COANALYTIC RANKS AND THE REFLECTION THEOREMS

CHRISTIAN ROSENDAL

1. Coanalytic ranks

A *rank* on a set C is a simply a map $\phi\colon C \to \mathrm{Ord}$. Similarly, a *prewellordering* $\preccurlyeq$ on C is a reflexive, transitive, and total relation for which $\prec$, defined by

$$x \prec y \iff x \preccurlyeq y \;\&\; y \not\preccurlyeq x,$$

is wellfounded. From a rank $\phi\colon C \to \mathrm{Ord}$ we can define a canonical prewellordering $\preccurlyeq_\phi$ on C by letting

$$x \preccurlyeq_\phi y \iff \phi(x) \leqslant \phi(y),$$

and, conversely, from a prewellordering $\preccurlyeq$ we can define a rank $\phi_\preccurlyeq$ by setting

$$\begin{cases} \phi_\preccurlyeq(x) = 0 & \text{if } x \preccurlyeq y \text{ for all } y \in C,\\ \phi_\preccurlyeq(x) = \sup\{\phi_\preccurlyeq(y)+1 \mid y \prec x\} & \text{otherwise.}\end{cases}$$

Definition 1. Suppose X is a Polish space and $C \subseteq X$ is a coanalytic subset. Then $\phi\colon C \to \mathrm{Ord}$ is said to be a $\mathbf{\Pi}^1_1$*-rank* provided that the following relations $\leqslant^*_\phi$ and $<^*_\phi$, defined by

$$x \leqslant^*_\phi y \iff \big(x \in C \;\&\; y \in X \setminus C\big) \text{ or } \big(x, y \in C \;\&\; \phi(x) \leqslant \phi(y)\big),$$
$$x <^*_\phi y \iff \big(x \in C \;\&\; y \in X \setminus C\big) \text{ or } \big(x, y \in C \;\&\; \phi(x) < \phi(y)\big),$$

are $\mathbf{\Pi}^1_1$ as subsets of $X \times X$.

Theorem 2 (Y. N. Moschovakis)**.** *Let C be a coanalytic subset of a Polish space X. Then C admits a $\mathbf{\Pi}^1_1$-rank $\phi\colon C \to \omega_1$.*

Proof. By composing with a Borel reduction of $C \subseteq X$ to $WO \subseteq 2^{\mathbb{Q}}$, we may suppose that actually $X = 2^{\mathbb{Q}}$ and $C = WO$. So set $\phi(x) = \mathrm{ordertype}(x, <_{\mathbb{Q}})$ and let $E \subseteq \mathbb{Q}^{\mathbb{Q}} \times 2^{\mathbb{Q}} \times 2^{\mathbb{Q}}$ be the Borel set defined by

$$\begin{aligned}(f, x, y) \in E &\iff \forall p, q \in \mathbb{Q} \Big(\text{ if } p, q \in x \;\&\; p < q, \text{ then } f(p), f(q) \in y \;\&\; f(p) < f(q)\Big)\\ &\iff f \text{ order-embeds } (x, <_{\mathbb{Q}}) \text{ into } (y, <_{\mathbb{Q}}).\end{aligned}$$

We then have

$$\begin{aligned} x <^*_\phi y &\iff (x, <_{\mathbb{Q}}) \text{ is wellordered and } (y, <_{\mathbb{Q}}) \text{ does not order-embed into } (x, <_{\mathbb{Q}})\\ &\iff x \in WO \text{ and } \forall f \in \mathbb{Q}^{\mathbb{Q}}\ (f, y, x) \notin E,\end{aligned}$$

which is $\mathbf{\Pi}^1_1$. Similarly,

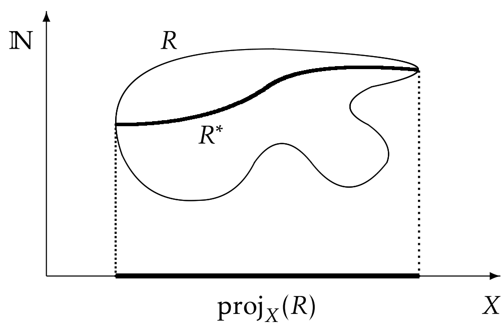
$$\begin{aligned}
x \leqslant^*_\phi y \iff & \text{ either } x <^*_\phi y \text{ or } (x, <_{\mathbb{Q}}) \text{ and } (y, <_{\mathbb{Q}}) \text{ are wellordered with} \\
& \text{ the same ordertype} \\
\iff & \text{ either } x <^*_\phi y \text{ or } x, y \in WO, \text{ any order-embedding of } (x, <_{\mathbb{Q}}) \\
& \text{ into } (y, <_{\mathbb{Q}}) \text{ is cofinal and vice versa} \\
\iff & \text{ either } x <^*_\phi y \text{ or } x, y \in WO, \forall f \in \mathbb{Q}^{\mathbb{Q}} \left(E(f, x, y) \to \forall p \in y\, \exists q \in x \;\; p \leqslant f(q) \right) \\
& \text{ and } \forall f \in \mathbb{Q}^{\mathbb{Q}} \left(E(f, y, x) \to \forall p \in x\, \exists q \in y \;\; p \leqslant f(q) \right),
\end{aligned}$$

which also is $\mathbf{\Pi}^1_1$. □

Exercise 3. Show that the usual rank of trees is a $\mathbf{\Pi}^1_1$ rank on the set of wellfounded trees $WF \subseteq \mathrm{Tr}_{\mathbb{N}}$.

We can now deduce some classical separation and reduction theorems directly from the existence of $\mathbf{\Pi}^1_1$-ranks.

Theorem 4. *Let X be a Polish space and $R \subseteq X \times \mathbb{N}$ be a coanalytic subset. Then there is a coanalytic subset $R^* \subseteq R$* uniformising R, *i.e., such that for all $x \in X$, $\exists n\, (x, n) \in R \Leftrightarrow \exists! n\, (x, n) \in R^*$.*



Proof. Suppose $R \subseteq X \times \mathbb{N}$ is $\mathbf{\Pi}^1_1$ and let $\phi \colon R \to \omega_1$ be a coanalytic rank. To uniformise R, for every $x \in \mathrm{proj}_X(R)$, we look for an n such that $(x, n) \in R$ and $\phi(x, n)$ is least possible. If there are several such n, we then choose the smallest of these. That is, we set

$$(x, n) \in R^* \iff (x, n) \in R \;\&\; \forall m\, (x, n) \leqslant^*_\phi (x, m) \;\&\; \forall m \left((x, n) <^*_\phi (x, m) \text{ or } n \leqslant m \right),$$

which is clearly $\mathbf{\Pi}^1_1$. □

Corollary 5 (K. Kuratowski, Generalised reduction for $\mathbf{\Pi}^1_1$)**.** *If (C_n) is a sequence of coanalytic subsets of a Polish space X, then there is a sequence (C^*_n) of pairwise disjoint coanalytic sets such that $C^*_n \subseteq C_n$ and $\bigcup_n C^*_n = \bigcup_n C_n$.*

Proof. Set $(x, n) \in R \iff x \in C_n$ and let $R^* \subseteq R$ be a $\mathbf{\Pi}^1_1$-uniformisation. Then $C^*_n = \{x \in X \mid (x, n) \in R^*\}$ is a reducing sequence. □

Corollary 6 (P. S. Novikov, Generalised separation for $\mathbf{\Sigma}^1_1$)**.** *If (A_n) is a sequence of analytic subsets of a Polish space X with $\bigcap_n A_n = \emptyset$, then there is a sequence (B_n) of Borel sets such that $A_n \subseteq B_n$ and $\bigcap_n B_n = \emptyset$.*

Proof. Given (A_n), we let $C_n = \sim A_n$ and find a reducing sequence $C_n^* \subseteq C_n$ of coanalytic sets. Thus, $\bigcup_n C_n^* = \bigcup_n C_n = \sim \bigcap_n A_n = X$ and thus the C_n^* partition X. Since $C_n^* = \sim \bigcup_{m \neq n} C_m^*$, it follows that the $B_n = C_n^*$ are also analytic and hence Borel. □

Theorem 7 (Boundedness for coanalytic ranks)**.** *Let X be a Polish space, $C \subseteq X$ a coanalytic subset and $\phi\colon C \to \omega_1$ a coanalytic rank on C. Then of $A \subseteq C$ is an analytic subset, we have*

$$\sup\{\phi(x) \mid x \in A\} < \omega_1.$$

Also, for all $\xi < \omega_1$,

$$D_\xi = \{x \in C \mid \phi(x) < \xi\}, \text{ and}$$
$$E_\xi = \{x \in C \mid \phi(x) \leqslant \xi\}$$

are Borel subsets of X.

Proof. Note that since $A \subseteq C$ is analytic in X, the relation

$$x \prec y \iff x, y \in A \,\&\, \phi(x) < \phi(y) \iff x, y \in A \,\&\, y \not\leqslant_\phi^* x$$

is analytic and wellfounded on X. It follows by the Kunen–Martin Theorem that $\rho(\prec) < \omega_1$ and thus $\sup\{\phi(x) \mid x \in A\} < \omega_1$.

Since $D_\xi = \bigcup_{\zeta<\xi} E_\xi$, it suffices to show that E_ξ is Borel for all $\xi < \omega_1$. Also, if $\alpha = \sup\{\phi(x) \mid x \in C\}$, we have $E_\xi = E_\alpha$ for all $\alpha < \xi < \omega_1$, so it suffices to consider $\xi \leqslant \alpha$. Now, suppose first that there is $x_0 \in C$ such that $\phi(x_0) \geqslant \xi$ and pick such an x_0 of minimal rank. Then, for any $y \in X$,

$$y \in E_\xi \iff y \in C \,\&\, \phi(y) \leqslant \xi \iff y \leqslant_\phi^* x_0 \iff x_0 \not<_\phi^* y,$$

showing that E_ξ is $\mathbf{\Delta}_1^1$ and thus Borel. On the other hand, if there is no such x_0 for $\xi \leqslant \alpha$, then $\xi = \alpha$ and $E_\xi = E_\alpha = \bigcup_{\zeta<\alpha} E_\zeta$, which, by the previous case, is a countably union of Borel sets. □

Exercise 8 (D. H. Blackwell)**.** Let X be a Polish space and $A, B \subseteq X$ be analytic subsets. Fix continuous surjections $f\colon \mathbb{N}^\mathbb{N} \to A$ and $g\colon \mathbb{N}^\mathbb{N} \to B$ and set $A_s = \overline{f[N_s]}$ and $B_s = \overline{g[N_s]}$ for all $s \in \mathbb{N}^{<\mathbb{N}}$. For every $x \in A_\emptyset \cup B_\emptyset = \overline{A} \cup \overline{B}$, we partition $\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ into three sets as follows.

$$U_x = \{(a,b) \in \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \mid \exists k \quad x \in A_{a_0 \ldots a_k} \setminus B_{b_0 \ldots b_k}\},$$
$$V_x = \{(a,b) \in \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \mid \exists k \quad x \in B_{b_0 \ldots b_{k-1}} \setminus A_{a_0 \ldots a_k}\},$$
$$C_x = \{(a,b) \in \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \mid \forall k \quad x \in A_{a_0 \ldots a_k} \cap B_{b_0 \ldots b_k}\}.$$

Clearly, U_x and V_x are open, while C_x is closed. Consider now the game in which players **I** and **II** alternate in choosing natural numbers with **I** beginning.

$$\begin{array}{lccccccc} \textbf{I} & a_0 & & a_1 & & a_2 & & \cdots \\ \textbf{II} & & b_0 & & b_1 & & b_2 & \cdots \end{array}$$

The *outcome* of the game is then the pair $(a,b) = (a_0 a_1 \ldots, b_0 b_1 \ldots) \in \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$. Finally, consider the sets

$$A^* = \{x \in A_\emptyset \cup B_\emptyset \mid \textbf{I} \text{ has a strategy to play in } U_x \cup C_x\},$$
$$B^* = \{x \in A_\emptyset \cup B_\emptyset \mid \textbf{II} \text{ has a strategy to play in } V_x \cup C_x\},$$

and note that since A^* is the projection of the Borel set

$$\{(x,\sigma) \mid \sigma \text{ is a strategy for } \mathbf{I} \text{ to play in the closed set } U_x \cup C_x = \sim V_x\},$$

we see that A^* is analytic. Similarly for B^*.

Show that (i) $A \subseteq A^*$ and $B \subseteq B^*$, (ii) $A^* \cap B^* = A \cap B$ and, using the determinacy of open games, (iii) $A^* \cup B^* = A_\emptyset \cup B_\emptyset = \overline{A} \cup \overline{B}$. Use this to give a proof of the reduction theorem for coanalytic sets.

2. Reflection theorems

Definition 9. Suppose X is a Polish space and Φ is a property of subsets of X. We say that Φ is $\mathbf{\Pi}^1_1$ *on* $\mathbf{\Pi}^1_1$ if for all Polish spaces Y and $\mathbf{\Pi}^1_1$ sets $C \subseteq Y \times X$, the set

$$C_\Phi = \{y \in Y \mid \Phi(C_y)\}$$

is $\mathbf{\Pi}^1_1$.

Theorem 10 (First reflection theorem). *Suppose X is a Polish space and Φ is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Pi}^1_1$ property of subsets of X. Then for any $\mathbf{\Pi}^1_1$ subset $C \subseteq X$ such that $\Phi(C)$, there is a Borel subset $B \subseteq C$ satisfying $\Phi(B)$.*

Proof. Without loss of generality, we may assume that C is not itself Borel and hence that $\sim C$ is not $\mathbf{\Pi}^1_1$ either. Fix a $\mathbf{\Pi}^1_1$-rank $\phi \colon C \to \omega_1$ and let $D = \{(x,y) \in X \times X \mid y <^*_\phi x\}$. Since D is $\mathbf{\Pi}^1_1$, so is the set

$$D_\Phi = \Big\{x \in X \;\Big|\; \Phi\Big(\{y \in X \mid y <^*_\phi x\}\Big)\Big\}$$

whereby $\sim C \neq D_\Phi$. Note also that if $x \in \sim C$, then $C = \{y \in X \mid y <^*_\phi x\}$, whereby $\Phi\Big(\{y \in X \mid y <^*_\phi x\}\Big)$ and so $x \in D_\Phi$. In other words, $\sim C \subseteq D_\Phi$, and thus there must be some $x \in D_\Phi \cap C$, whence Φ holds of the Borel set $B = \{y \in X \mid y <^*_\phi x\} \subseteq C$. □

If Φ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Pi}^1_1$, we can think of Φ as a largeness condition as illustrated by the following example.

Example 11. For (A_n) a sequence of subsets of a Polish space X, we let $\limsup A_n = \{x \in X \mid \exists^\infty n\; x \in A_n\}$. Define $\Phi \subseteq \mathbb{N} \times X$ by

$$\Phi(A) \iff \limsup A_n = X,$$

where $A_n = \{x \in X \mid (n,x) \in A\}$. Then Φ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Pi}^1_1$. For if $C \subseteq Y \times \mathbb{N} \times X$ is $\mathbf{\Pi}^1_1$, then

$$y \in C_\Phi \iff \forall x \in X\, \exists^\infty n \in \mathbb{N}\; (y,n,x) \in C,$$

which is $\mathbf{\Pi}^1_1$ in the variable y.

It follows that if $A \subset \mathbb{N} \times X$ is a coanalytic set with $\limsup A_n = X$, there is a Borel subset $B \subseteq A$ satisfying $\limsup B_n = X$.

It is also possible to give a dual formulation of the first reflection theorem, which is equally useful. For this we say that a property Ψ of subsets of a Polish space X is $\mathbf{\Pi}^1_1$ on Σ^1_1 if for all Polish spaces Y and Σ^1_1 sets $A \subseteq Y \times X$, the set

$$A_\Psi = \{y \in Y \mid \Psi(A_y)\}$$

is $\mathbf{\Pi}^1_1$. So Ψ is $\mathbf{\Pi}^1_1$ on Σ^1_1 if and only if Φ, defined by $\Phi(C) \iff \Psi(\sim C)$, is $\mathbf{\Pi}^1_1$ on $\mathbf{\Pi}^1_1$. It follows that if Ψ is $\mathbf{\Pi}^1_1$ on Σ^1_1 and A is an analytic set with $\Psi(A)$, then there is a Borel set $B \supseteq A$ with $\Psi(B)$.

Exercise 12. Suppose that $\leqslant$ is a coanalytic partial order on a Polish space X. Show that any analytic set $A \subseteq X$ that is linearly ordered by $\leqslant$ is contained in a Borel set with the same property.

However, for many applications we need a more delicate argument, which is encapsulated by the following definition.

Definition 13. A property Φ of pairs (A, C) of subsets of a Polish space X is said to be $\mathbf{\Pi}^1_1$ *on* $\mathbf{\Pi}^1_1$ if for all Polish spaces Y and $\mathbf{\Pi}^1_1$ sets $A, C \subseteq Y \times X$, the set

$$(A, C)_\Phi = \{(x, y) \in Y \times Y \mid \Phi(A_x, C_y)\}$$

is $\mathbf{\Pi}^1_1$.

Also, Φ is *monotone* provided if for all $A \subseteq A'$ and $C \subseteq C'$, if $\Phi(A, C)$ then also $\Phi(A', C')$.

Finally, Φ is *downwards continuous in the second variable* if for all $A, C_0 \supseteq C_1 \supseteq \ldots$, if $\Phi(A, C_n)$ for all $n \in \mathbb{N}$, we also have $\Phi(A, \bigcap_n C_n)$.

Theorem 14 (Second reflection theorem)**.** *Let X be a Polish space and Ψ a property of pairs of subsets of X that is $\mathbf{\Pi}^1_1$ on $\mathbf{\Pi}^1_1$, monotone and continuous downwards in the second variable. Assume that $C \subseteq X$ is a coanalytic set such that $\Psi(C, \sim C)$. Then there is a Borel set $B \subseteq C$ with $\Psi(B, \sim B)$.*

Proof. Note first that if $A \subseteq C$ is a Borel set, then there is a Borel set $A \subseteq \hat{A} \subseteq C$ such that $\Psi(\hat{A}, \sim A)$. To see this, we define a property Φ of subsets of X by letting

$$\Phi(D) \iff A \subseteq D \,\&\, \Psi(D, \sim A),$$

and note that Φ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Pi}^1_1$ and, as $\sim C \subseteq \sim A$ and Ψ is monotone, also $\Phi(C)$. By the first reflection theorem, there is therefore a Borel set $\hat{A} \subseteq C$ satisfying Φ, i.e., $A \subseteq \hat{A} \subseteq C$ and $\Psi(\hat{A}, \sim A)$

With this in hand, we can now inductively construct a sequence of Borel sets $\emptyset = A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq C$ such that for all n, $\Psi(A_{n+1}, \sim A_n)$. Set $B = \bigcup_n A_n$ and note that by monotonicity we have $\Psi(B, \sim A_n)$ for all n. Finally, by downwards continuity in the second variable, it follows that $\Psi(B, \bigcap_n \sim A_n)$, i.e., $\Psi(B, \sim B)$ as required. $\square$

As for the first reflection theorem, we can reformulate the second reflection theorem as follows. If $\Psi(\cdot, \cdot)$ is *hereditary*, i.e., closed under passing to subsets, continuous upwards in the second variable and $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, then whenever $A \subseteq X$ is an analytic sets such that $\Psi(A, \sim A)$, there is a Borel set $B \supseteq A$ with $\Psi(B, \sim B)$.

Example 15 (J. P. Burgess)**.** Suppose E is an equivalence relation on a Polish space X that is analytic seen as a subset of $X \times X$. Let $A \subseteq X \times X$ be an analytic set disjoint from E and define Ψ by

$$\begin{aligned}\Psi(F, D) \iff\ & F \cap A = \emptyset \ \& \ \forall x\, (x, x) \notin D \ \& \ \forall x, y\, \big((x, y) \notin F \text{ or } (y, x) \notin D\big)\\ & \& \ \forall x, y\, \big((x, y) \notin F \text{ or } (y, z) \notin F \text{ or } (x, z) \notin D\big).\end{aligned}$$

Since $E \cap A = \emptyset$ and E is an equivalence relation, we have $\Psi(E, \sim E)$. Moreover, Ψ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, hereditary and continuous upwards in the second variable. So by the second reflection theorem there is Borel set $F \supseteq E$ with $\Psi(F, \sim F)$. It follows that F is a Borel equivalence relation containing E and disjoint from A.

Now, writing $\sim E$ as the increasing union of ω_1 many Borel sets, we can inductively construct a decreasing family of Borel equivalence relations $(F_\xi)_{\xi<\omega_1}$ such that $E = \bigcap_\xi F_\xi$.

Exercise 16. Suppose E is an analytic equivalence relation on a Polish space X and let D_ξ denote the constituents of $\sim E$ as defined in Theorem 7. Show that there is a cofinal set of $\xi < \omega_1$ such that $F_\xi = \sim D_\xi$ is a Borel equivalence relation containing E. In particular, any analytic set disjoint from E is also disjoint from some F_ξ.

Exercise 17. A collection I of subsets of $\mathbb{N}$ is said to be an *ideal* if $x \subseteq y \in I \Rightarrow x \in I$ and $x, y \in I \Rightarrow x \cup y \in I$. Show that if I is an ideal on $\mathbb{N}$ that is analytic seen as a subset of $2^{\mathbb{N}}$, then I is the intersection of a decreasing family of ω_1 many Borel ideals.

Exercise 18. Let X be a separable Banach space and $A, C \subseteq X$ disjoint analytic subsets with A convex. Show that there is a convex Borel set B containing A and disjoint from C.

Theorem 19 (A. Louveau)**.** *Let $\leqslant$ be a closed quasiorder, i.e., reflexive and transitive relation, on a Polish space X. Then there is a Borel function $\phi\colon X \to 2^{\mathbb{N}}$ such that $x \leqslant y \Leftrightarrow \phi(x) \subseteq \phi(y)$.*

It follows that any closed partial order on a Polish space Borel embeds into $\subseteq$ on $2^{\mathbb{N}}$.

Proof. Note that since $\leqslant$ is closed, we can write $\nleqslant$ as a countable union of open rectangles $\nleqslant = \bigcup_n U_n \times V_n$. Also, for all n, we define the following property Ψ_n that is hereditary, continuous upwards in the second variable and $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.

$$\begin{aligned}\Psi_n(A,C) &\Leftrightarrow \forall x\,(x \notin A \text{ or } x \notin V_n) \;\&\; \forall x,y\,(x \nleqslant y \text{ or } x \notin A \text{ or } y \notin C)\\ &\Leftrightarrow A \cap V_n = \emptyset \;\&\; \forall x \leqslant y\,(x \in A \to y \notin C).\end{aligned}$$

Setting $\check{U}_n = \{y \in X \mid \exists x \in U_n\ x \leqslant y\}$, we note that $\check{U}_n$ is analytic and upwards closed with respect to $\leqslant$. Moreover, since no point in U_n is below a point in V_n, we have $\check{U}_n \cap V_n = \emptyset$, whereby $\Psi(\check{U}_n, \sim\check{U}_n)$. By dual version of the second reflection theorem, there is therefore a Borel set B_n containing $\check{U}_n$ and thus also U_n such that $\Psi(B_n, \sim B_n)$. By the definition of Ψ_n, it follows that B_n is closed upwards in $\leqslant$ and $B_n \cap V_n = \emptyset$. Therefore, for any $x, y \in X$, we have

$$x \nleqslant y \;\Rightarrow\; \exists n\; x \in U_n \;\&\; y \in V_n \;\Rightarrow\; \exists n\; x \in B_n \;\&\; y \notin B_n \;\Rightarrow\; x \nleqslant y,$$

i.e.,

$$x \leqslant y \quad\Leftrightarrow\quad \{n \in \mathbb{N} \mid x \in B_n\} \subseteq \{n \in \mathbb{N} \mid y \in B_n\}.$$

Setting $\phi(x) = \{n \in \mathbb{N} \mid x \in B_n\}$ the first statement of the theorem follows. For the second part, just note that if $\leqslant$ is a patial order, then ϕ must be injective. □

Corollary 20. *Let E be a closed equivalence relation on a Polish space X. Then E is* smooth *, i.e., there is a Borel function $\phi\colon X \to 2^{\mathbb{N}}$ such that $xEy \Leftrightarrow \phi(x) = \phi(y)$.*

3. Uniformisation of Borel sets with compact sections

Theorem 21 (K. Kunugui & P. S. Novikov)**.** *Let X and Y be Polish spaces and $A \subseteq X \times Y$ a Borel set all of whose vertical sections $A_x = \{y \in Y \mid (x,y) \in A\}$ are open. Then there is a finer Polish topology τ on X such that A is open in $(X,\tau) \times Y$.*

Proof. Fix a countable basis $\{V_n\}$ for the topology on Y consisting of non-empty open sets and let $X_n = \{x \in X \mid V_n \subseteq A_x\}$, which is coanalytic. Since every section A_x can be written as a union of V_n's, we see that $A = \bigcup_n X_n \times V_n$. By the generalised reduction theorem for coanalytic sets (or the generalised separation theorem applied to the complement) there are disjoint coanalytic sets $B_n \subseteq X_n \times V_n$ such that $A = \bigcup_n B_n$. Since A is Borel it follows that each $B_n = A \setminus \bigcup_{m \neq n} B_m$ is also analytic, whereby the projection $R_n = \mathrm{proj}_X(B_n)$ is analytic too. It follows that R_n can be separated from the disjoint analytic set $\sim X_n$ by a Borel set $R_n \subseteq D_n \subseteq X_n$. Therefore, $B_n \subseteq D_n \times V_n \subseteq X_n \times V_n$, whereby $A = \bigcup_n D_n \times V_n$. Letting τ be a finer Polish topology in which all the D_n are open, the result follows. □

Exercise 22 (G. Mokobodzki)**.** Give an alternative direct proof of Novikov's generalised separation theorem for analytic sets as follows.

Let (A_n) be a sequence of analytic sets in a Polish space X with $\bigcap_n A_n = \emptyset$ and fix continuous surjections $f_n \colon \mathbb{N}^{\mathbb{N}} \to A_n$. We say that a sequence $(s_0, s_1, s_2, \ldots)$ of $s_i \in \mathbb{N}^{<\mathbb{N}}$ is *bad* if there are no Borel sets $B_n \supseteq f[N_{s_n}]$ with $\bigcap_n B_n = \emptyset$. Using only Lusin's separation theorem, show that if $(s_0, s_1, s_2, \ldots)$ is bad and n is given, then there is a proper extension $s_n' \supset s_n$ such that also $(s_0, s_1, \ldots, s_{n-1}, s_n', s_{n+1}, s_{n+2}, \ldots)$ is bad. Assuming that $(\emptyset, \emptyset, \emptyset, \ldots)$ is bad, use this to construct $x_0, x_1, x_2, \ldots \in \mathbb{N}^{\mathbb{N}}$ such that for all n, $(x_0|_n, x_1|_n, \ldots, x_n|_n, \emptyset, \emptyset, \ldots)$ is bad. Finally, arrive at a contradiction by separating two terms of the sequence $(f(x_0), f(x_1), f(x_2), \ldots)$ by disjoint open sets.

Theorem 23. *Let X and Y be Polish spaces and $A \subseteq X \times Y$ a Borel set all of whose vertical sections A_x are compact. Then $\mathrm{proj}_X(A)$ is Borel and the map $x \in X \to A_x \in \mathcal{K}(Y)$ is Borel. Moreover, A admits a Borel uniformisation $A^* \subseteq A$.*

Proof. Replacing Y by a compactification, we may assume that Y is compact without altering the compactness of the sections A_x. Also, by the theorem of Kunugui and Novikov, modulo changing the topology on X, we can assume that A is a closed subset of $X \times Y$, from which it follows that $\mathrm{proj}_X(A)$ is closed in the new topology on X and thus Borel in the original topology.

To see that $x \mapsto A_x$ is Borel, we write $\sim A$ as a union $\bigcup_n B_n \times V_n$ for Borel sets $B_n \subseteq X$ and open sets $V_n \subseteq Y$. Letting K_n be the compact set $\sim V_n$, we see that

$$x \mapsto A_x = \bigcap_{x \in B_n} K_n$$

is Borel. Composing $x \mapsto A_x$ with the map on $\mathcal{K}(Y)$ given by the theorem of Kuratowski and Ryll-Nardzewski, we obtain a Borel function $f \colon X \to Y$ such that $f(x) \in A_x$ for all $x \in \mathrm{proj}_X(A)$. In particular, $A^* = \mathrm{graph}(f) \cap A$ is a Borel uniformisation of A. □

Though requiring somewhat stronger tools for the proof, the Kunugui–Novikov theorem admits a wide ranging generalisation as follows.

Theorem 24 (A. Louveau)**.** *Let X and Y be Polish spaces, $1 \leqslant \xi < \omega_1$ and $A \subseteq X \times Y$ a Borel set all of whose vertical sections $A_x = \{y \in Y \mid (x,y) \in A\}$ belong to $\mathbf{\Sigma}^0_\xi$. Then there is a finer Polish topology τ on X such that A is $\mathbf{\Sigma}^0_\xi$ in $(X, \tau) \times Y$.*

We shall not prove Theorem 24 but instead use it to uniformise more complicated Borel sets.

Theorem 25 (V. Ya. Arsenin, K. Kunugui, J. Saint-Raymond)**.** *Let X and Y be Polish spaces and $A \subseteq X \times Y$ a Borel set all of whose vertical sections A_x are K_σ. Then $A = \bigcup_n A_n$, where each $A_n \subseteq X \times Y$ is Borel with compact vertical sections. It follows that* $\mathrm{proj}_X(A)$ *is Borel and A admits a Borel uniformisation $A^* \subseteq A$.*

Proof. Replacing Y by a compactification, we may assume that Y is compact. Moreover, since all vertical sections A_x are K_σ and thus also F_σ, by the theorem of Louveau, there is a finer Polish topology τ on X making A an F_σ set. Thus, $A = \bigcup_n A_n$, where each A_n is closed in $(X,\tau) \times Y$. So, since Y is compact, every section $(A_n)_x$ is compact. Thus, $\mathrm{proj}_X(A) = \bigcup_n \mathrm{proj}_X(A_n)$ is Borel and if, for all n, $A_n^* \subseteq A_n$ is a Borel uniformisation, we can define $A^* \subseteq A$ by

$$(x,y) \in A^* \iff \exists n \Big((x,y) \in A_n^* \ \& \ \forall m < n \ \ x \notin \mathrm{proj}_X(A_m) \Big).$$

Then A^* is Borel and uniformises A. □