$. Eg.,

$$A_{\bigwedge_n \phi_n, k} = \{(x,s) \in X_L \times \mathbb{N}^k \mid \mathcal{M}_x \models (\bigwedge_n \phi_n)(s_0, \ldots, s_{k-1})\}$$

$$= \bigcap_n \{(x,s) \in X_L \times \mathbb{N}^k \mid \mathcal{M}_x \models \phi_n(s_0, \ldots, s_{k-1})\}$$

and

$$A_{\exists v \phi, k} = \{(x,s) \in X_L \times \mathbb{N}^k \mid \mathcal{M}_x \models (\exists v\, \phi)(s_0, \ldots, s_{k-1})\}$$

$$= pr_{X_L \times \mathbb{N}^k}\left(\{(x, si) \in X_L \times \mathbb{N}^{k+1} \mid \mathcal{M}_x \models \phi(s_0, \ldots, s_{k-1}, i)\}\right). \blacksquare$$

Recall that a <u>sentence</u> is a formula with no free variables. So if $\sigma$ is a sentence, then

$$A_{\sigma,0} = \{x \in X_L \mid \mathcal{M}_x \models \sigma\}$$

is an isomorphism invariant Borel set.

**Theorem** (López-Escobar) The invariant Borel sets of $X_L$ are exactly the sets of the form $A_{\sigma,0}$ for $\sigma$ an $L_{\omega_1\omega}$ formula.

**Pf** We write $[u] = \{g \in S_\infty \mid u \subseteq g^{-1}\}$, where $u \in \mathbb{N}^{<\mathbb{N}}$ and we have identified $g \in S_\infty$ with the sequence $(g(0), g(1), g(2), \ldots) \in \mathbb{N}^{\mathbb{N}}$. Note that unless $u = (u_0, \ldots, u_{n-1})$ is *injective*, i.e., $u_i \neq u_j$ when $i \neq j$, we have $[u] = \emptyset$.

Now, for $A \subseteq X_L$, $k \in \mathbb{N}$ set

$$A^{*k} = \{(x, u) \in X_L \times \mathbb{N}^k \mid u \text{ is injective } \& \ x \in A^{*[u]}\}$$

$$A^{\Delta k} = \{(x, u) \in X_L \times \mathbb{N}^k \mid u \text{ is injective } \& \ x \in A^{\Delta [u]}\}$$

We claim that for any Borel $A \subseteq X_L$ and $k \in \mathbb{N}$, there is some $L_{\omega_1\omega}$ formula $\phi(v_0, \ldots, v_{k-1})$ st.

$$A^{*k} = A_{\phi, k}.$$

Note that this would be enough, for if then $A$ is *invariant*:

$$A = A^* = A^{*0} = A_{\phi,0}, \quad \text{for some sentence } \phi.$$

The proof of the claim is by induction on the complexity of $A$.

First, suppose $A$ is basic open. Then we can easily find a formula $\theta(v_0, \dots, v_{p-1}) \in L_{\omega_1\omega}$ s.t.

$$A = \{x \in X_L \mid \mathcal{M}_x \models \theta(0, 1, \dots, p-1)\}.$$

Then $(*) \iff (x, u) \in A^{*k} \iff u$ inj. & $x \in A^{*[u]}$

$\iff u$ inj. & $\forall^* g \in [u] \;\; gx \in A$

$\iff u$ inj. & $\forall^* g \in [u] \;\; \mathcal{M}_{gx} \models \theta(0, \dots, p-1)$

$\iff u$ inj. & $\forall^* g \in [u] \;\; \mathcal{M}_x \models \theta(g^{-1}(0), \dots, g^{-1}(p-1))$

Now, if $k > p$, then for $g \in [u]$,

$(g^{-1}(0), \dots, g^{-1}(p-1)) = (u(0), \dots, u(p-1))$ and so

$(*) \iff u$ inj. & $\mathcal{M}_x \models \theta(u(0), \dots, u(p-1))$

$\iff \mathcal{M}_x \models \left(\theta(u(0), \dots, u(p-1)) \wedge \bigwedge_{i \neq j} u_i \neq u_j\right)$

Setting $\phi = \theta \wedge \bigwedge_{i \neq j} u_i \neq u_j$, we see $A^{*k} = A_{\phi,k}$.

On the other hand, if $k \leq p$, then

$(*) \iff u$ inj. & $\forall w \supseteq u$, $w \in \mathbb{N}^p$ inj

$\mathcal{M}_x \models \theta(w(0), \dots, w(p-1))$

$\iff \mathcal{M} \models \forall w_k, \dots, w_{p-1} \left(\bigwedge_{i \neq j} (w_i \neq w_j \;\&\; u_i \neq w_j \;\&\; u_i \neq u_j\right)$

$$\longrightarrow \Theta(u(0), \ldots, u(k-1), w_k, \ldots, w_{p-1}) \Big) .$$

Again we find that $A^{*k} = A_{\phi,k}$ for some formula $\phi$.

Now, suppose that $A^{*k} = A_{\phi_k, k}$ for every $k$. Then

$$(x,u) \in (\sim A)^{*k} \iff x \in (\sim A)^{*[u]}$$

$$\iff \forall w \exists u \text{ inj. } x \notin A^{*[w]}$$

$$\iff \forall l \geq k \ \forall w \exists u, w \in \mathbb{N}^l \text{ inj } (x,w) \notin A_{\phi_l, l}$$

$$\iff \mathcal{M}_x \models \bigwedge_{l \geq k} \forall w_k, \ldots w_{l-1} \Big( \big( \bigwedge_{i \neq j} w_i \neq w_j \ \& \ u_i \neq w_j \ \& \ u_i \neq u_j \big)$$

$$\longrightarrow \neg \phi_l (u(0), \ldots, u(k-1), w_k, \ldots, w_{l-1}) \Big)$$

We leave the case of ctbl intersections as an exercise. $\square$

<u>Corollary</u> (Scott) Let $L$ be a ctbl. relational language. Then for any ctbl. $L$-structure $\mathcal{M}$ there is an $L_{\omega_1 \omega}$-sentence $\sigma$ s.t. for any $L$-structure $\mathcal{N}$, $\mathcal{N} \cong \mathcal{M} \iff \mathcal{N} \models \sigma$.

<u>The interpolation theorem</u> (López-Escobar)
Let $L$ be a ctbl relational language and $R, S$ be two distinct relational symbols not in $L$. Suppose $\sigma, \varphi$ are $L \cup \{R\}$ and $L \cup \{S\}$ sentences resp. st. $\sigma \models^* \varphi$. Then for some $L$-sentence $\tau$ we have $\sigma \models^* \tau$ and $\tau \models^* \varphi$.

Here $\sigma \models^* \varrho$ denotes that for any ctbl ~~structure~~,
if $\mathcal{M} \models \sigma$ then also $\mathcal{M} \models \varrho$.

<u>Pf</u> Suppose $R$ has arity $n$ and $S$ arity $m$. Then

$$A = \{x \in X_L \mid \exists a \in 2^{\mathbb{N}^n} \ \mathcal{M}_{(x,a)} \models \sigma\}$$

$$B = \{x \in X_L \mid \forall b \in 2^{\mathbb{N}^m} \ \mathcal{M}_{(x,b)} \models \varrho\}$$

are analytic, resp. coanalytic.

Here, $(x,a) \in X_L \times 2^{\mathbb{N}^n}$ is the expanded structure in which $R^{\mathcal{M}(x,a)} = a$. Similarly for $(x,b) \in X_L \times 2^{\mathbb{N}^m}$.

The assumption says:

$$\mathcal{M}_{(x,a,b)} \models \sigma \implies \mathcal{M}_{(x,a,b)} \models \varrho .$$

But as $\mathcal{M}_{(x,a,b)} \models \sigma \iff \mathcal{M}_{(x,a)} \models \sigma$ and
$\mathcal{M}_{(x,a,b)} \models \varrho \iff \mathcal{M}_{(x,b)} \models \varrho$, this gives

$A \subseteq B$. Now both $A$ and $\sim B$ are $\underset{\sim}{\Sigma}^1_1$,
and invariant, so if we can separate $A$ from $\sim B$ by an invariant Borel set
$D = \{x \in X_L \mid \mathcal{M}_x \models \tau\}$, $\tau$ an $L_{\omega_1 \omega}$-sentence,
then we see that $\sigma \models^* \tau$, $\tau \models^* \varrho$. $\square$

<u>Proposition</u> Two disjoint invariant $\underset{\sim}{\Sigma}^1_1$ sets can be separated by an invariant Borel set.