COUNTABLE SECTIONS FOR LOCALLY COMPACT GROUP ACTIONS

CHRISTIAN ROSENDAL

Theorem 1 (A. S. Kechris)**.** *Suppose G is a locally compact Polish group acting continuously on a Polish space X. Then there is a Borel set $Z \subseteq X$ intersecting every G-orbit in a non-empty countable set.*

Proof. Fix a compact symmetric neighbourhood K of 1 and let $V = \mathrm{int}\, K$. We define a closed, reflexive, symmetric relation R on X by letting

$$xRy \iff \exists k \in K \;\; kx = y.$$

Though R need not be transitive, we shall construct a covering of X by countably many Borel sets X_m such that R is transitive and thus an equivalence relation when restricted to each X_m.

Since R is closed, we see that, for any $\epsilon > 0$,

$$C_\epsilon = \{(g,x) \in G \times X \mid d(x,gx) \leqslant \epsilon \;\&\; g \in K^2 \;\&\; gx \not R x\}$$

is F_σ and so $\mathrm{proj}_X(C_\epsilon)$ is F_σ too. Moreover, $\bigcap_{\epsilon>0} \mathrm{proj}_X(C_\epsilon) = \emptyset$. For if x belongs to the intersection, we can choose $g_n \in K^2$ such that $d(x, g_n x) \underset{n\to\infty}{\longrightarrow} 0$ and $g_n x \not R x$. It follows, in particular, that $g_n \notin KG_x \supseteq VG_x$, where G_x is the stabiliser of x and VG_x is open. Passing to a subsequence, we can suppose that the limit $g = \lim_{n\to\infty} g_n \in K^2 \setminus VG_x$ exists, contradicting that also $gx = \lim_{n\to\infty} g_n x = x$.

It follows that the sets

$$A_\epsilon = X \setminus \mathrm{proj}_X(C_\epsilon) = \{x \in X \mid \forall g \in K^2 \; (d(x,gx) \leqslant \epsilon \to gxRx)\}$$

are G_δ and that $X = \bigcup_{\epsilon>0} A_\epsilon$. Moreover, if $B \subseteq A_\epsilon$ has diameter $\leqslant \epsilon$, then R is transitive on B. To see this, note that if $x,y,z \in B$ with $xRyRz$, there are $k,k' \in K$ such that $kx = y, k'y = z$. So $g = k'k \in K^2$ and $d(x,gx) \leqslant \epsilon$, which by the definition of A_ϵ implies that $z = gxRx$. Therefore, covering each $A_{\frac{1}{n}}$ by countably many Borel sets of diameter $\leqslant \frac{1}{n}$, we obtain a covering of X by Borel sets X_m, $m \in \mathbb{N}$, such that R is an equivalence relation on each X_m.

Finally, let $Y_m = \{y \in X_m \mid \exists^* g \in G \;\; gyRy \;\&\; gy \in X_m\}$, which is Borel. Then $Y = \bigcup_m Y_m$ intersects any orbit $G\cdot x$. To see this, pick an open neighbourhood $U \ni 1$ such that $UU^{-1} \subseteq K$ and find some m such that $\{g \in U \mid gx \in X_m\}$ is non-meagre in G. Choose also $u \in U$ such that $y = ux \in X_m$ and note that

$$\{g \in U \mid gx \in X_m\}u^{-1} \subseteq \{g \in K \mid gy \in X_m\} \subseteq \{g \in G \mid gyRy \;\&\; gy \in X_m\},$$

whence $y \in Y_m$. A similar argument also shows that Y_m is R-invariant in X_m.

Now fix m and, by refining the topology on X_m, assume without loss of generality that X_m is Polish. Since R is a closed equivalence relation on X_m, it is smooth, so we can find a Borel function $\phi\colon X_m \to 2^{\mathbb{N}}$ such that $xRy \iff \phi(x) = \phi(y)$ for all $x,y \in X_m$. By further refining the topology on X_m, we can suppose that ϕ is continuous and thus has closed graph $F \subseteq X_m \times 2^{\mathbb{N}}$.

Choose now a Souslin scheme $(F_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of closed sets such that $F = F_\emptyset$, $F_s = \bigcup_n F_{sn}$ and $\mathrm{diam}_d(F_s) < \frac{1}{|s|+1}$ for all $s \in \mathbb{N}^{<\mathbb{N}}$ and some compatible complete metric d on $X_m \times 2^{\mathbb{N}}$.

Define an R-invariant Borel mapping $x \in X_m \mapsto T^x \in \mathrm{PTr}_{\mathbb{N}}$ by

$$T^x = \{s \in \mathbb{N}^{<\mathbb{N}} \mid \exists^* g \in G\ (gx, \phi(x)) \in F_s\}.$$

Clearly $x \mapsto T_x$ is Borel and to see that it is R-invariant, suppose that $x, kx \in X_m$ for some $k \in K$. Then $(gx, \phi(x)) = (gk^{-1} \cdot kx, \phi(kx))$, whereby

$$\{g \in G \mid (gx, \phi(x)) \in F_s\}k^{-1} = \{g \in G \mid (g \cdot kx, \phi(kx)) \in F_s\}$$

and $T^x = T^{kx}$. Moreover, each T^x is pruned, for if $s \in T^x$, then $\exists^* g \in G\ (gx, \phi(x)) \in F_s = \bigcup_n F_{sn}$ and so for some $n \in \mathbb{N}$, also $\exists^* g \in G\ (gx, \phi(x)) \in F_{sn}$, i.e., $sn \in T^x$. Finally, for every $y \in Y_m$, we have $\exists^* g \in G\ \ gyRy\ \&\ gy \in X_m$, i.e., $\emptyset \in T^y$.

It follows that for $y \in Y_m$ we can choose the leftmost branch $b(y)$ of T^y in a Borel manner and then let $f(y) \in X_m$ be defined by

$$\{f(y)\} = \bigcap_{s \subseteq b(y)} F_s^{\phi(y)}.$$

It follows that $yRf(y)$, whence, as Y_m is R-invariant in X_m, actually $f(y) \in Y_m$. Moreover, $f\colon Y_m \to Y_m$ is Borel and thus $T_m = \{y \in Y_m \mid f(y) = y\}$ is a Borel transversal for R on Y_m. It follows that $T = \bigcup_m T_m$ is a Borel set intersecting all G-orbits in a countable non-empty set. □