1. THE $G_0$ DICHTOMY

A digraph (or directed graph) on a set $X$ is a subset $G \subseteq X^2 \setminus \Delta$. Given a digraph $G$ on a set $X$ and a subset $A \subseteq X$, we say that $A$ is $G$-discrete if for all $x, y \in A$ we have $(x, y) \not\in G$.

Now let $s_n \in 2^n$ be chosen for every $n \in \mathbb{N}$ such that $\forall s \in 2^{<\omega} \exists n \ s \subseteq s_n$. Then we can define a digraph $G_0$ on $2^\omega$ by

$$G_0 = \{(s_n0x, s_n1x) \in 2^\omega \times 2^\omega \mid n \in \mathbb{N} \& x \in 2^\omega\}.$$

**Exercise 1.** Show that if $x, y \in 2^\omega$ differ in only finitely many coordinates, then there is a path $x_0 = x, x_1, \ldots, x_n = y$ such that for all $i$, either $(x_i, x_{i+1}) \in G_0$ or $(x_{i+1}, x_i) \not\in G_0$.

**Hint:** The proof is by induction on the last coordinate in which they differ.

**Lemma 2.** If $f : 2^\omega \to X$ is a continuous function into a Polish space $X$ such that $xG_0x \Rightarrow f(x) = f(y)$, then $f$ is constant.

**Proof.** If not, by continuity, we can find basic open sets $N_s, N_t \subseteq 2^\omega$ such that $f[N_s] \cap f[N_t] = \emptyset$. Extending $s$ or $t$, we can suppose that $|s| = |t|$, and thus for any $x \in 2^\omega$, $f(sx) \neq f(tx)$. On the other hand, such $sx$ and $tx$ differ only in finitely many coordinates, so by Exercise 1 they are connected by a path in $G_0$, which contradicts the properties of $f$.

**Lemma 3.** If $B \subseteq 2^\omega$ has the Baire property and is non-meagre, then $B$ is not $G_0$-discrete.

**Proof.** By assumption on $B$, we can find some $s \in 2^{<\omega}$ such that $B$ is comeagre in $N_s$. Also, by choice of $(s_n)$, we can find some $n$ such that $s \subseteq s_n$, whereby $B$ is comeagre in $N_{s_n}$. By the characterisation of comeagre subsets of $2^\omega$, we see that for some $x \in 2^\omega$, we have $s_n0x, s_n1x \in B$, showing that $B$ is not $G_0$-discrete.

Suppose $G$ and $H$ are digraphs on sets $X$ and $Y$ respectively. A homomorphism from $G$ to $H$ is a function $h : X \to Y$ such that for all $x, y \in X$,

$$(x, y) \in G \Rightarrow (h(x), h(y)) \in H.$$ 

Also, if $Z$ is any set, a $Z$-colouring of a digraph $G$ on $X$ is a homomorphism from $G$ to the digraph $\neq$ on $Z$, i.e., a function $h : X \to Z$ such that for all $x, y \in X$,

$$(x, y) \in G \Rightarrow h(x) \neq h(y).$$

**Proposition 4.** There is no Baire measurable $\mathbb{N}$-colouring of $G_0$.

**Proof.** Note that if $h : 2^\omega \to \mathbb{N}$ is a Baire measurable function, then for some $n \in \mathbb{N}$, $B = h^{-1}(n)$ is non-meagre with the Baire property and hence not $G_0$-discrete. So $h$ cannot be a homomorphism from $G_0$ to $\neq$ on $\mathbb{N}$. 

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**Theorem 5** (Kechris–Solecki–Todorcevic). Suppose $G$ is an analytic digraph on a Polish space $X$. Then exactly one of the following holds:
- there is a continuous homomorphism from $G_0$ to $G$,
- there is a Borel $\mathbb{N}$-colouring of $G$.

**Proof.** (B. Miller) If $X$ is countable, the result is trivial. So if not, let $f : \mathbb{N}^\mathbb{N} \to P$ be a continuous bijection onto the perfect kernel $P$ of $X$. By replacing $G$ with $(f \times f)^{-1}[G]$, there is no loss of generality in assuming that $X = \mathbb{N}^\mathbb{N}$.

So suppose $F \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ is a closed set such that
\[(x, y) \in G \Leftrightarrow \exists z (x, y, z) \in F.\]

In order to produce a continuous homomorphism $h$ from $G_0$ to $G$ it suffices to find monotone Lipschitz functions $u, v^m : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, $m \in \mathbb{N}$, such that for all $m < k$ and $t \in 2^{k-m-1}$,
\[\{N_u(s_m0w) \times N_u(s_m1t) \times N_{v^m}(t)\} \cap F \neq \emptyset.\]

In this case, we can define $h, \tilde{v}^m : 2^{\mathbb{N}} \to \mathbb{N}^\mathbb{N}$ by $h(w) = \bigcup_n u(w|_n)$ and $\tilde{v}^m(w) = \bigcup_n v^m(w|_n)$. For then if $m \in \mathbb{N}$ and $w \in 2^\mathbb{N}$ are given, there are $x_k, y_k, z_k \in \mathbb{N}^\mathbb{N}$ such that $x_k \to h(s_m0w)$, $y_k \to h(s_m1w)$ and $z_k \to \tilde{v}^m(w)$ such that for all $k$, $(x_k, y_k, z_k)$. So, as $F$ is closed, also
\[(h(s_m0w), h(s_m1w), \tilde{v}^m(w)) \in F,\]

whence $(h(s_m0w), h(s_m1w)) \in G$, showing that $h$ is a homomorphism from $G_0$ to $G$.

An $n$-approximation is a pair $(u, v)$ of functions $u : 2^n \to \mathbb{N}^n$ and $v : 2^{\mathbb{N}} \to \mathbb{N}^\mathbb{N}$. Also, if $(u, v)$ is an $n$-approximation and $(u', v')$ is an $n + 1$-approximation, we say that $(u', v')$ extends $(u, v)$ if $u(s) \subseteq u'(si)$ and $v(t) \subseteq v'(ti)$ for all $s \in 2^n$, $t \in 2^{\mathbb{N}}$ and $i = 0, 1$.

Suppose $A \subseteq X$ and $(u, v)$ is an $n$-approximation. We define the set of $A$-realisations, $\mathbb{R}(A, u, v)$, to be the set of pairs of tuples $(x_s)_{s \in 2^n} \in \prod_{s \in 2^n} (A \cap N_u(s))$ and $(z_t)_{t \in 2^{\mathbb{N}}}$ in $\prod_{t \in 2^{\mathbb{N}}} N_{v^m}(t)$ such that
\[(x_s, x_{s, 1t}, z_t) \in F\]
for all $s \in 2^n$, $m \in \mathbb{N}$ and $t \in 2^{n-m-1}$.

So if $(u_0, v_0)$ is the unique 0-approximation (i.e., $u(\emptyset) = \emptyset$ and $v$ is the function with empty domain), we have $\mathbb{R}(A, u_0, v_0) = \{x_{s_0} \mid x_{s_0} \in A\} = A$. If $(u, v)$ has no $A$-realised extension, we say that $(u, v)$ is $A$-terminal.

**Lemma 6.** Suppose $(u, v)$ is an $A$-terminal $n$-approximation, then
\[\mathbb{D}(A, u, v) = \{x_{s_n} \mid ((x_s)_{s \in 2^n}, (z_t)_{t \in 2^{\mathbb{N}}}) \in \mathbb{R}(A, u, v)\}\]
is $G$-discrete.

**Proof.** Suppose toward a contradiction that
\[((x_s^0)_{s \in 2^n}, (z_t^0)_{t \in 2^{\mathbb{N}}}), ((x_s^1)_{s \in 2^n}, (z_t^1)_{t \in 2^{\mathbb{N}}}) \in \mathbb{R}(A, u, v)\]
satisfy $(x_{s_n}^0, x_{s_n}^1) \in G$. Then for some $z_{s_0} \in \mathbb{N}^\mathbb{N}$, we have
\[x_{s_0}^0, x_{s_0}^1, z_{s_0} \in F,\]
and hence, setting $x_{si} = x_{si}^1$ and $z_{ti} = z_{ti}^1$ for all $s i \in 2^{n+1}$ and $t i \in 2^{n+1} \setminus \{\emptyset\}$, we get an $A$-realisation $((x_s)_{s \in 2^{n+1}}, (z_t)_{t \in 2^{\mathbb{N}+1}})$ of an extension of $(u, v)$, contradicting that $(u, v)$ is $A$-terminal.
Now define $\Phi \subseteq P(X)$ by

$$\Phi(A) \leftrightarrow A \text{ is } G\text{-discrete.}$$

Since $G$ is analytic, $\Phi$ is $\Pi^1_1$ on $\Sigma^1_1$, and so, by the First Reflection Theorem, any $G$-discrete analytic set $A$ is contained in a $G$-discrete Borel set $A'$. Using this, we can define a function $D$ assigning to each Borel set $A \subseteq X$ a Borel subset given by

$$D(A) = A \setminus \bigcup \{ B(A,u,v) \mid (u,v) \text{ is } A\text{-terminal} \}. $$

Note that, as there are only countably many approximations $(u,v)$, the set $A \setminus D(A)$ is a countable union of $G$-discrete Borel sets.

**Lemma 7.** Suppose $(u,v)$ is an $n$-approximation all of whose extensions are $A$-terminal. Then $(u,v)$ is $D(A)$-terminal.

**Proof.** Note that if $(u,v)$ is not $D(A)$-terminal, there is some extension $(u',v')$ of $(u,v)$ and some realisation $(x_\xi)_{\xi \in 2^{n+1}} \in \mathbb{R}(D(A),u,v') \subseteq \mathbb{R}(A,u',v')$. But since $(u',v')$ is $A$-terminal, we have $D(A,u',v') \cap D(A) = \emptyset$, contradicting that $\phi(x_{n+1}) \in \mathbb{R}(A,u',v') \cap D(A)$. \hfill \Box

Now define, by transfinite induction, $D^0(X) = X$, $D^{\xi+1}(X) = D(D^\xi(X))$ and $D^\xi(X) = \bigcap_{\zeta < \xi} D^\zeta(X)$, whenever $\lambda$ is a limit ordinal. Then $(D^\xi(X))_{\xi < \omega_1}$ is a well-ordered, decreasing sequence of Borel subsets of $X$, so the sets $T_\xi$ of approximations $(u,v)$ that are $D^\xi(X)$-terminal is an increasing sequence of subsets of the countable set of all approximations. It follows that for some $\xi < \omega_1$, we have $T_\xi = T_{\xi+1}$.

Now if $(u,v) \notin T_{\xi+1}$, then $(u,v)$ is not $D(D^\xi(X))$-terminal and hence admits an extension $(u',v')$ that is not $D^\xi(X)$-terminal either, whereby $(u',v') \notin T_\xi = T_{\xi+1}$. So if $(u_0,v_0)$ denotes the unique 0-approximation and $(u_0,v_0) \notin T_{\xi+1}$, we can inductively construct $(u_n,v_n) \notin T_{\xi+1}$ extending each other. Setting

$$u = \bigcup_n u_n$$

and for $t \in 2^n$

$$v^m(t) = v_{n+m+1}(t),$$

we have the required monotone Lipschitz functions $u,v : 2^{<\omega} \to \mathbb{N}^{<\omega}$ to produce a continuous homomorphism from $G_0$ to $G$.

Conversely, if $(u_0,v_0) \in T_{\xi+1}$, then $(u_0,v_0)$ is $D^{\xi+1}(X)$-terminal and hence $D^{\xi+2}(X) \subseteq D^{\xi+1}(X) \setminus D(D^{\xi+1}(X),u_0,v_0)$. But, since $(u_0,v_0)$ is the unique 0-approximation, we have

$$D(D^{\xi+1}(X),u_0,v_0) = \mathbb{R}(D^{\xi+1}(X),u_0,v_0) = D^{\xi+1}(X),$$

whereby $D^{\xi+2}(X) = \emptyset$. It follows that

$$X = \bigcup_{\zeta < \xi+2} D^\zeta(X) \setminus D^{\xi+1}(X)$$

is a countable union of $G$-discrete Borel sets. We can then define a Borel $\mathbb{N}$-colouring of $G$ by letting $c(x)$ be a code for the discrete Borel subset of $X$ to which $x$ belongs. \hfill \Box

**Exercise 8.** By inspection of the proof of Theorem 5, show that if $G$ is a $\kappa$-Souslin digraph on $\mathbb{N}^\kappa$, then one of the following holds

- there is a continuous homomorphism from $G_0$ to $G$,
- there is a $\kappa$-colouring of $G$.
2. The Mycielski, Silver and Burgess dichotomies

**Theorem 9** (Mycielski's Independence Theorem). Suppose $X$ is a perfect Polish space and $R \subseteq X^2$ is a comeagre set. Then there is a continuous injection $\varphi: 2^\mathbb{N} \to X$ such that for all distinct $x, y \in 2^\mathbb{N}$ we have $(\varphi(x), \varphi(y)) \in R$.

**Proof.** Let $d \leq 1$ be a compatible complete metric on $X$ and choose a decreasing sequence of dense open subsets $U_n \subseteq X^2$ such that $\bigcap_{n \in \mathbb{N}} U_n \subseteq R$. We construct a Cantor scheme $(C_s)_{s \in 2^\mathbb{N}}$ of non-empty open subsets of $X$ by induction on the length of $s$ such that for all distinct $s, t \in 2^n$ and $i = 0, 1$, we have

$$C_{si} \subseteq C_s, \quad \text{diam}(C_s) \leq \frac{1}{|s|+1}, \quad \text{and} \quad C_s \times C_t \subseteq U_{n-1}.$$

To see how this is done, suppose that $C_s$ has been defined for all $s \in 2^n$. Since $X$ is perfect, we can find disjoint, non-empty open subsets $D_{s0}$ and $D_{s1}$ of $C_s$ for every $s \in 2^n$. Now, as $U_n$ is dense, $U_n \cap (D_{s1} \times D_{t1}) \neq \emptyset$ for all distinct $t, t' \in 2^{n+1}$ and so we can inductively shrink the $D_t$ to open subsets $C_t$ such that whenever $t, t' \in 2^{n+1}$ are distinct, we have $C_t \cap C_{t'} \subseteq U_n$. By further shrinking the $C_{si}$ if necessary, we can ensure that $C_{si} \subseteq C_s$ and diam$(C_s) \leq \frac{1}{|s|+1}$. Now letting $\varphi: 2^\mathbb{N} \to X$ be defined by $(\varphi(s)) = \bigcap_{n \in \mathbb{N}} C_{x|_n}$, we see that $\varphi$ is continuous. Also, if $x, y \in 2^\mathbb{N}$ are distinct, then for all but finitely many $n$ we have $(\varphi(x), \varphi(y)) \in C_{x|_n} \times C_{y|_n} \subseteq U_{n-1}$, so, since the $U_n$ are decreasing, we have $(x, y) \in \bigcap_{n \in \mathbb{N}} U_n \subseteq R$. \hfill $\Box$

**Theorem 10** (J. Silver). Suppose $E$ is a conalytic equivalence relation on a Polish space $X$. Then exactly one of the following holds

- $E$ has at most countably many classes,
- there is a continuous injection $\varphi: 2^\mathbb{N} \to X$ such that for distinct $x, y \in 2^\mathbb{N}$, $\varphi(x)E\varphi(y)$.

**Proof.** We define an analytic digraph $G$ on $X$ by setting $G = X^2 \setminus E$. Notice first that if $c: X \to \mathbb{N}$ is a Borel $\mathbb{N}$-colouring of $G$, then for all $x, y \in X$,

$$\neg xEy \Rightarrow (x, y) \in G \Rightarrow c(x) \neq c(y).$$

So for any $n \in \mathbb{N}$, $c^{-1}(n)$ is contained in a single equivalence class of $E$. Moreover, as $X = \bigcup_{n \in \mathbb{N}} c^{-1}(n)$, this shows that $X$ is covered by countably many $E$-equivalence classes.

So suppose instead that there is no Borel $\mathbb{N}$-colouring of $G$. Then by Theorem 5 there is a continuous homomorphism $h: 2^\mathbb{N} \to X$ from $G_0$ to $G$. Now let $F = \{(x, y) \in 2^\mathbb{N} \times 2^\mathbb{N} \mid h(x)Eh(y)\}$. Then $F$ is meagre. For otherwise, by the Kuratowski–Ulam Theorem, there is some $x \in 2^\mathbb{N}$ such that $F_x$ is non-meagre and hence, by Lemma 3, there are $y, z \in F_x$ such that $(y, z) \in G_0$. As $h$ is a homomorphism it follows that $(h(y), h(z)) \in G = X^2 \setminus E$, which contradicts that $h(y)Eh(x)Eh(z)$. Therefore, applying Mycielski’s Theorem to the meagre set $F$, we get a continuous function $f: 2^\mathbb{N} \to 2^\mathbb{N}$ such that for distinct $x, y \in 2^\mathbb{N}$, $(f(x), f(y)) \notin F$, i.e., $\neg h \circ f(x)Eh \circ f(y)$. Letting $\varphi = h \circ f$, we have the result. \hfill $\Box$

By the same proof, using instead the $G_0$-dichotomy for $\omega_1$-Souslin sets, we deduce the following result.

**Theorem 11** (J. Burgess, L. A. Harrington–S. Shelah). Let $E$ be a $\Sigma^1_2$ equivalence relation on a Polish space $X$. Then one of the following holds

- $E$ has at most $\aleph_1$ classes,
there is a continuous injection $\phi : 2^\mathbb{N} \to X$ such that for distinct $x, y \in 2^\mathbb{N}$, $\neg \phi(x)E\phi(y)$.

Now as the isomorphism relation between the countable models of an $L_{\omega_1\omega}$-sentence is an analytic equivalence relation, we have the following corollary, initially proved by analysing the space of complete types.

**Corollary 12** (M. Morley). Suppose $L$ is a countable language and $\sigma$ is a $L_{\omega_1\omega}$ sentence. Then there are either a continuum of non-isomorphic countable models of $\sigma$ or at most $\aleph_1$ non-isomorphic models of $\sigma$.

**Theorem 13** (Lusin–Novikov). Suppose $X$ and $Y$ are Polish spaces and $A \subseteq X \times Y$ a Borel subset. Assume that for every $x \in X$, the vertical section $A_x$ is countable. Then there are Borel sets $F_n$ such that $|(F_n)_x| \leq 1$ for every $x \in X$ and $A = \bigcup_{n \in \mathbb{N}} F_n$.

*Proof.* Define a Borel digraph $G$ on $X \times Y$ by

$$(x, y)G(x', y') \Leftrightarrow x = x' \& y \neq y' \& (x, y) \in A \& (x', y') \in A.$$ 

Assume first that $f : X \times Y \to \mathbb{N}$ is a Borel $\mathbb{N}$-coloring of $G$. Then for every $n \in \mathbb{N}$, $F_n = A \cap f^{-1}(n)$ is a $G$-discrete Borel subset of $X \times Y$ and $A = \bigcup_{n \in \mathbb{N}} F_n$. Moreover, since $F_n$ is $G$-discrete, we see that $|(F_n)_x| \leq 1$ for all $x \in X$.

Now, by Theorem 5, if there is no such colouring, then there is a continuous homomorphism $h : 2^\mathbb{N} \to X \times Y$ from $G_0$ to $G$. Composing with the coordinate projections, we obtain continuous functions $h_X : 2^\mathbb{N} \to X$ and $h_Y : 2^\mathbb{N} \to Y$ such that

$$aG_0b \Rightarrow h_X(a) = h_X(b).$$

By Lemma 2, $h_X$ is constant with some value $x_0 \in X$ and so

$$aG_0b \Rightarrow h_Y(a) \neq h_Y(b) \& h_Y(a) \in A_{x_0} \& h_Y(b) \in A_{x_0}.$$ 

Since $A_{x_0}$ is countable, there is an injection $\pi : A_{x_0} \to \mathbb{N}$, and thus $\pi \circ h_Y : 2^\mathbb{N} \to \mathbb{N}$ is a continuous $\mathbb{N}$-colouring of $G_0$, contradicting Proposition 4. So the first option holds. \qed