A CLOSURE PROPERTY FOR THE SOUSLIN OPERATION

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Definition 1. Suppose $(X,\mathscr{S})$ is a measurable space. A *cover* of a subset $D\subseteq X$ with respect to $\mathscr{S}$ is a measurable set $\hat{D}\supseteq D$ such that if $C\subseteq \hat{D}\setminus D$ is measurable, then any subset of C is also measurable. If every set has a cover, we say that $(X,\mathscr{S})$ admits covers.

Example 2. Suppose X is a topological space and $\mathrm{BP}(X)$ is the σ-algebra of sets with the Baire property. Then $(X,\mathrm{BP}(X))$ admits covers. To see this, for any $D\subseteq X$, we let $U(\sim D)$ be the largest open set in which $\sim D$ is comeagre or, equivalently, where D is meagre and set

$$\hat{D}=\sim U(\sim D)\cup D=\underbrace{\sim U(\sim D)}_{\text{closed}}\cup\underbrace{\big(D\cap U(\sim D)\big)}_{\text{meagre}}\in \mathrm{BP}(X).$$

Note also that if $\hat{D}\setminus D=\sim U(\sim D)\cap \sim D$ is comeagre in an open set V, then so is $\sim D$, whereby $V\subseteq U(\sim D)$ and $\big(\hat{D}\setminus D\big)\cap V=\emptyset$, from which it follows that V is meagre. In other words, any subset of $\hat{D}\setminus D$ with the Baire property must be meagre, whereby all its subsets belong to $\mathrm{BP}(X)$. So $\hat{D}$ is a cover of D with respect to $\mathrm{BP}(X)$.

Example 3. Suppose X is a standard Borel space and μ is a Borel probability measure on X. We let MEAS_μ denote the σ-algebra of all μ-measurable sets, i.e., the class of all sets that differ from a Borel set by a set of outer μ-measure 0. Then (X,MEAS_μ) admits covers. To see this, for every set $D\subseteq X$, let $\hat{D}\supseteq D$ be a Borel set realising the outer measure. Then clearly any measurable subset of $\hat{D}\setminus D$ is a null set, whence all its subsets belong to MEAS_μ.

Theorem 4 (E. Szpilrajn-Marczewski)**.** *Let $(X,\mathscr{S})$ be a measurable space admitting covers. Then $\mathscr{S}$ is closed under the Souslin operation $\mathscr{A}$.*

Proof. Suppose (A_s) is a Souslin scheme of measurable sets and let $B_s=\bigcap_{t\subseteq s}A_t\in\mathscr{S}$. Then (B_s) is a regular Souslin scheme with $\mathscr{A}(B_s)=\mathscr{A}(A_s)$. Define

$$D_s=\bigcup_{x\in N_s}\bigcap_n B_{x|n}\subseteq B_s$$

and note that $\mathscr{A}(B_s)=D_\emptyset$ and $D_s=\bigcup_{n\in\mathbb{N}}D_{sn}$ for all $s\in\mathbb{N}^{<\mathbb{N}}$. Let now $\hat{D}_s$ be the cover of D_s and note that, by intersecting $\hat{D}_s$ with B_s, we can suppose that $\hat{D}_s\subseteq B_s$.

Consider the measurable error term $E_s=\hat{D}_s\setminus\bigcup_n\hat{D}_{sn}\subseteq\hat{D}_s\setminus\bigcup_n D_{sn}=\hat{D}_s\setminus D_s$. By the defining property of covers, all subsets of E_s and thus also of $\bigcup_{s\in\mathbb{N}^{<\mathbb{N}}}E_s$ are measurable. Observe now that, for any $s\in\mathbb{N}^{<\mathbb{N}}$, we have

$$\hat{D}_s\setminus\bigcup_{t\in\mathbb{N}^{<\mathbb{N}}}E_t\ \subseteq\ \bigcup_n\hat{D}_{sn}\setminus\bigcup_{t\in\mathbb{N}^{<\mathbb{N}}}E_t\ \subseteq\ \bigcup_n\big(\hat{D}_{sn}\setminus\bigcup_{t\in\mathbb{N}^{<\mathbb{N}}}E_t\big).$$

Thus, for any $x\in\hat{D}_\emptyset\setminus\bigcup_{t\in\mathbb{N}^{<\mathbb{N}}}E_t$, we can inductively define $n_0,n_1,n_2,\ldots\in\mathbb{N}$ such that

$$x\in\hat{D}_{(n_0,\ldots,n_k)}\setminus\bigcup_{t\in\mathbb{N}^{<\mathbb{N}}}E_t$$

for all k, from which it follows that $\hat{D}_\emptyset \setminus \bigcup_{t\in\mathbb{N}^{<\mathbb{N}}} E_t \subseteq \mathscr{A}(\hat{D}_s) \subseteq \mathscr{A}(B_s) = \mathscr{A}(A_s)$. Therefore, $\hat{D}_\emptyset \setminus \mathscr{A}(A_s) \subseteq \bigcup_{t\in\mathbb{N}^{<\mathbb{N}}} E_t$, showing that $\hat{D}_\emptyset \setminus \mathscr{A}(A_s)$ and thus also $\mathscr{A}(A_s)$ are measurable. □

Corollary 5 (O. M. Nikodým)**.** *The class of sets with the Baire property in any topological space is closed under the Souslin operation.*

Corollary 6 (N. Lusin & W. Sierpiński)**.** *Let μ be a Borel probability measure on a standard Borel space X. Then the class of μ-measurable sets is closed under the Souslin operation.*