Algebraic curves of GL_2-type

Jerome William Hoffman
Atkin Memorial Conference

Louisiana State University

April 29-May 1, 2011

For any number field K let $G_K = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$. Let

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}(V_\ell) \cong \mathrm{GL}_{2d}(\mathbb{Q}_\ell)$$

be a representation. Mostly we will be concerned with $d = 2$.

Assume that

1. There is a number field $E/\mathbb{Q}$ of degree d and a homomorphism $E \to \mathrm{End}(V_\ell)$.
2. There is a finite extension $K/\mathbb{Q}$ such that $\rho(G_K)$ commutes with the image of E.

Then we have a factorization:

$$\rho_K := \rho \mid G_K \to \mathrm{GL}_{E\otimes\mathbb{Q}_\ell}(V_\ell) \cong \mathrm{GL}_2(E\otimes\mathbb{Q}_\ell) \subset \mathrm{GL}(V_\ell) \cong \mathrm{GL}_{2d}(\mathbb{Q}_\ell).$$

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Example: Abelian varieties of GL_2-type. For an abelian variety A/k let $V_\ell(A)$ be its Tate module: a 2 dim A-dimenional representation of G_k.

Theorem. (Ribet + Serre's conjecture)

1. Let $A/\mathbb{Q}$ be a $\mathbb{Q}$-simple abelian variety. Suppose that $E = \mathbb{Q} \otimes \mathrm{End}_{\mathbb{Q}}(A)$ is a number field of degree $= \dim A$. Then the Tate module $V_\ell(A)$ defines a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with values in $\mathrm{GL}_2(E \otimes \mathbb{Q}_\ell)$. Moreover A is isogenous to a $\mathbb{Q}$-simple factor of $J_1(N)$ for some $N \geq 1$.
2. Let $C/\overline{\mathbb{Q}}$ be a elliptic curve. Then C is a quotient of $J_1(N)_{\overline{\mathbb{Q}}}$ for some $N \geq 1$ if and only if C is a $\mathbb{Q}$-curve, i.e., C is isogenous to each of its conjugates ${}^\sigma C$, $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Problem: Explicitly construct $\mathbb{Q}$-curves.

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Theorem. (Shimura) Let $f = \sum a_n q^n$ be a normalized cuspidal Hecke eigenform of weight 2 on $\Gamma_1(N)$ for some $N \geq 1$. Then there is an abelian variety A_f defined over $\mathbb{Q}$ with an action of the field $E = \mathbb{Q}(\ldots, a_n, \ldots)$. This is a quotient of $J_1(N)$. We have $\dim A_f = [E : \mathbb{Q}]$, $E = \mathrm{End}_{\mathbb{Q}}(A_f) \otimes \mathbb{Q}$, and thus $V_\ell(A_f)$ is of GL_2-type over $\mathbb{Q}$.

Example. $N = 29$. There exist Hecke eigenforms $f, \overline{f} \in S_2(29, (*/29))$ with field of coefficients $E = \mathbb{Q}(\sqrt{-5})$. Shimura's A_f is two-dimensional, and in fact it is isogenous to $C \times {}^\sigma C$ for an elliptic $\mathbb{Q}$-curve $C/\mathbb{Q}(\sqrt{5})$. Also we have an isogeny $A_f \cong \mathrm{Jac}(X)$ where X is the genus 2 curve

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Note that for a given ρ, there can be more than one pair E, K such that ρ_K has an E-linear structure.

Examples are given by quaternion structures. Here $d = 2$. That is, assume:

1. there is a quaternion algebra $B/\mathbb{Q}$ and a homomorphism $B \to \mathrm{End}(V_\ell)$ and
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From now on, we take $d = 2$

Suppose ρ has an E-linear structure defined over K (so $[E : \mathbb{Q}] = 2$).

Let $\lambda, \overline{\lambda}$ be the places of E lying over the prime number ℓ (possibly $\lambda = \overline{\lambda}$). Then we get a decomposition $E \otimes \mathbb{Q}_\ell = \oplus_{\lambda,\overline{\lambda}} E_\lambda$ hence

$$\rho_K \otimes E = \rho_{K,\lambda} \oplus \overline{\rho}_{K,\lambda}, \quad L(\rho_K, s) = L(\rho_{K,\lambda}, s) L(\overline{\rho}_{K,\lambda}, s).$$

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Special case: Suppose that:

1. $K/\mathbb{Q}$ is also a quadratic extension.
2. The generator $s \in \mathrm{Gal}(K/\mathbb{Q})$ induces an isomorphism $s : \rho_{K,\lambda} \cong \overline{\rho}_{K,\lambda}$.

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$$\rho = \mathrm{Ind}_K^{\mathbb{Q}}(\rho_{K,\lambda})$$

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Let X be a projective nonsingular curve of genus 2 defined over $\mathbb{Q}$. We assume an equation $y^2 = f(x)$ where $f(x) \in \mathbb{Q}[x]$ has degree 5 or 6 with distinct roots.

Then

$$V_\ell = H^1(X \otimes \overline{\mathbb{Q}}, \mathbb{Q}_\ell) = H^1(Jac(X) \otimes \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$$

gives a four dimensional representation of $G_{\mathbb{Q}}$. These are of GL_2-type if the Jacobian $Jac(X)$ has extra endomorphisms. For any abelian variety A defined over a field k we let $\mathrm{End}^0_k(A) := \mathrm{End}_k(A) \otimes \mathbb{Q}$ be the semisimple $\mathbb{Q}$-algebra of endomorphisms defined over k; $\mathrm{End}^0(A) = \mathrm{End}^0_{\bar{k}}(A)$

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Then if A is a two dimensional absolutely simple abelian variety defined over a number field k then $\mathrm{End}^0(A)$ is one of the following:

1. $\mathbb{Q}$;
2. a real quadratic field $E/\mathbb{Q}$;
3. an indefinite quaternion algebra $B/\mathbb{Q}$;
4. a quartic CM field $K/\mathbb{Q}$.

Items 2 (RM) and 3 (QM) lead to representations of GL_2-type.

How to write down genus two curves over $\mathbb{Q}$ such that $Jac(X)$ has RM or QM?

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The familiar invariants of an elliptic curve, e.g., j, g_2, g_3 arise as in- and covariants of the action of $\mathrm{PGL}(2)$ on binary quartic forms $f(x, y) = a_0x^4 + a_1x^3y + a_2x^2y^2 + a_3xy^3 + a_4y^4$. These were worked out in the 19th century.

Reason: every genus 1 curve can be expressed as a double cover of $\mathbf{P}^1$ with four branch points. Then $f(x, y) = 0$ give the coordinates of the branch points. This leads to coordinate systems on the moduli spaces of elliptic curves: these are the modular curves.

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Let X be a projective nonsingular curve of genus 2 defined over a field k. Then X has a model in the form

$$y^2 = f(x) \quad \deg f = 6, \text{ with distinct roots.}$$

As for elliptic curves, the moduli for genus 2 curves then can be expressed via the in- and covariants of the action of $\mathrm{PGL}(2)$ on binary sextic forms. The expression of the moduli of genus 2 curves via projective invariants of binary sextics was done by Clebsch, and in more modern times by Igusa and Mestre.

Note that $\dim \mathcal{M}_2 = 3$, so there are three "J-invariants" j_1, j_2, j_3. Recall also that the map $X \mapsto \mathrm{Jac}(X)$: $\mathcal{M}_2 \to \mathcal{A}_2$ to the moduli space of principally polarized abelian varieties of dimension 2 is a birational correspondence.

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Analytic moduli.

Let

$$\mathfrak{H}_2 = \{\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) \mid \mathrm{Im}(\tau) > 0\}$$

the Siegel space of genus 2.

As an analytic space, $\mathcal{A}_2^{an} = \Gamma \backslash \mathfrak{H}_2$ where

$$\Gamma = \{g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z})\}$$

acting via $\tau \mapsto (A\tau + B)(C\tau + D)^{-1}$.

We get coverings by taking subgroups of finite index in Γ (congruence subgroups!)

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Algebraic moduli. A natural set of coordinates on the covering of $\mathcal{M}_2$ given by level 2 structure is gotten by taking the cross ratios of the roots e_i, $i = 1, ..., 6$ of

$$y^2 = f(x) = (x - e_1)...(x - e_6).$$

level 2 structure = an ordering of the 6 roots $e_1, ..., e_6$: $\mathrm{Sp}_4(\mathbb{Z}/2) = S_6$.

So we can describe subvarieties of $\mathcal{M}_2$ by equations $H(e_1, ..., e_6) = 0$, etc. Cross-ratios in the e_i can be expressed by certain Siegel modular functions (ratios of thetanullwerte.)

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$$y^2 = f(x) = (x - e_1)...(x - e_6).$$

level 2 structure = an ordering of the 6 roots $e_1, ..., e_6$:
$\mathrm{Sp}_4(\mathbb{Z}/2) = S_6$.
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Explicit equations for Humbert surfaces were written down for $\Delta = 5, 8$ by G. Humbert. Modern treatment given by P. Bending, Hashimoto, Hirzebruch, Murabayashii, R. M. Wilson, Sakai, Shephard-Barron, R. Taylor, van der Geer.

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The analytic equations of Humbert surfaces are quite simple: Each point of H_Δ is $\mathrm{Sp}_4(\mathbb{Z})$-equivalent to a point $\tau \in \mathfrak{H}_2$ which satisfies

$$a\tau_1 + b\tau_2 + \tau_3 = 0,\ \ a, b, \in \mathbb{Z},\ \ b^2 - 4a = \Delta,\ \ b = 0 \text{ or } 1.$$

We want equations in the algebraic moduli. Humbert's construction is based on Poncelet's theorem.

Given two projective plane conics C and D, if an n-gon can be inscribed in C in such a way that each edge of the polygon is tangent to D (i.e., the n-gon is circumscribed about D) then, given any point $P \in C$, there is an n-gon inscribed in C and circumscribed about D which passes through P.

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Proof (Cayley): Consider the dual projective plane:
$(\mathbf{P}^2)^* =$ the variety of lines in $\mathbf{P}^2$.
Let $D^* \subset (\mathbf{P}^2)^*$ be the variety of lines tangent to D. This is also a conic.
Define

$$E(C, D) := \{(P, \ell) \in C \times D^* \mid P \in \ell\},$$

the incidence correspondence.
Then $E(C, D)$ is an elliptic curve, and a Poncelet n-gon corresponds to a point of order n on $E(C, D)$.

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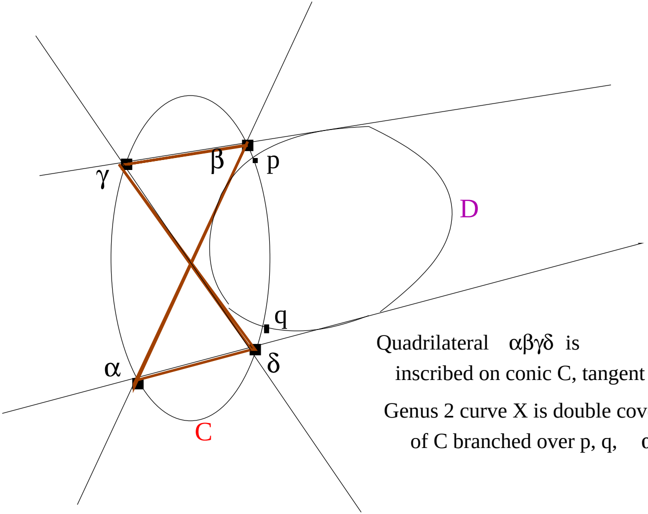


Quadrilateral αβγδ is
inscribed on conic C, tangent to conic D

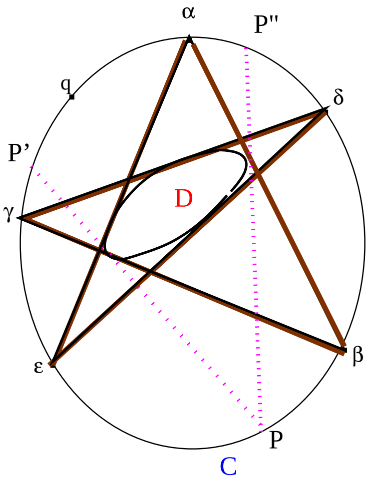
Genus 2 curve X is double cover
of C branched over p, q, α, β, γ, δ.

Humbert 8 = Poncelet 4

This configuration leads to the explicit equation for $H_8 = 0$ in terms of the roots e_i of the sextic $f(x)$ where the genus 2 curve is represented by $y^2 = f(x)$.

$$
\begin{aligned}
&H_8(e_1, \ldots, e_8) = \\
&(e_3 - e_1)(e_3 - e_2)(e_3 - e_4)(e_4 - e_2)^2(e_3 - e_5)(e_6 - e_1)(e_6 - e_2)(e_6 - e_4)(e_6 - e_5)(e_1 - e_5)^2 \\
&+ (e_1 - e_2)(e_1 - e_3)(e_1 - e_4)(e_2 - e_4)^2(e_3 - e_5)^2(e_6 - e_2)(e_6 - e_3)(e_6 - e_4)(e_6 - e_5)(e_1 - e_5) \\
&+ (e_3 - e_1)^2(e_4 - e_1)(e_4 - e_2)(e_4 - e_3)(e_2 - e_5)^2(e_4 - e_5)(e_6 - e_1)(e_6 - e_2)(e_6 - e_3)(e_6 - e_5) \\
&+ (e_2 - e_1)(e_1 - e_3)^2(e_2 - e_3)(e_2 - e_4)(e_2 - e_5)(e_4 - e_5)^2(e_6 - e_1)(e_6 - e_3)(e_6 - e_4)(e_6 - e_5) \\
&+ 16\,(e_1 - e_2)(e_2 - e_3)(e_1 - e_4)(e_3 - e_4)(e_2 - e_5)(e_3 - e_5) \\
&\qquad (e_4 - e_5)(e_1 - e_6)(e_2 - e_6)(e_3 - e_6)(e_4 - e_6)(e_1 - e_5)
\end{aligned}
$$

$\Delta = 5$: $-\tau_1 + \tau_2 + \tau_3 = 0$. This corresponds to a Poncelet 5-gon.



Humbert 5 = Poncelet 5

Pentagon αβγδε
inscribes conic C
circumscribes conic D

Genus 2 curve X is the
double cover of C branched
above α, β, γ, δ, ε and
a point q in C intersect D.

The correspomdence

P –> P'+P''

lifts to a correspondence
ϕ of X with $\phi^2 + \phi - 1 = 0$
in Jac(X).

This configuration leads to the explicit equation for $H_5 = 0$ in terms of the roots e_i of the sextic $f(x)$ where the genus 2 curve is represented by $y^2 = f(x)$.

$$H_5(e_1, \ldots, e_6) =$$
$$(e_2 - e_3)^2(e_5 - e_1)(e_5 - e_2)(e_5 - e_3)(e_5 - e_4)(e_1 - e_6)(e_2 - e_6)(e_3 - e_6)(e_4 - e_6)(e_1 - e_4)^2$$
$$+ (e_1 - e_2)(e_1 - e_3)(e_3 - e_4)^2(e_1 - e_5)(e_2 - e_5)^2(e_2 - e_6)(e_3 - e_6)(e_4 - e_6)(e_5 - e_6)(e_1 - e_4)$$
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Let $B/\mathbb{Q}$ be an indefinite quaternion algebra. Two-dimensional principally polarized abelian varieties A with $B \subset \mathrm{End}(A)$ are parametrized by the points of the Shimura curve S_B associated to B. There is a universal family $X \to S_B$ of genus two curves which have Jacobians that have QM by B.

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If a simple abelian surface A has $\mathrm{End}^0(A) \supset \mathbb{Q}(\Delta_1), \mathbb{Q}(\Delta_2)$ for two different real quadratic fields, then $\mathrm{End}^0(A)$ is a quaternion algebra. Thus

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Example 1 (Brumer/Hashimoto) Let

$$f(X;a,b,c) = X^6 - (4+2b+3c)X^5 + (2+2b+b^2-ac)X^4 - (6+4a+6b-2b^2+5c+2ac)X^3 + (1+b^2-ac)X^2 + (2-2b)X + (c+1)$$

Let $C(a,b,c) : Y^2 = f(X;a,b,c)$, assume $f(X,a,b,c)$ has 6 distict roots. Then $C(a,b,c)$ is a genus two curve with RM by $\mathbb{Q}(\sqrt{5})$. These endomorphisms are defined over $\mathbb{Q}(a,b,c)$. Hence if $a,b,c,\in\mathbb{Q}$, the curve is modular.

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Example 2 (P. Bending) Let $K \subset \mathbb{C}$ be a field. Let $A, P \neq 0, Q, D \neq 0$ in K. Define

$$B = \frac{Q(PA - Q) + 4P^2 + 1}{P^2}, \quad C = \frac{4(PA - Q)}{P}.$$

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Example 4 (QM by B_{10}) Let

$$C : y^2 = x^6 - 16x^5 + 40x^4 + 140x^3 + 80x^2 - 64x + 64.$$

Then $\mathrm{Jac}(C)$ has multiplications by the quaternion division ring of discriminant 10. We have

$$\det(X - \rho(\mathrm{Frob}_p)) = g_p(X)\overline{g_p(X)}$$

for $g_p(X) \in \mathbb{Q}(\sqrt{5})[X]$.

p	$g_p(X)$
3	$X^2 + \sqrt{5}X + 3$
11	$X^2 + 3X + 11$
13	$X^2 + 3\sqrt{5}X + 13$
17	$X^2 + \sqrt{5}X + 17$
19	$X^2 + 19$
23	$X^2 + 23$
29	$X^2 + 9X + 29$
31	$X^2 + 31$
37	$X^2 + 37$
41	$X^2 + 41$
43	$X^2 + 43$
47	$X^2 + 5\sqrt{5}X + 47$

Example 5 (RM by $\mathbb{Q}(\sqrt{2})$) Let

$$C : y^2 = 21x^5 + 50x^4 + 5x^3 - 20x^2 + 4x.$$

Then $\mathrm{Jac}(C)$ has multiplications by $\mathbb{Q}(\sqrt{2})$, but the endomorphisms are defined over $\mathbb{Q}(\sqrt{2})$. The ℓ-adic representation here is not obviously automorphic. The conductor is $2^?3^27^2$. This is special case of a 1-parameter family of Hashimoto/Sakai.

p	$\det(X - \rho(\mathrm{Frob}_p))$
5	$(X^2 - 5)^2$
11	$(X^2 + 4\sqrt{2}X + 11)(X^2 - 4\sqrt{2}X + 11)$
13	$X^4 + 6X^2 + 13^2$
17	$(X^2 + (4\sqrt{2} - 2)X + 17)(X^2 + (-4\sqrt{2} - 2)X + 17)$
19	$X^4 - 10X^2 + 19^2$
23	$(X^2 + 4\sqrt{2}X + 23)(X^2 - 4\sqrt{2}X + 23)$
29	$(X^2 + 6\sqrt{2}X + 29)(X^2 - 6\sqrt{2}X + 29)$
31	$(X^2 + 31)^2$
37	$(X^2 + 10X + 37)(X^2 - 10X + 37)$
41	$(X^2 + (4\sqrt{2} - 2)X + 41)(X^2 + (-4\sqrt{2} - 2)X + 41)$
43	$X^4 + 3X^2 + 43^2$
47	$(X^2 - 8X + 47)^2$

Example 6 (RM by $\mathbb{Q}(\sqrt{2})$) Let

$$C : y^2 = 9x^5 - 210x^4 + 165x^3 + 16740x^2 - 74844x$$

Then $\mathrm{Jac}(C)$ has multiplications by $\mathbb{Q}(\sqrt{2})$, but the endomorphisms are defined over $\mathbb{Q}(\sqrt{-2})$. The ℓ-adic representation here is not obviously automorphic. The conductor is $2^?3^?7^?11^?$. This is special case of a 1-parameter family of Hashimoto/Sakai.

p	$\det(X - \rho(\mathrm{Frob}_p))$
5	$(X^2 - 5)^2$
13	$X^4 - 6X^2 + 13^2$
17	$(X^2 + (4\sqrt{2} - 2)X + 17)(X^2 + (-4\sqrt{2} - 2)X + 17)$
19	$(X^2 + 4\sqrt{2}X + 19)(X^2 - 4\sqrt{2}X + 19)$
23	$(X^2 + 4\sqrt{2}X + 23)(X^2 - 4\sqrt{2}X + 23)$
29	$SX^4 - 10X^2 + 29^2$
31	$(X^2 + 4\sqrt{2}X + 31)(X^2 - 4\sqrt{2}X + 31)$
37	$(X^2 + 6X + 37)(X^2 - 6X + 37)$
41	$(X^2 + (4\sqrt{2} + 6)X + 41)(X^2 + (-4\sqrt{2} + 6)X + 41)$
43	$(X^2 + 8X + 43)^2$
47	$(X^2 + 8X + 47)(X^2 - 8X + 47)$

Thanks to Ramin Takloo-Bighash and Winnie Li!