Rational Equivariant Forms

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This is joint work with Abdellah Sebbar.

Notation

Let us fix some notation:

$$\mathfrak{H} := \{z \in \mathbb{C};\ \Im(z) > 0\}\ ,\ \mathfrak{H}^* := \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})\ ,$$

$$\mathsf{SL}_2(\mathbb{Z}) := \text{the modular group},$$

$$\alpha \cdot z := \frac{az+b}{cz+d},\ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}),\ z \in \mathbb{C}.$$

Γ is a subgroup of $\mathsf{SL}_2(\mathbb{Z})$ of finite index, which we call a modular subgroup.
(We would like to mention that all what we present here, in fact, holds for any discrete subgroup of $\mathsf{SL}_2(\mathbb{R})$).

Preliminaries: The Schwarz derivative:

For a meromorphic function on a domain (open and connected) of $\mathbb{C}$, the Schwarz derivative, denoted $\{f, z\}$, is defined by

$$\begin{aligned} \{f, z\} &= 2\left(\frac{f''}{f'}\right)' - \left(\frac{f''}{f'}\right)^2 \\ &= \frac{2f'f''' - 3f''^2}{f'^2} . \end{aligned} \tag{1}$$

It satisfies the following rules.

- *Chain rule:* If w is a function of z then $\{f, z\} = (dw/dz)^2\{f, w\} + \{w, z\}$.
- Consequently, for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$, we have

$$\{w, z\} = \frac{\det \alpha}{(cz + d)^4}\{w, \alpha \cdot z\} \tag{2}$$

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$$\{w, z\} = \frac{\det \alpha}{(cz + d)^4}\{w, \alpha \cdot z\} \qquad (2)$$

Preliminaries: The Schwarz derivative:

- $\{f, z\} = 0$ if and only if f is a linear fractional transform of z.
- $\{w_1, z\} = \{w_2, z\}$ if and only if each function (w_i) is a linear fraction of the other.
- Inversion formula: If $w'(z_0) \neq 0$ for some point z_0, then in a neighborhood of z_0,

$$\{z, w\} \;=\; -(dz/dw)^2\{w, z\}\,. \tag{3}$$

Lastly, given a meromorphic function f on a domain D of $\mathbb{C}$, then one deduces the following

Proposition

The Schwarz derivative $\{f, z\}$ of f has a double pole at the critical points of f and is holomorphic elsewhere including at simple poles of f.

This is due to J. McKay and A. Sebbar.
Suppose now that f is a modular function for some modular subgroup Γ.

Proposition (M-S)

i. *If f is a modular function for Γ then $\{f, z\}$ is a (meromorphic) weight 4 modular form for Γ.*
ii. *If in addition Γ is of genus 0 and f is a Hauptmodul for Γ, then $\{f, z\}$ is weight 4 (holomorphic if Γ is torsion free) modular form for the normalizer of Γ inside $SL_2(\mathbb{R})$.*

What about the converse?
In other words, given meromorphic function f on $\mathfrak{H}$ such that $\{f, z\}$ is a weight 4 modular form on a modular subgroup Γ, what can be said about the invariance group G_f of f.

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In other words, given meromorphic function f on $\mathfrak{H}$ such that $\{f, z\}$ is a weight 4 modular form on a modular subgroup Γ, what can be said about the invariance group G_f of f.

In fact, using properties of the Schwarz derivative, we have for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$

$$\begin{aligned} \{f(\alpha\cdot z), \alpha\cdot z\} &= (cz+d)^4\{f(z), z\} \text{ (modularity)} \\ &= (cz+d)^4\{f(\alpha\cdot z), z\} \text{ (prpty the Schz. der.)} \end{aligned}$$

Hence

$$\{f(\alpha\cdot z), z\} = \{f(z), z\}$$

and therefore

$$f(\alpha\cdot z) = \Phi_\alpha\cdot f(z), \text{ for some } \Phi_\alpha \in \mathsf{GL}_2(\mathbb{C}) \;.$$

In particular, we have

$$\begin{aligned}\Phi \quad : \quad \Gamma &\longrightarrow \mathsf{GL}_2(\mathbb{C}) \\ \alpha &\longmapsto \Phi_\alpha\end{aligned}$$

is a group homomorphism. Moreover, $G_f = \operatorname{Ker}\Phi$.

Theorem (S)

If f is as above and is such that $f(z+n) = f(z),\ n \in \{1,2,3,4,5\}$, then f is a modular function for $\Gamma(n) = G_f$.

There are some cases where G_f, for instance $\log(f)$, f a Hauptmodul of $\Gamma(n)$ with n as above, is no larger than $< T^m >$, for some m ($T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$).

Question: when G_f is trivial?

This is the case if Φ is injective.

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Equivariant "functions": first examples

For the rest of this talk, Φ is the natural injection of Γ into $\mathrm{GL}_2(\mathbb{C})$. Hence, our functions are of the following type.

Definition

A meromorphic function h on $\mathfrak{H}$ is called an equivariant function for Γ if it satisfies
$h(\alpha \cdot z) = \alpha \cdot h(z),$ for all $\alpha \in \Gamma$.

A first, obvious, example is $f(z) = z$.
A larger class comes from

Proposition

Let f be a weight k, $k \in \mathbb{Z}$, modular form for Γ. Then the function

$$h_f(z) = z + \frac{kf(z)}{f'(z)} \qquad (\star)$$

is an equivariant function.

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The general definition

We define a "slash operator" on equivariant functions via the following action of $\mathrm{SL}_2(\mathbb{R})$ on meromorphic functions on $\mathfrak{H}$. For a meromorphic function f and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, we let

$$f||[\gamma](z) = j_\gamma(z)^{-2} f(\gamma \cdot z) - c j_\gamma(z)^{-1} \;,\; j_\gamma(z) = cz + d \;.$$

For a meromorphic function h on $\mathfrak{H}$, set $H(z) = (h(z) - z)^{-1}$. Then we have

Proposition

The function h is an equivariant function for Γ if and only if $H||[\gamma](z) = H(z)$ for all $\gamma \in \Gamma$. Furthermore, $H||[-\gamma](z) = H||[\gamma](z)$ if $-1_2 \in \Gamma$.

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An equivariant function for Γ is called an equivariant form for Γ if it satisfies

1. h is meromorphic on $\mathfrak{H}$;
2. h is meromorphic at the cusps, meaning that the function $H|||[\gamma](z)$ is meromorphic at infinity for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Example

For $f \in M_k^m(\Gamma)$, $k \neq 0$, the equivariant function $h_f = z + kf/f'$ (as in $(\star)$) is therefore an equivariant form.

Proposition

Suppose that Γ_1 and Γ_2 are conjugate subgroups of $\mathrm{SL}_2(\mathbb{Z})$, say $\Gamma_1 = \alpha\Gamma_2\alpha^{-1}$, for $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Then if h_1 is an equivariant form on Γ_1, the function $h_2(z) = \alpha^{-1} \circ h_1 \circ \alpha(z)$ is an equivariant form on Γ_2.

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Rational Equivariant forms

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An equivariant form is called a rational equivariant form if it arises from a modular form (of weight $k \neq 0$).

Proposition

For $c \in \mathbb{C}$ and $n \in \mathbb{Z}^\times$, we have
$h_{f^n} = h_{cf} = h_f = z + kf/f'\,, f \in M_k^m(\Gamma)\,.$

Proposition

For conjugate subgroups $\Gamma_1,\ \Gamma_2$, if $h_1(z) = z + kf(z)/f'(z)$, $f \in M_k^m(\Gamma_1)$, is a rational equivariant form then so is $h_2(z) = \alpha^{-1} \circ h_1 \circ \alpha(z)$, and we have $h_2(z) = z + \dfrac{k(f|_k[\alpha])(z)}{(f|_k[\alpha])'(z)}$.

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Example

A fundamental example corresponds to the modular discriminant Δ
$h_{\Delta}(z) = z + 12\Delta(z)/\Delta'(z) = z + 6/\pi i E_2(z)$.

Remarks

- The trivial example $h_t(z) = z$ does not fit in the above setting, however we can "associate" it to modular functions.
- The zeros and poles of the function $H_f(z) = (h_f(z) - z)^{-1}$ are all simple and have rational residues. Indeed, if n is the multiplicity of f at z_0 (a pole or a zero), then z_0 is a simple pole of H_f with residue n/k. At a cusp $\mathfrak{s}$ of Γ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \cdot \mathfrak{s} = \infty$, one can see that
$$\frac{1}{2i\pi} \lim_{z \to i\infty} H_f|| [\gamma^{-1}](z) = \frac{n}{kl_{\mathfrak{s}}} \in \mathbb{Q}\,,$$
where n is the order of infinity in the $q_{\mathfrak{s}}$-expansion of $f|_k[\gamma^{-1}](z)$ and $l_{\mathfrak{s}}$ is the cusp width at $\mathfrak{s}$ of Γ. This justifies the use of "rational".

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Rational Equivariant Forms: The converse

Question: When an equivariant form is a rational equivariant form?

The following proves that the conditions of the remark are in fact also sufficient.

Theorem

Let Γ be a modular subgroup and let h be an equivariant form for Γ. Then h is rational if and only if

1. The poles of H in $\mathfrak{H}$ are all simple with rational residues.
2. For each cusp $\mathfrak{s}$ of Γ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\gamma \cdot \mathfrak{s} = \infty$, we have

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To prove this one has to establish the following lemmas.

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Rational Equivariant Forms: The converse

Lemma

Let Γ be a modular subgroup, and let h be an equivariant form for Γ. Then H has only a finite number of poles in the closure of a fundamental domain of Γ.

Lemma

Suppose that h is equivariant for a modular subgroup Γ such that H has only simple poles in $\mathfrak{H}$, then the set of the residues at these poles is finite.

The theorem then follows by associating to h the function

$$f(z) = \exp\left(\int_{z_0}^{z} kH(u)\mathrm{d}\,u\right),$$

where $z_0 \in \mathfrak{H}$ not a zero of H and k is chosen to make disappear the denominators of the (finitely many) rational residues. The function f is in fact a modular form of wght k.

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A particular case

In the case of genus zero subgroups

Theorem

Let h be an equivariant form on a subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ of genus 0 such that $h(z) \neq z$ for all $z \in \mathfrak{H}$. Suppose also that, for every cusp s, if $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ is such that $\gamma \cdot s = \infty$, we have

$$\lim_{z \mapsto i\infty} H|||[\gamma^{-1}](z) = a_s$$

is finite and satisfies $(a_s/6) \in \pi i\mathbb{Z}$. Then $h(z) = h_{\Delta}(z) = z + \dfrac{6}{\pi i E_2(z)}$.

Are there other examples?

Yes!

Theorem

Let Γ be a modular subgroup and let f and g be modular forms of weights k and $k+2$ respectively, then

$$h(z) = z + k\,\frac{f(z)}{f'(z) + g(z)}$$

is an equivariant form for Γ.

A complete classification will appear in a joint work with Abdellah Sebbar, with a complete structure (an affine space) and geometric "correspondences".

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The Effect of the Schwarz Derivative and the Cohen-Rankin Bracket

Proposition

If h is an equivariant form for Γ, then $\{h, z\}$ is a weight 4 modular form on Γ.

In the case of rational equivariant forms, this is connected to the Cohen-Rankin bracket, which is defined for $n \geq 0$ with $D^j = \frac{d^j}{dz^j}$ by $[f, g]_n = \sum_{r+s=n} \binom{k+n-1}{s}\binom{l+n-1}{r} D^r f \; D^s g \;, \quad r, \; s \geq 0 \;.$
It is known that for $f \in \mathfrak{M}_k^m(\Gamma)$ and $g \in \mathfrak{M}_l^m(\Gamma)$ and for every $n \geq 0$, the function $[f, g]_n$ belongs to $\mathfrak{M}_{k+l+2n}^m(\Gamma)$.

Proposition

Let f be a modular form of weight k on Γ and h_f the corresponding rational equivariant form. Then $f'^2 h'_f$ is a weight $2k + 4$ modular form on Γ. Moreover, the poles of $\{h_f, z\}$ are located at the zeros of the second Cohen-Rankin bracket $[f, f]_2$ of f.

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Theorem 6

The Eisenstein series E_2 has infinitely many zeros in the half-strip $\mathfrak{S} = \{\tau \in \mathfrak{H},\ -\frac{1}{2} < \mathrm{Re}(\tau) \leq \frac{1}{2}\}$.

The proof is as follows. One first observes that, for each $z \in \mathfrak{H}$ such that $h_\triangle(z) \in \mathbb{Q}$, the $\mathrm{SL}_2(\mathbb{Z})$-equivalence class of z contains a zero of E_2. Fix such a point z_0. Then, in any neighborhood U of z_0, the function $h_\triangle$ takes infinitely many rational values, and so any neighborhood of z_0 contains infinitely many points that are equivalent to a zero of E_2. If U is sufficiently small then, each equivalence class of a point of U meets U in at most 3 points (about the vertices of a fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$). In fact, as z_0 cannot be an elliptic point, one can choose U such that U is in the interior of some fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$, and then the points of U are two by two inequivalent.

An Application: The zeros of E_2

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The Eisenstein series E_2 has infinitely many zeros in the half-strip $\mathfrak{S} = \{\tau \in \mathfrak{H},\ -\frac{1}{2} < \mathrm{Re}(\tau) \leq \frac{1}{2}\}$.

The proof is as follows. One first observes that, for each $z \in \mathfrak{H}$ such that $h_\Delta(z) \in \mathbb{Q}$, the $\mathrm{SL}_2(\mathbb{Z})$-equivalence class of z contains a zero of E_2. Fix such a point z_0. Then, in any neighborhood U of z_0, the function h_Δ takes infinitely many rational values, and so any neighborhood of z_0 contains infinitely many points that are equivalent to a zero of E_2. If U is sufficiently small then, each equivalence class of a point of U meets U in at most 3 points (about the vertices of a fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$). In fact, as z_0 cannot be an elliptic point, one can choose U such that U is in the interior of some fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$, and then the points of U are two by two inequivalent.

So, U contains infinitely many points in the equivalence class of distinct zeros of E_2.
Note that the strict monotonicity of E_2 on the imaginary axis provides us with the point z_0.
Finally, shift all the zeros to the strip $\mathfrak{S}$ and notice that E_2 is invariant under translation and has integer coefficients.

The Cross-ratio of Equivariant Forms

Proposition

Let $h_1,\ h_2,\ h_3,\ h_4$ be four equivariant forms on a modular subgroup Γ. Define a function f as the cross-ratio of these four elements

$$f = (h_1, h_2; h_3, h_4) = \frac{(h_1 - h_3)(h_2 - h_4)}{(h_2 - h_3)(h_1 - h_4)} \ .$$

Then, if $h_2 \neq h_3$ and $h_1 \neq h_4$, the function f is a modular function on Γ.

This actually gives a parametrization of equivariant forms by modular functions.

Proposition

We have

$$1728(h_t,\ h_{E_6};\ h_{E_4},\ h_{\Delta}) = j \ .$$

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Thank you for your attention!

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Special thanks to Ramin!