BAIRE CATEGORY

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1. THE BAIRE CATEGORY THEOREM

Theorem 1 (The Baire category theorem)**.** *Let $(D_n)_{n\in\mathbb{N}}$ be a countable family of dense open subsets of a Polish space X. Then $\bigcap_{n\in\mathbb{N}} D_n$ is dense in X.*

Proof. Fix a compatible complete metric d on X and let $V \subseteq X$ be an arbitrary non-empty open subset. We need to show that $V \cap \bigcap_{n\in\mathbb{N}} D_n \neq \emptyset$. So choose inductively open sets $\emptyset \neq U_n \subseteq D_n$ with $\mathrm{diam}(U_n) < \frac{1}{n}$ such that

$$V \supseteq U_0 \supseteq \overline{U_1} \supseteq U_1 \supseteq \overline{U_2} \supseteq \ldots.$$

Picking $x_n \in U_n$ for all n, we see that (x_n) is a Cauchy sequence and thus converges to some $x \in \bigcap_{n\in\mathbb{N}} \overline{W_n} = \bigcap_{n\in\mathbb{N}} W_n \subseteq V \cap \bigcap_{n\in\mathbb{N}} D_n$. So $V \cap \bigcap_{n\in\mathbb{N}} D_n \neq \emptyset$. □

Exercise 2. Show that the Baire category theorem remains valid in locally compact Hausdorff spaces.

Exercise 3. Show that for any subset A of a topological space X once has

$$\mathrm{int}\,\mathrm{cl}\,\mathrm{int}\,\mathrm{cl}\,A = \mathrm{int}\,\mathrm{cl}\,A.$$

Sets of the form $\mathrm{int}\,\mathrm{cl}\,A$ are called the *regular open* subsets of X.

A subset A of a topological space X is said to be *somewhere dense* if there is a non-empty open set $U \subseteq X$ such that A is dense in U and *nowhere dense* otherwise, that is, if $\mathrm{int}\,\mathrm{cl}\,A = \emptyset$. Note that since $\mathrm{int}\,\mathrm{cl}\,A = \mathrm{int}\,\mathrm{cl}\,\mathrm{cl}\,A$, one sees that A is nowhere dense if and only if $\mathrm{cl}\,A$ is nowhere dense. For example, the boundary ∂U of any open set U is nowhere dense.

$A \subseteq X$ is said to be *meagre* if it is the union of countably many nowhere dense subsets of X and *comeagre* if $X \setminus A$ is meagre. So A is meagre if and only if it can be covered by countably many closed nowhere dense sets and comeagre if and only if it contains the intersection of countably many dense open subsets of X.

Note now that the following are equivalent for a topological space X.

- Every comeagre set is dense,
- meagre sets have empty interior.

A space satisfying these properties is said to be a *Baire* space. E.g., Polish and locally compact Hausdorff spaces are Baire. On the other hand, $\mathbb{Q}$ with the topology induced from $\mathbb{R}$ is not a Baire space, since the whole space is meagre. Thus, a subset of a Baire space is comeagre if and only of it contains a dense G_δ subset.

Exercise 4. Suppose X is a topological space and $Y \subseteq X$ a subspace.
(i) Show that if Y is a non-meagre subset of X, then Y is non-meagre in its relative topology.
(ii) Give an example to show that the converse implication fails.

The meagre subsets of a topological space X form a σ-ideal $\mathscr{M}$, that is,

- if $N \subseteq M \in \mathscr{M}$, then also $N \in \mathscr{M}$,
- if $N_n \in \mathscr{M}$ for all $n \in \mathbb{N}$, then also $\bigcup_{n\in\mathbb{N}} N_n \in \mathscr{M}$.

So, analogously to the null sets of a σ-additive measure, meagreness is a notion of smallness of a topological space, while comeagreness is a notion of largeness.

Exercise 5. Fix a set X and let $\mathscr{P}(X)$ denote the power set of X and $A\triangle B = (A \setminus B)\cup(B\setminus A)$ the symmetric difference of $A,B \subseteq X$.

Show that $\mathscr{P}(X)$ forms an abelian group with the group operation $\triangle$ and identify the group identity and the inverse of an element A. [Hint: Identify $\mathscr{P}(X)$ with $(\mathbb{Z}_2)^X$.]

A subset A of a topological space X has the *Baire property* if there is an open set $U \subseteq X$ such that $M = A\triangle U$ is meagre or, equivalently, there are an open set $U \subseteq X$ and a meagre set $M \subseteq X$ such that $A = U\triangle M$. We also let $\mathbf{BP}(X)$ denote the class of subsets of X having the Baire property.

Lemma 6. *If X is a topological space, $\mathbf{BP}(X)$ is a σ-algebra of subsets of X.*

Proof. Suppose that $A_n \subseteq X$ have the Baire property and $U_n \subseteq X$ are open subsets such that $A_n\triangle U_n$ are meagre. Then

$$\big(\bigcup_n A_n\big)\triangle\big(\bigcup_n U_n\big) \subseteq \bigcup_n A_n\triangle U_n$$

is meagre too, showing that $\bigcup_n A_n$ has the Baire property. Moreover,

$$(\sim A_0)\triangle(\sim \overline{U}_0) = A_0\triangle\overline{U}_0 \subseteq (A_0\triangle U_0)\cup\partial U_0$$

is meagre, whence also $\sim A_0$ has the Baire property. So $\mathbf{BP}(X)$ is closed under countable unions and complementation as required. □

Recall that if X is a topological space, the Borel algebra $\mathbf{B}(X)$ is the σ-algebra generated by the open subsets of X. Thus, by the preceding lemma, $\mathbf{B}(X) \subseteq \mathbf{BP}(X)$, meaning that any Borel set has the Baire property.

Almost all the sets encountered in analysis will have the Baire property and, in fact, to produce sets without the Baire property, one needs the Axiom of Choice in the follwoing precise sense.

Theorem 7. *(S. Shelah – R. M. Solovay) Assuming the consistency of ZF, there is a model of ZF in which every set of reals $A \subseteq \mathbb{R}$ has the Baire property.*

Suppose X is a topological space and $A \subseteq X$ is any subset. We say that A is *comeagre* in an open set U if $U\setminus A$ is meagre. Let also

$$U(A) = \bigcup\{U \subseteq X \text{ open } \big| A \text{ is comeagre in U }\}.$$

Theorem 8. *$U(A)$ is the largest open set in which A is comeagre.*

Proof. Let $(U_i)_{i\in I}$ be a maximal family of pairwise disjoint open sets in which A is comeagre. Then, for each $i \in I$, there are dense open sets $D^i_n \subseteq U_i$ such that $\bigcap_{n\in\mathbb{N}} D^i_n \subseteq A$, whereby, as $D^i_n \cap D^j_m = \emptyset$ for $i \neq j$,

$$\bigcap_{n\in\mathbb{N}}\bigcup_{i\in I} D^i_n = \bigcup_{i\in I}\bigcap_{n\in\mathbb{N}} D^i_n \subseteq A.$$

Since each $\bigcup_{i\in I} D^i_n$ is dense in $\bigcup_{i\in I} U_i$ and $\bigcup_{i\in I} U_i$ is dense in $U(A)$, it follows that each $\bigcup_{i\in I} D^i_n$ is dense in $U(A)$, whereby A is comeagre in $U(A)$. □

Corollary 9. *$A \subseteq X$ has the Baire property if and only if $A\triangle U(A)$ is meagre.*

Proof. If A has the Baire property, then there is an open set $V \subseteq X$ such that $A \triangle V$ is meagre, whence A is comeagre in V, and thus $V \subseteq U(A)$. It follows that $A \setminus U(A) \subseteq A \setminus V$ is meagre too, whence $A \triangle U(A)$ is meagre. $\square$

Exercise 10. In the following, we identify a subset $A \subseteq \mathbb{N}$ with its characteristic function $\chi_A \in 2^{\mathbb{N}}$.

Show that if $G \subseteq 2^{\mathbb{N}}$ is comeagre there is a partition $(I_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$ into disjoint finite intervals, $I_0 < I_1 < \ldots$ and subsets $a_n \subseteq I_n$ such that for all $A \subseteq \mathbb{N}$,

$$(\exists^\infty n \ A \cap I_n = a_n) \Rightarrow A \in G.$$

Exercise 11 (Vertical and horizontal sections)**.** Suppose X and Y are Hausdorff. If $A \subseteq X \times Y$ and $x_0 \in X$, $y_0 \in Y$, we define the *vertical* and *horizontal sections* at x_0 and y_0 respectively by

$$A_{x_0} = \{y \in Y \mid (x_0, y) \in A\},$$
$$A^{y_0} = \{x \in X \mid (x, y_0) \in A\}.$$

(i) Show that

$$(\sim A)_{x_0} = \sim A_{x_0}$$

and that for any family $(A_i)_{i\in I}$ of subsets of $X \times Y$,

$$\Big(\bigcap_{i\in I} A_i\Big)_{x_0} = \bigcap_{i\in I} (A_i)_{x_0}$$

and

$$\Big(\bigcup_{i\in I} A_i\Big)_{x_0} = \bigcup_{i\in I} (A_i)_{x_0}.$$

(ii) Show also that if A is open or compact in $X \times Y$, then A_{x_0} is open, respectively compact in Y. Conclude that the same holds with closed, G_δ, F_σ and K_σ in place of open.

Exercise 12 (Direct and inverse images)**.** Let $f : X \to Y$ be a function between two sets.

(i) Decide which of the set theoretical operations $\sim$, $\bigcap$ and $\bigcup$ commute with taking direct images, i.e.,

$$A \mapsto f(A).$$

(ii) Decide which of the set theoretical operations $\sim$, $\bigcap$ and $\bigcup$ commute with taking inverse images, i.e.,

$$B \mapsto f^{-1}(B).$$

Exercise 13 (Coordinate projections)**.** Let X and Y be topological spaces and define the *first coordinate projection* $\pi_X : X \times Y \to X$ by $\pi_X(x, y) = x$.

(i) Show that for any family $(A_i)_{i\in I}$ of subsets $A_i \subseteq X \times Y$, we have

$$\pi_X\Big(\bigcup_{i\in I} A_i\Big) = \bigcup_{i\in I} \pi_X(A_i)$$

and find examples to show that π_X does not preserve complement $\sim$ nor intersection $\bigcap$ under forward images.

(ii) Show that π_X is continuous and *open*, i.e., for any open set $U \subseteq X \times Y$, $\pi_X(U)$ is open in X.

(iii) Show that if moreover Y is compact and $F \subseteq X \times Y$ is closed, then $\pi_X(F)$ is closed. Conclude that if instead Y is K_σ and $F \subseteq X \times Y$ is F_σ, then also $\pi_X(F)$ is F_σ.

Exercise 14 (D. Montgomery, P. S. Novikov)**.** Let X and Y be Polish spaces and $A \subseteq X \times Y$ a Borel subset. Show that, for any open $U \subseteq Y$, the set

$$\{x \in X \mid A_x \text{ is comeagre in } U\}$$

is Borel. [Hint: Show it first for basic open rectangles $A = V \times W$ and then note that the class of A with the above property is closed under countable unions and complementation.]

If X is a topological space and P is a property of points in X, we write $\forall^* x\, P(x)$ to denote that the set $\{x \in X \mid P(x)\}$ is comeagre in X and similarly $\exists^* x\, P(x)$ to denote that the set $\{x \in X \mid P(x)\}$ is non-meagre in X.

Theorem 15 (K. Kuratowski–S. Ulam)**.** *Let X and Y be second countable topological spaces and $A \subseteq X \times Y$ a set with the Baire property. Then the following are equivalent*

(1) *A is comeagre in $X \times Y$,*
(2) $\forall^* x \in X \;\; A_x$ *is comeagre in Y.*
(3) $\forall^* y \in Y \;\; A^y$ *is comeagre in X.*

Proof. By symmetry, it suffices to show the equivalence of (1) and (2).

To see the implication from (1) to (2), fix a basis $\{U_n\}_{n \in \mathbb{N}}$ for the topology on Y consisting of non-empty open sets and find dense open sets $D_n \subseteq X \times Y$ such that $\bigcap_{n \in \mathbb{N}} D_n \subseteq A$. Then, for any $x \in X$,

$$\begin{aligned} x \in \bigcap_{n,m \in \mathbb{N}} \pi_X\big(D_n \cap (X \times U_m)\big) &\Rightarrow \forall n\; \forall m\; \exists y \in Y\; (x,y) \in D_n \cap (X \times U_m) \\ &\Rightarrow \forall n\; \forall m\; \exists y \in Y\; y \in U_m \cap (D_n)_x \\ &\Rightarrow \forall n\; (D_n)_x \text{ is dense in } Y \\ &\Rightarrow \Big(\bigcap_{n \in \mathbb{N}} D_n\Big)_x = \bigcap_{n \in \mathbb{N}} (D_n)_x \text{ is comeagre in } Y. \end{aligned}$$

Now, since the projection π_X maps a dense open subset of $X \times U_m$ to a dense open subset of X, the countable intersection

$$\bigcap_{n,m \in \mathbb{N}} \pi_X\big(D_n \cap (X \times U_m)\big)$$

is comeagre in X, whence $\big(\bigcap_{n \in \mathbb{N}} D_n\big)_x$, and a fortiori A_x, is comeagre in Y for a comeagre set of $x \in X$, i.e., $\forall^* x \in X\; A_x$ is comeagre in Y.

To see that (2) implies (1), suppose that $A \subseteq X \times Y$ is not comeagre and find a non-empty basic open set $U \times V \subseteq X \times Y$ such that $B = (U \times V) \setminus A$ is comeagre in $U \times V$. Applying (1)⇒(2) to the set B in the product space $U \times V$, we see that

$$\forall^* x \in U \quad B_x = V \setminus A_x \text{ is comeagre in } V,$$

whence

$$\forall^* x \in U \quad A_x \text{ is not comeagre in } Y,$$

contradicting (2). □

Note that the statement

$$\forall^* x \in X \quad A_x \text{ is comeagre in } Y$$

is equivalent to

$$\forall^* x \in X\; \forall^* y \in Y \;\; (x,y) \in A.$$

So the Kuratowski–Ulam Theorem can be reformulated as stating that for $A \subseteq X \times Y$ with the Baire property,

$$\begin{aligned}\forall^*(x,y)\ \ (x,y) \in A &\Leftrightarrow \forall^* x\, \forall^* y\ \ (x,y) \in A \\ &\Leftrightarrow \forall^* y\, \forall^* x\ \ (x,y) \in A.\end{aligned}$$

This an analogy of the Fubini Theorem that states that a measurable subset of a product of two probability spaces has full measure if and only if almost all vertical sections have full measure.

Note also that if $A_n \subseteq X$ for every $n \in \mathbb{N}$, then

$$\forall n\, \forall^* x\ \ x \in A_n \Leftrightarrow \forall^* x\, \forall n\ \ x \in A_n.$$

This follows on the one hand from the Kuratowski–Ulam Theorem or simply from the fact that a countable intersection of comeagre sets is comeagre.

Exercise 16. Let X and Y be Polish spaces and $A \subseteq X \times Y$ a subset with the Baire property.
(i) Deduce from the Kuratowski–Ulam Theorem that

$$\begin{aligned}\text{A is meagre } &\Leftrightarrow \forall^* x\ A_x \text{ is meagre in } Y \\ &\Leftrightarrow \forall^* y\ A^y \text{ is meagre in } X.\end{aligned}$$

(ii) Show that

$$\begin{aligned}\text{A is non-meagre } &\Leftrightarrow \exists^* x\ A_x \text{ is non-meagre in } Y \\ &\Leftrightarrow \exists^* y\ A^y \text{ is non-meagre in } X.\end{aligned}$$

(ii) Show that

$$\forall^* x\ A_x \text{ has the Baire property.}$$

Proposition 17 (A zero-one law)**.** *Suppose $(X_n)_{n\in\mathbb{N}}$ is a sequence of Polish spaces and $A \subseteq \prod_{n\in\mathbb{N}} X_n$ is a* tail set *with the Baire property, i.e., if (x_n) and (y_n) differ in only finitely many coordinates, then*

$$(x_n) \in A \Leftrightarrow (y_n) \in A.$$

Then A is either meagre or comeagre.

Proof. Suppose that A is non-meagre and find a non-empty basic open set $U_0 \times \ldots \times U_{n-1} \times X_n \times X_{n+1} \times \ldots$ in which A is comeagre, i.e.,

$$\forall^*(x,y) \in \big(U_0 \times \ldots \times U_{n-1}\big) \times \big(X_n \times X_{n+1} \times \ldots\big)\ \ (x,y) \in A,$$

whence, by the Kuratowski–Ulam Theorem,

$$\forall^* y \in \big(X_n \times X_{n+1} \times \ldots\big)\, \forall^* x \in \big(U_0 \times \ldots \times U_{n-1}\big)\ \ (x,y) \in A.$$

However, since A is a tail set, for any $y \in X_n \times X_{n+1} \times \ldots$ and $x, x' \in X_0 \times \ldots \times X_{n-1}$, we have $(x,y) \in A$ if and only if $(x',y) \in A$, from which it follows that

$$\forall^* y \in \big(X_n \times X_{n+1} \times \ldots\big)\, \forall x \in \big(X_0 \times \ldots \times X_{n-1}\big)\ \ (x,y) \in A.$$

Using Kuratowski–Ulam once more, we obtain

$$\forall^*(x,y) \in \big(X_0 \times \ldots \times X_{n-1}\big) \times \big(X_n \times X_{n+1} \times \ldots\big)\ \ (x,y) \in A,$$

and thus A is comeagre. □

A function $f: X \to Y$ between Polish spaces X and Y is said to be *Baire measurable* if $f^{-1}(U)$ has the Baire property for any open subset $U \subseteq Y$. Note that, since Y is second countable, it is enough to consider preimages of basic open sets in Y.

Exercise 18. Suppose $f: X \to Y$ is a Baire measurable function between topological spaces X and Y, where Y is second countable. Show that there is a comeagre G_δ-set $A \subseteq X$ such that $f|_A$ is continuous.

Exercise 19 (Non-smoothness of E_0)**.** Let E_0 be the equivalence relation defined on $2^{\mathbb{N}}$ by

$$(x_n)E_0(y_n) \Leftrightarrow \exists N\ \forall n \geqslant N\ \ x_n = y_n.$$

Suppose that $f: 2^{\mathbb{N}} \to X$ is a Baire measurable function into a Polish space X satisfying

$$xE_0y \Rightarrow f(x) = f(y).$$

Show that there is a comeagre subset of $2^{\mathbb{N}}$ on which f is constant. [Hint: Consider inverse images of basic open sets. Alternatively, show that f is constant on the comeagre set $A \subseteq 2^{\mathbb{N}}$ given by Exercise 18.]

Suppose G is a group of homeomorphisms of a topological space X. We say that G is *topologically transitive* if for any non-empty open subsets $U, V \subseteq X$ there is $g \in G$ such that $g \cdot U \cap V \neq \emptyset$.

Proposition 20. *The following are equivalent for a group G of homeomorphisms of a Polish space X.*

(i) *G is topologically transitive,*
(ii) *there is a dense orbit $G \cdot x$,*
(iii) *there is a comeagre set of points with dense orbits.*

Proof. That (iii)⇒(ii)⇒(i) is trivial. Also, if $\{U_n\}_{n\in\mathbb{N}}$ is a basis for the topology on X consisting of non-empty open sets and G is topologically transitive, then $G \cdot U_n$ is dense for every n, whereby

$$\bigcap_{n\in\mathbb{N}} G \cdot U_n = \bigcap_{n\in\mathbb{N}} \{x \in X \mid G \cdot x \cap U_n \neq \emptyset\} = \{x \in X \mid G \cdot x \text{ is dense }\}$$

is comeagre, showing (i)⇒(iii). □

2. Mycielski's independence Theorem

Theorem 21 (J. Mycielski)**.** *Let X be a non-empty perfect Polish space and $R \subseteq X \times X$ a comeagre set. Then there is a homeomorphic copy $C \subseteq X$ of Cantor space such that $(x, y) \in R$ for all $x, y \in C$, $x \neq y$.*

Proof. Fix a compatible complete metric d on X. Note first that if $V_0, V_1 \subseteq X$ are non-empty open subsets and $D \subseteq X \times X$ is dense open, then there are disjoint, non-empty open subsets $U_0 \subseteq V_0$, $U_1 \subseteq V_1$ such that $U_0 \times U_1 \subseteq D$. Moreover, by shrinking the U_i further, one can ensure that the U_i have arbitrarily small diameter.

Now, suppose that $R \subseteq X \times X$ is comeagre and find a decreasing sequence $(D_n)_{n\in\mathbb{N}}$ of dense open subsets of $X \times X$ such that $R \supseteq \bigcap_{n\in\mathbb{N}} D_n$. We then define a Cantor scheme $(U_s)_{s\in 2^{<\mathbb{N}}}$ on X satisfying

(1) each U_s is a non-empty open set of diameter $< \frac{1}{|s|+1}$,
(2) $\overline{U_{s0}} \cap \overline{U_{s1}} = \emptyset$ for all $s \in 2^{<\mathbb{N}}$,
(3) $U_s \times U_t \subseteq D_n$ for all $s, t \in 2^n$, $s \neq t$.

Letting $f\colon 2^{\mathbb{N}} \to X$ be defined by $\{f(x)\} = \bigcap_{n\in\mathbb{N}} U_{x|n}$, we see that f is continuous, injective and that $(f(x), f(y)) \in \bigcap_{n\in\mathbb{N}} D_n \subseteq R$ for all $x \neq y$. Let now $C = f[2^{\mathbb{N}}]$. □

Exercise 22. Generalise Theorem 21 to the following statement: If X is a non-empty perfect Polish space and, for every n, $R_n \subseteq X^n$ is a comeagre subset, then there is a homeomorphic copy of Cantor space $C \subseteq X$ such that for every n and all $x_1, \ldots, x_n \in C$ with $x_i \neq x_j$, we have $(x_1, \ldots, x_n) \in R_n$.

Exercise 23. Show that there is a homeomorphic copy $C \subseteq \mathbb{R}$ of Cantor space that is linearly independent over $\mathbb{Q}$.

3. Polish groups

Lemma 24 (S. Banach and B. J. Pettis)**.** *Suppose G is a Polish group and $A, B \subseteq G$ are subsets. Then*

$$U(A) \cdot U(B) \subseteq AB.$$

Proof. We note that if $x \in U(A)U(B)$, then the open set

$$V = xU(B)^{-1} \cap U(A) = U(xB^{-1}) \cap U(A)$$

is non-empty and so xB^{-1} and A are comeagre in V. It follows that $xB^{-1} \cap A \neq \emptyset$, whereby $x \in AB$. □

Theorem 25 (S. Banach–B. J. Pettis)**.** *Any Baire measurable homomorphism $\pi\colon G \to H$ between Polish groups is continuous.*

Proof. It is enough to prove that π is continuous at 1_G, i.e., that for any open $V \ni 1_H$ in H, $\pi^{-1}(V)$ is a neighbourhood of 1_G in G. So suppose $1_H \in V \subseteq H$ is given and find an open set $W \in 1_H$ such that $WW^{-1} \subseteq V$. Then $\pi^{-1}(W)$ is non-meagre, as it covers G by countably many left translates, and also has the Baire property. Thus, $U(\pi^{-1}(W))$ is non-empty open, and hence by the Banach-Pettis Theorem

$$1_G \in U(\pi^{-1}(W))U(\pi^{-1}(W))^{-1} \subseteq \pi^{-1}(W)\pi^{-1}(W)^{-1} \subseteq \pi^{-1}(V),$$

whereby $1_G \in \mathrm{Int}(\pi^{-1}(V))$. □

4. Baire category and the orbit structure of Polish group actions

Lemma 26. *Suppose G is a Polish group acting continuously on a Polish space X and let $x \in X$. Then the following are equivalent:*

1. *For every neighbourhood of the identity $V \subseteq G$, $V \cdot x$ is comeagre in a neighbourhood of x.*
2. *For each neighbourhood of the identity $V \subseteq G$, $V \cdot x$ is somewhere dense.*
3. *The orbit $G \cdot x$ is non-meagre.*

Proof. (1)⇒(3) is trivial. Also, for (3)⇒(2), suppose $G \cdot x$ is non-meagre and $V \subseteq G$ is a neighbourhood of 1. Then we can find $g_n \in G$ such that $G = \bigcup_n g_n V$, whence $G \cdot x = \bigcup_n g_n V \cdot x$. So some $g_n V \cdot x$, and therefore also $V \cdot x$, is non-meagre and hence somewhere dense.

Finally, for (2)⇒(1), suppose that $V \cdot x$ is somewhere dense for every neighbourhood $V \subseteq G$ of 1. Suppose towards a contradiction that for some neighbourhood $U \subseteq G$ of 1, $U \cdot x$ is meagre, whence there are closed nowhere dense sets $F_n \subseteq X$ covering $U \cdot x$. But then the sets $K_n = \{g \in G \mid g \cdot x \in F_n\}$ are closed and cover U and thus, by the Baire category theorem, some K_n contains a non-empty open set gV, where V is a neighbourhood of 1 and $g \in G$. So $gV \cdot x \subseteq F_n$ and $V \cdot x$ must be nowhere dense, which is a contradiction.

Now, if $V \subseteq G$ is any neighbourhood of 1, let $U \subseteq V$ be a smaller neighbourhood such that $U^{-1}U \subseteq V$. Then $U \cdot x$ is comeagre in some neighbourhood of a point $g \cdot x$, where $g \in U$, and thus $g^{-1}U \cdot x \subseteq V \cdot x$ is comeagre in a neighbourhood of x. □

Lemma 27. *Suppose G is a Polish group acting continuously on a Polish space X and let $x \in X$ have a non-meagre orbit $G \cdot x$. Then for any open subset $\emptyset \neq V \subseteq G$, $V \cdot x$ is relatively open in $G \cdot x$.*

Proof. Let $g \in V$ be given and pick an open set $W \ni 1$ such that $W^{-1}Wg \subseteq V$. By Lemma 26, $z \in U(Wz)$ for any $z \in G \cdot x$. So, for any $z \in U(Wg \cdot x) \cap G \cdot x$, $z \in U(W \cdot z) \cap U(Wg \cdot x)$, whence $W \cdot z$ and $Wg \cdot x$ are both comeagre in a neighourhood of z and hence must intersect, i.e., $z \in W^{-1}Wg \cdot x \subseteq V \cdot x$. In other words, for any $g \in V$, $gx \in U(Wg \cdot x) \cap G \cdot x \subseteq Vx$, showing that $V \cdot x$ is a relative neighbourhood of gx in $G \cdot x$. □

Lemma 28. *Suppose G is a Polish group acting continuously on a Polish space X. Then the following are equivalent:*

(1) *There is a non-meagre orbit $\mathcal{O} \subseteq X$.*
(2) *There is a non-empty open set $O \subseteq X$ with the following property: For all open $\emptyset \neq V \subseteq O$ and neighbourhood $U \subseteq G$ of 1, there is a smaller open $\emptyset \neq W \subseteq V$ such that the action of U on W is topologically transitive, i.e., for any non-empty open $W_0, W_1 \subseteq W$ there is $g \in U$ such that $gW_0 \cap W_1 \neq \emptyset$.*

Moreover, if $\mathcal{O}$ is an orbit comeagre in an open set $O \subseteq X$, then (2) holds for O.

Proof. (1)⇒(2): If $\mathcal{O} \subseteq X$ is a non-meagre orbit, let $O \subseteq X$ be a non-empty open set in which $\mathcal{O}$ is comeagre. Now, if $V \subseteq O$ is non-empty open and $U \subseteq G$ is a neighbourhood of 1, pick $x \in V \cap \mathcal{O}$ and choose an open neighbourhood $U_0 \subseteq U$ of 1 such that $U_0U_0^{-1} \subseteq U$. Then, by the preceding lemma, $U_0 \cdot x$ is dense in some open neighbourhood $W \subseteq V$ of x and it follows that the action of U on W is topologically transitive.

(2)⇒(1): Suppose $O \subseteq X$ is an open set satisfying the assumption in (2). Fix a neighbourhood basis $\{U_n\}_{n\in\mathbb{N}}$ at $1 \in G$ and a basis $\{V_n\}_{n\in\mathbb{N}}$ for the induced topology on O consisting of non-empty open sets. Now, for every n and m, let $W_{n,m} \subseteq V_n$ be a non-empty open subset such that the action of U_m^{-1} on $W_{n,m}$ is topologically transitive. Then $W_m = \bigcup_n W_{n,m}$ is open dense in O since it intersects every V_n. Also, for any $V_k \subseteq W_{n,m}$, $W_{n,m} \cap \left(U_m^{-1} \cdot V_k\right)$ is open dense in $W_{n,m}$, and so

$$D_{n,m} = W_{n,m} \cap \bigcap_{V_k \subseteq W_{n,m}} \left(U_m^{-1} \cdot V_k\right)$$

is comeagre in $W_{n,m}$. Note also that if $x \in D_{n,m}$, then for any $V_k \subseteq W_{n,m}$, $U_m \cdot x \cap V_k \neq \emptyset$, showing that $U_m \cdot x$ is dense in $W_{n,m}$. We notice that $D_m = \bigcup_n D_{n,m}$ is comeagre in O and that for any $x \in D_m$, $U_m \cdot x$ is somewhere dense. It follows that for any x belonging to the comeagre subset $\bigcap_m D_m \subseteq O$, and for any k, $U_k \cdot x$ is somewhere dense, which by the previous lemma implies that $G \cdot x$ is non-meagre. □

Combining Lemmas 20 and 28, we have the following characterisation of the existence of comeagre orbits.

Proposition 29. *Suppose G is a Polish group acting continuously on a Polish space X. Then there is a comeagre orbit on X if and only if*

(1) *the action of G is topologically transitive, and*
(2) *for any non-empty open $V \subseteq X$ and neighbourhood $U \ni 1$, there is a smaller non-empty open set $W \subseteq V$ on which the action of U is topologically transitive.*