GAMES AND LEBESGUE MEAUSURABILITY

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Theorem 1. *If all games on $\mathbb{N}$ are determined, then all sets of reals are Lebesgue measurable.*

Proof. Instead of Lebesgue measure on $\mathbb{R}$, we shall work with the equivalent situation of Lebesgue measure μ on $2^{\mathbb{N}}$, i.e., the unique Borel measure such that $\mu(N_s) = 2^{-|s|}$ for all $s \in 2^{<\mathbb{N}}$. We denote by μ_* and μ^* the corresponding inner and outer measures defined for all subsets $A \subseteq 2^{\mathbb{N}}$ by

$$\mu_*(A) = \sup\{\mu(B) \bigm| B \subseteq A \;\&\; B \text{ is Borel }\},$$

and

$$\mu^*(A) = \inf\{\mu(B) \bigm| A \subseteq B \;\&\; B \text{ is Borel }\}.$$

Now fix $A \subseteq 2^{\mathbb{N}}$. To see that a set A is Lebesgue measurable, we need to show that $\mu_*(A) = \mu^*(A)$ or equivalently that for all $r \in \mathbb{Q}\cap]0,1[$, we have

$$\mu_*(A) \geqslant r \Leftrightarrow \mu^*(A) \geqslant r.$$

So fix such an r and define a game $G_r(A)$ between two players I and II as follows:

I	h_0		h_1		h_2		h_3		$\dots$
II		i_0		i_1		i_2		i_3	$\dots$

Here $i_n \in \{0,1\}$ and $h_n\colon \{0,1\} \to \mathbb{Q} \cap [0,1[$. Moreover, the h_n are subject to the conditions $h_n(i) \leqslant \frac{1}{2^{n+1}}$, $h_0(0) + h_0(1) = r$, and $h_{n+1}(0) + h_{n+1}(1) \geqslant h_n(i_n)$, while the i_n are subject to $h_n(i_n) > 0$. The outcome of an infinite run of the game is the infinite binary sequence $x = (i_0, i_1, i_2, \dots) \in 2^{\mathbb{N}}$, and we say that I wins the game in case $x \in A$. If not, II wins the game.

Claim 2. *If I has a winning strategy in $G_r(A)$, then $\mu_*(A) \geqslant r$.*

Proof. Suppose σ is a winning strategy for I in $G_r(A)$. We define a pruned tree T on 2 such that $[T] \subseteq A$ and $\mu([T]) \geqslant r$, from which $\mu_*(A) \geqslant r$ follows. $T \cap 2^n$ is defined by induction on n along with an auxiliary function ψ from T to the set of positions in $G_r(A)$:

- $\emptyset \in T$ and $\psi(\emptyset) = \emptyset$,
- Suppose $s = (i_0, i_1, \dots, i_{n-1}) \in T$ and

  $$\psi(s) = (h_0, i_0, h_1, i_1, \dots, h_{n-1}, i_{n-1}) \in \sigma$$

  has been defined. Then if h_n is such that

  $$(h_0, i_0, h_1, i_1, \dots, h_{n-1}, i_{n-1}, h_n) \in \sigma,$$

  we let $si \in T$ if and only if $h_n(i) > 0$, in which case we set

  $$\psi(si) = (h_0, i_0, h_1, i_1, \dots, h_{n-1}, i_{n-1}, h_n, i).$$

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We now see that if $x \in [T]$, then $\psi(\emptyset) \sqsubseteq \psi(x|_1) \sqsubseteq \psi(x|_2) \sqsubseteq \ldots$ is a coherent sequence of positions in $G_r(A)$ in which I has played according to σ with outcome x. So, as σ is winning for I, $x \in A$, which shows that $[T] \subseteq A$.

Also, as T is pruned, we have $[T] \cap N_s \neq \emptyset \Leftrightarrow s \in T$. Therefore, by outer regularity of μ and compactness of $[T]$, to see that $\mu([T]) \geqslant r$ it is enough to show that

$$\frac{|T \cap 2^n|}{2^n} \geqslant r,$$

for all $n \in \mathbb{N}$.

So to verify this, if $s \in T$ with $\psi(s) = (h_0, i_0, h_1, i_1, \ldots, h_{n-1}, i_{n-1})$, we let $\delta(s) = h_{n-1}(i_{n-1})$. Then we see that $\sum_{s \in T \cap 2^n} \delta(s) \geqslant r$ and $\delta(s) \leqslant \frac{1}{2^n}$ for all $s \in T \cap 2^n$. It follows that

$$\frac{|T \cap 2^n|}{2^n} \geqslant r,$$

for all $n \in \mathbb{N}$. □

Claim 3. *If II has a winning strategy τ in $G_r(A)$, then $\mu^*(A) \leqslant r$.*

Proof. We show that for any $\epsilon > 0$, $\mu^*(A) \leqslant r + \epsilon$. Again, for this, we define a pruned tree T on 2 such that $[T] \cap A = \emptyset$ and $\mu(\sim [T]) \leqslant r + \epsilon$. $T \cap 2^n$ is defined by induction on n along with an auxiliary function ψ from T to the set of positions in $G_r(A)$ and a real valued function δ defined on all si for $s \in T$ and $i = 0, 1$.

- First put $\emptyset \in T$ and set $\psi(\emptyset) = \emptyset$.
- Now, suppose that $s = (i_0, i_1, \ldots, i_{n-1}) \in T$ and

  $$\psi(s) = (h_0, i_0, h_1, i_1, \ldots, h_{n-1}, i_{n-1}) \in \tau$$

  has been defined. For $i = 0, 1$, if there is some h such that

  $$(h_0, i_0, h_1, i_1, \ldots, h_{n-1}, i_{n-1}, h, i) \in \tau,$$

  we let

  $$\delta(si) = \inf(h(i) \mid (h_0, i_0, h_1, i_1, \ldots, h_{n-1}, i_{n-1}, h, i) \in \tau),$$

  and otherwise set $\delta(si) = 2^{n+1}$. Then, by checking cases, one verifies that

  $$\delta(s0) + \delta(s1) \leqslant h_{n-1}(i_{n-1})$$

  or $\delta(s0) + \delta(s1) \leqslant r$ in case $s = \emptyset$.

  Now put $si \in T$ if and only if $\delta(si) < 2^{n+1}$, and in this case let

  $$\psi(si) = (h_0, i_0, h_1, i_1, \ldots, h_{n-1}, i_{n-1}, h, i)$$

  for some h such that $h(i) < \delta(si) + \frac{\epsilon}{4^{n+1}}$ and

  $$(h_0, i_0, h_1, i_1, \ldots, h_{n-1}, i_{n-1}, h, i) \in \tau.$$

  This finishes the construction of T.

Now, define $f \colon 2^{<\mathbb{N}} \to [0, 1]$ by $f(\emptyset) = r$ and for $s \in T$ with

$$\psi(s) = (h_0, i_0, h_1, i_1, \ldots, h_{n-1}, i_{n-1}),$$

let $f(s) = h_{n-1}(i_{n-1})$, while for $s \notin T$, $f(s) = \frac{1}{2^{|s|}}$.

Then for $s \in T \cap 2^n$

$$\begin{aligned} f(s0) + f(s1) &< \delta(s0) + \frac{\epsilon}{4^{n+1}} + \delta(s1) + \frac{\epsilon}{4^{n+1}} \\ &= f(s) + \frac{2\epsilon}{4^{n+1}}, \end{aligned}$$

while for $s \in 2^n \setminus T$,

$$f(s0) + f(s1) = \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n} < f(s) + \frac{2\epsilon}{4^{n+1}}.$$

As in the proof of Claim **??**, we see that $[T] \subseteq \sim A$. Also, by induction on n, one sees that

$$\sum_{s \in 2^n} f(s) \leqslant f(\emptyset) + \epsilon = r + \epsilon.$$

But since for all s, $f(s) \geqslant 0$ and $f(s) = \frac{1}{2^{|s|}}$ whenever $s \notin T$, it follows that

$$\frac{|2^n \setminus T|}{2^n} \leqslant r + \epsilon,$$

for all $n \in \mathbb{N}$ and hence $\mu^*(A) \leqslant \mu(\sim [T]) \leqslant r + \epsilon$. □

Thus, if every game $G_r(A)$ is determined, we see that for every rational number $0 < r < 1$, we have either $\mu^*(A) \leqslant r$ or $r \leqslant \mu_*(A)$ from which it follows that $\mu^*(A) \leqslant \mu_*(A)$. □