THE BAIRE CLASSES

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Proposition 1. *Let X be metrisable and $1 \leqslant \xi < \zeta$. Then $\Sigma^0_\xi \cup \Pi^0_\xi \subseteq \Delta^0_\zeta$. It follows that we have the following diagram where every class is contained in all the classes to the right of it.*

$$\begin{array}{ccccccccc} & \Sigma^0_1 & & \Sigma^0_2 & & & \Sigma^0_\omega & & \\ \Delta^0_1 & & \Delta^0_2 & & \cdots & \Delta^0_\omega & & \Delta^0_{\omega+1} & \cdots \\ & \Pi^0_1 & & \Pi^0_2 & & & \Pi^0_\omega & & \end{array}$$

Proof. Note that, as $\Delta^0_\zeta = \Sigma^0_\zeta \cap \Pi^0_\zeta$ is closed under complementation, it suffices to prove that $\Sigma^0_\xi \subseteq \Delta^0_\zeta = \Sigma^0_\zeta \cap \Pi^0_\zeta$. Also, as $\Sigma^0_\xi \subseteq \left\{ \bigcup_{n\in\mathbb{N}} A_n \;\middle|\; A_n \in \bigcup_{\eta<\zeta} \Sigma^0_\eta \right\} = \Pi^0_\zeta$, we need only verify that $\Sigma^0_\xi \subseteq \Sigma^0_\zeta$.

Suppose first that $A \in \Sigma^0_\xi$ and $\xi \geqslant 2$. Then we can write $A = \bigcup_{n\in\mathbb{N}} A_n$ for some $A_n \in \bigcup_{\eta<\xi} \Pi^0_\eta \subseteq \bigcup_{\eta<\zeta} \Pi^0_\eta$, showing that also $A \in \Sigma^0_\zeta$.

If instead $A \in \Sigma^0_\xi$ for $\xi = 1$, then A is open and thus also F_σ, i.e., $A \in \Sigma^0_2 \subseteq \Sigma^0_\zeta$. So $\Sigma^0_\xi \subseteq \Sigma^0_\zeta$. □

Since thus the classes Σ^0_ξ, Π^0_ξ and Δ^0_ξ are increasing with ξ and the supremum of a countable sequence of countable ordinals is $< \omega_1$, one easily checks that their unions over $\xi < \omega_1$ are σ-algebras, from which we get the following result.

Corollary 2. $\mathscr{B}(X) = \bigcup_{\xi<\omega_1} \Sigma^0_\xi = \bigcup_{\xi<\omega_1} \Pi^0_\xi = \bigcup_{\xi<\omega_1} \Delta^0_\xi$.