**The Baire Classes**

**Christian Rosendal**

**Proposition 1.** Let \(X\) be metrisable and \(1 \leq \xi < \zeta\). Then \(\Sigma^0_\xi \cup \Pi^0_\xi \subseteq \Delta^0_\zeta\). It follows that we have the following diagram where every class is contained in all the classes to the right of it.

\[
\begin{array}{cccccccc}
\Delta^0_1 & \Sigma^0_1 & \Delta^0_2 & \Sigma^0_2 & \cdots & \Delta^0_\omega & \Sigma^0_\omega & \Delta^0_{\omega+1} & \cdots \\
\Pi^0_1 & \Pi^0_2 & \Pi^0_3 & \Pi^0_4 & \cdots & \Pi^0_\omega & \Pi^0_{\omega+1} & \cdots \\
\end{array}
\]

**Proof.** Note that, as \(\Delta^0_\xi = \Sigma^0_\xi \cap \Pi^0_\xi\) is closed under complementation, it suffices to prove that \(\Sigma^0_\xi \subseteq \Delta^0_\zeta = \Sigma^0_\zeta \cap \Pi^0_\zeta\). Also, as \(\Sigma^0_\xi \subseteq \{ \bigcup_{n \in \mathbb{N}} A_n \mid A_n \in \bigcup_{\eta < \xi} \Sigma^0_\eta \} = \Pi^0_\xi\), we need only verify that \(\Sigma^0_\xi \subseteq \Sigma^0_\zeta\).

Suppose first that \(A \in \Sigma^0_\xi\) and \(\xi \geq 2\). Then we can write \(A = \bigcup_{n \in \mathbb{N}} A_n\) for some \(A_n \in \bigcup_{\eta < \xi} \Pi^0_\eta \subset \bigcup_{\eta < \xi} \Pi^0_\eta\), showing that also \(A \in \Sigma^0_\zeta\).

If instead \(A \in \Sigma^0_\xi\) for \(\xi = 1\), then \(A\) is open and thus also \(F_\sigma\), i.e., \(A \in \Sigma^0_2 \subseteq \Sigma^0_\zeta\). So \(\Sigma^0_\xi \subseteq \Sigma^0_\zeta\). \(\square\)

Since thus the classes \(\Sigma^0_\xi, \Pi^0_\xi\) and \(\Delta^0_\xi\) are increasing with \(\xi\) and the supremum of a countable sequence of countable ordinals is \(< \omega_1\), one easily checks that their unions over \(\xi < \omega_1\) are \(\sigma\)-algebras, from which we get the following result.

**Corollary 2.** \(\mathcal{B}(X) = \bigcup_{\xi < \omega_1} \Sigma^0_\xi = \bigcup_{\xi < \omega_1} \Pi^0_\xi = \bigcup_{\xi < \omega_1} \Delta^0_\xi\).