COUNTABLE SECTIONS FOR LOCALLY COMPACT GROUP ACTIONS

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Theorem 1 (A. S. Kechris). Suppose $G$ is a locally compact Polish group acting continuously on a Polish space $X$. Then there is a Borel set $Z \subseteq X$ intersecting every $G$-orbit in a non-empty countable set.

Proof. Fix a compact symmetric neighbourhood $K$ of 1 and let $V = \text{int} K$. We define a closed, reflexive, symmetric relation $R$ on $X$ by letting

$$xRy \iff \exists k \in K \ kx = y.$$ 

Though $R$ need not be transitive, we shall construct a covering of $X$ by countably many Borel sets $X_{m}$ such that $R$ is transitive and thus an equivalence relation when restricted to each $X_{m}$.

Since $R$ is closed, we see that, for any $\epsilon > 0$,

$$C_{\epsilon} = \{(x, y) \in G \times X \mid d(x, gx) \leq \epsilon \ \& \ g \in K^{2} \ \& \ gx \ R x\}$$

is $F_{\sigma}$ and so $\text{proj}_{X}(C_{\epsilon})$ is $F_{\sigma}$ too. Moreover, $\cap_{\epsilon > 0} \text{proj}_{X}(C_{\epsilon}) = \emptyset$. For if $x$ belongs to the intersection, we can choose $g_{n} \in K^{2}$ such that $d(x, g_{n}x) \to 0$ and $g_{n}x \ R x$. It follows, in particular, that $g_{n} \in KG_{x} \supseteq VG_{x}$, where $G_{x}$ is the stabiliser of $x$ and $VG_{x}$ is open. Passing to a subsequence, we can suppose that the limit $g = \lim_{n \to \infty} g_{n} \in K^{2} \setminus VG_{x}$ exists, contradicting that also $gx = \lim_{n \to \infty} g_{n}x = x$.

It follows that the sets

$$A_{\epsilon} = X \setminus \text{proj}_{X}(C_{\epsilon}) = \{x \in X \mid \forall g \in K^{2} \ (d(x, gx) \leq \epsilon \to gxRx)\}$$

are $G_{\delta}$ and that $X = \bigcup_{\epsilon > 0} A_{\epsilon}$. Moreover, if $B \subseteq A_{\epsilon}$ has diameter $\leq \epsilon$, then $R$ is transitive on $B$. To see this, note that if $x, y, z \in B$ with $xRyRz$, there are $k, k' \in K$ such that $kx = y, k'x = z$. So $g = k'k \in K^{2}$ and $d(x, gx) \leq \epsilon$, which by the definition of $A_{\epsilon}$ implies that $z = gxRx$. Therefore, covering each $A_{\frac{1}{n}}$ by countably many Borel sets of diameter $\leq \frac{1}{n}$, we obtain a covering of $X$ by Borel sets $X_{m}, m \in \mathbb{N}$, such that $R$ is an equivalence relation on each $X_{m}$.

Finally, let $Y_{m} = \{y \in X_{m} \mid \exists^{*} g \in G \ gyRy \ \& \ gy \in X_{m}\}$, which is Borel. Then $Y = \bigcup_{m} Y_{m}$ intersects any orbit $G \cdot x$. To see this, pick an open neighbourhood $U \ni 1$ such that $UU^{-1} \subseteq K$ and find some $m$ such that $\{g \in U \mid gx \in X_{m}\}$ is non-meagre in $G$. Choose also $u \in U$ such that $y = ux \in X_{m}$ and note that

$$\{g \in U \mid gx \in X_{m}\}u^{-1} \subseteq \{g \in K \mid gy \in X_{m}\} \subseteq \{g \in G \mid gyRy \ \& \ gy \in X_{m}\},$$

whence $y \in Y_{m}$. A similar argument also shows that $Y_{m}$ is $R$-invariant in $X_{m}$.

Now fix $m$ and, by refining the topology on $X_{m}$, assume without loss of generality that $X_{m}$ is Polish. Since $R$ is a closed equivalence relation on $X_{m}$, it is smooth, so we can find a Borel function $\phi : X_{m} \to 2^{\mathbb{N}}$ such that $xRy \iff \phi(x) = \phi(y)$ for all $x, y \in X_{m}$. By further refining the topology on $X_{m}$, we can suppose that $\phi$ is continuous and thus has closed graph $F \subseteq X_{m} \times 2^{\mathbb{N}}$. 

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Choose now a Souslin scheme \((F_s)_{s \in \mathbb{N}^{<\mathbb{N}}} \) of closed sets such that \(F = F_\emptyset, F_s = \bigcup_n F_{sn} \) and \(\text{diam}_d(F_s) < \frac{1}{|s|+1} \) for all \(s \in \mathbb{N}^{<\mathbb{N}}\) and some compatible complete metric \(d\) on \(X_m \times 2^{\mathbb{N}}\).

Define an \(R\)-invariant Borel mapping \(x \in X_m \mapsto T^x \in \mathcal{P} Tr_N\) by

\[
T^x = \{ s \in \mathbb{N}^{<\mathbb{N}} \mid \exists^* g \in G \ (gx, \phi(x)) \in F_s \}.
\]

Clearly \(x \mapsto T^x \) is Borel and to see that it is \(R\)-invariant, suppose that \(x, kx \in X_m\) for some \(k \in K\). Then \((gx, \phi(x)) = (gk^{-1} \cdot kx, \phi(kx)),\) whereby

\[
\{ g \in G \mid (gx, \phi(x)) \in F_s \} k^{-1} = \{ g \in G \mid (g \cdot kx, \phi(kx)) \in F_s \}
\]

and \(T^x = T^{kx}\). Moreover, each \(T^x\) is pruned, for if \(s \in T^x\), then \(\exists^* g \in G \ (gx, \phi(x)) \in F_s = \bigcup_n F_{sn}\) and so for some \(n \in \mathbb{N}\), also \(\exists^* g \in G \ (gx, \phi(x)) \in F_{sn}\), i.e., \(sn \in T^x\). Finally, for every \(y \in Y_m\), we have \(\exists^* g \in G \ gyRy \& gy \in X_m\), i.e., \(\emptyset \in Ty\).

It follows that for \(y \in Y_m\) we can choose the leftmost branch \(b(y)\) of \(Ty\) in a Borel manner and then let \(f(y) \in X_m\) be defined by

\[
\{ f(y) \} = \bigcap_{s \subseteq b(y)} F_s^{\phi(y)}.
\]

It follows that \(yRf(y)\), whence, as \(Y_m\) is \(R\)-invariant in \(X_m\), actually \(f(y) \in Y_m\). Moreover, \(f : Y_m \to Y_m\) is Borel and thus \(T_m = \{ y \in Y_m \mid f(y) = y \}\) is a Borel transversal for \(R\) on \(Y_m\). It follows that \(T = \bigcup_m T_m\) is a Borel set intersecting all \(G\)-orbits in a countable non-empty set. □