Weakly Holomorphic Vector Valued Modular Forms

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Presented at:
Atkin Memorial Lecture and Workshop
On
Noncongruence modular forms and Galois representations
University of Illinois at Chicago
April 29 - May 1, 2011

May 1, 2011

Notation

- $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ - extended upper Half plane.
- $\overline{\Gamma(1)} = PSL_2(\mathbb{Z}) = \langle t, s \rangle$, where
$t = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, s = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

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- $\overline{\Gamma(1)} = PSL_2(\mathbb{Z}) = \langle t, s \rangle$, where
$$t = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, s = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
- Γ be any genus-0 finite index subgroup.
- $\mathfrak{C}_\Gamma$ is the set of all inequivalent cusps of Γ.

Multiplier System

Let $\rho : \overline{\Gamma(1)} \longrightarrow GL_d(\mathbb{C})$ be rank d representation of $\overline{\Gamma(1)}$. We say that ρ is an *admissible multiplier* of $\overline{\Gamma(1)}$ if $\rho(t)$ is a diagonal matrix, i.e. for some diagonal matrix $\Lambda \in M_d(\mathbb{C})$, $\rho(t) = e^{2\pi i \Lambda}$.

Remark

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- Here $t_{\mathfrak{c}}$ denote the generator of the stabilizer subgroup of cusp $\mathfrak{c}$.

Weakly Holomorphic Vector Valued Modular Form

Let ρ be an admissible multiplier for $\overline{\Gamma(1)}$ of rank d. A map $\mathbb{X} : \mathbb{H} \longrightarrow \mathbb{C}^d$ is said to be *weakly holomorphic vector valued modular form for* $\overline{\Gamma(1)}$ *of weight* w *and multiplier* ρ, if $\mathbb{X}$ is holomorphic throughout $\mathbb{H}$ and may have poles only at the cusps with following functional and cuspidal behaviour:

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Functional behaviour

$$\mathbb{X}(\gamma\tau) = \rho(\gamma)(c\tau + d)^w \mathbb{X}(\tau), \qquad \forall \gamma \in \Gamma \,\&\, \forall \tau \in \mathbb{H}.$$

Weakly Holomorphic Vector Valued Modular Form

Cuspidal behaviour

Since $\overline{\Gamma(1)}$ has only one cusp ∞, $q^{-\Lambda}\mathbb{X}(\tau)$ has periodicity 1 therefore it has Fourier expansion of the following form,

$$q^{-\Lambda}\mathbb{X}(\tau) = \sum_{n\in\mathbb{Z}} a_n q^n, \qquad \textit{where} \qquad q = e^{2\pi i\tau}$$

which has at most finitely many nonzero $a_n \in \mathbb{C}^d$ with $n < 0$.

- For any weight $w \in 2\mathbb{Z}$ and multiplier ρ of any Γ, $\mathcal{M}_w(\Gamma, \rho, d)$ denotes the $\mathbb{C}$-Vector Space of all Weakly Holomorphic Vector Valued Modular Forms.

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- More generally, weakly holomorphic vector valued modular forms for Γ will be holomorphic on $\mathbb{H}$ and may have poles at every cusp.
- So if Γ has ℓ inequivalent cusps then it has Fourier series expansion at every cusp.

Where we are heading to...

- $\mathcal{M}_w(\Gamma, \rho, d)$ is a free module of rank d over the ring of weakly holomorphic modular functions for Γ.

- The ring of weakly holomorphic modular functions for $\overline{\Gamma(1)}$ is $\mathbb{C}[J_{\overline{\Gamma(1)}}]$.

- $J_{\overline{\Gamma(1)}} = J = q^{-1} + 196884q + \cdots$ is the normalised hauptmodul for $\overline{\Gamma(1)}$.

Normalised Hauptmodul

If Γ is genus-0 finite index subgroup of $\overline{\Gamma(1)}$ and k be the cusp width of the cusp $\{\infty\}$ then the normalised hauptmodul of Γ is

$$J_\Gamma = q_k^{-1} + a_1 q_k^1 + a_2 q_k^2 + a_3 q_k^3 + \cdots$$

$a_i \in \mathbb{Q}, \forall i \geq 1$, and $q_k = e^{2\pi i\tau/k}$.

Normalised Hauptmodul

For any cusp $\mathfrak{c} \in \mathfrak{C}_\Gamma$ of cusp width $k_\mathfrak{c}$, we define the normalised hauptmodul at $\mathfrak{c}$ to be the modular function $J_\Gamma^\mathfrak{c}$, holomorphic everywhere on $\mathbb{H}^*$ except at the cusp $\mathfrak{c}$ where it has local $q-$series expansion of the form

$$J_\Gamma^\mathfrak{c} = \tilde{q}_{k_\mathfrak{c}}^{-1} + a_1^\mathfrak{c} \tilde{q}_{k_\mathfrak{c}}^1 + a_2^\mathfrak{c} \tilde{q}_{k_\mathfrak{c}}^2 + a_3^\mathfrak{c} \tilde{q}_{k_\mathfrak{c}}^3 + \cdots$$

where $\tilde{q}_{k_\mathfrak{c}} = e^{2\pi i A^{-1}(\tilde{\tau})/k_\mathfrak{c}}$ and $A \in \overline{\Gamma(1)}$ such that $A(\infty) = \mathfrak{c}$ and $A\tau = \tilde{\tau}$.

Normalised Hauptmodul For $\Gamma = \overline{\Gamma(2)}$

- Γ is generated by $\pm t^2$ and $\pm st^2 s$.
- Γ is *genus* $-$ 0 congruence subgroup of the modular group of index 6.
- Γ has three cusps, namely $\infty, 0\&1$.
- $\Gamma_\infty = \langle t_\infty = t^2 \rangle$; $\Gamma_0 = \langle t_0 = st^2 s \rangle$; $\Gamma_1 = \langle t_1 = (ts)t^2(st^{-1}) \rangle$.

Normalised Hauptmodul For $\overline{\Gamma(2)}$

- $J_\Gamma^{(\infty)} = q_2^{-1} + a_1 q_2 + a_2 q_2^2 + \cdots$, where $q_2 = e^{\frac{2\pi i \tau}{2}}$.
- $J_\Gamma^{(0)} = \tilde{q}_2^{\,-1} + b_1 \tilde{q}_2 + b_2 \tilde{q}_2^{\,2} + \cdots$ where $\tilde{q}_2 = e^{\frac{2\pi i (s\tau)}{2}}$.
- $J_\Gamma^{(1)} = \bar{q}_2^{\,-1} + c_1 \bar{q}_2 + c_2 \bar{q}_2^{\,2} + \cdots$ where $\bar{q}_2 = e^{\frac{2\pi i (st^{-1}\tau)}{2}}$.

Main Result

Let's denote the ring of weakly holomorphic modular functions for Γ by $R(\Gamma)$, where
$R(\Gamma) \leftrightarrow \mathbb{C}[J_\Gamma^{\mathfrak{c}_1}, J_\Gamma^{\mathfrak{c}_2}, \cdots, J_\Gamma^{\mathfrak{c}_l}]$.

Theorem (J.B.)

$\mathcal{M}_w(\Gamma, \rho, d)$ is a free $R(\Gamma)$-module of rank d.

$\overline{\Gamma(1)}$&$\overline{\Gamma(2)}$

- $\mathcal{M}_w(\overline{\Gamma(1)}, \rho, d)$ is a free $R(\overline{\Gamma(1)}) = \mathbb{C}[J]$-module of rank d.
- $\mathcal{M}_w(\overline{\Gamma(2)}, \rho, d)$ is a free $R(\overline{\Gamma(2)})$-module of rank d.
- In general $\mathcal{M}_w(\Gamma, \rho, d)$ is a free $R(\Gamma)$-module of rank d.

Dedekind η function

Recall that

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

$$\Delta = \eta^{24}$$

- Δ is a cusp form for $\overline{\Gamma(1)}$ of weight 12.
- η is a modular form of weight $1/2$ for $\overline{\Gamma(1)}$.

Dedekind η function

- For any w, η^{2w} is a modular form of weight w with some multiplier ν for $\overline{\Gamma(1)}$.
- η^w is holomorphic and nonvanishing on $\mathbb{H}$.

Two Simple but Important Observations...

First Observation:

$\mathcal{M}_w(\Gamma, \rho, d)$ and $\mathcal{M}_0(\Gamma, \rho \otimes \nu^{-2w}, d)$ are isomorphic as $R(\Gamma)$-modules.

- Where ν is a multiplier system of Dedekind η.
- $\mathbb{X} \mapsto \eta^{-2w}\mathbb{X}$.

Two Simple but Important Observations...

Second Observation:

If $[\Gamma(1):\Gamma] = m$ then there is a $\mathbb{C}[J]$-module isomorphism between $\mathcal{M}_w(\Gamma, \rho, d)$ and $\mathcal{M}_w(\Gamma(1), \tilde{\rho}, dm)$, where $\tilde{\rho} = Ind_{\Gamma}^{\Gamma(1)}\rho$. Where J is the normalised hauptmodul of $\overline{\Gamma(1)}$, i.e. $J(\tau) = q^{-1} + 196884q + \cdots$.

Two Relations between Scalar-Valued and Vector-Valued Modular Forms

- Restriction to $\mathrm{Ker}\rho$: vvmf for $\Gamma(1)$ will give svmf for $\mathrm{Ker}\rho$(as long as $\mathrm{Ker}\rho$ is a finite index subgroup of $\Gamma(1)$).
- Induction from Γ of finite index m in $\Gamma(1)$: Let ρ_Γ be the trivial representation of Γ then by inducing this to a representation of $\Gamma(1)$, any svmf of $\mathcal{M}_w(\Gamma, \rho_\Gamma, 1)$ will give a vvmf of $\mathcal{M}_w(\Gamma(1), \rho, m)$.

Relevance of Restriction Idea to noncongruence modular forms

- In dimension 2, ρ has finite image iff $\mathrm{Ker}\rho$ is congruence.
- In dimension $\geq$ 3, there are infinitely many inequivalent irreducble ρ where ρ has finite image and $\mathrm{Ker}\rho$ is finite index noncongruence subgroup of $\Gamma(1)$.
- So for example if we "understand" vvmf for 3$-$dimensional ρ, then we "understand" svmf for infinitely many different noncongruence subgroups.

Relevance of induction idea to noncongruence modular forms

Suppose f is a modular form for finite index noncongruence subgroup Γ of $\overline{\Gamma(1)}$ and for trivial multiplier ρ then f will induce to a vector valued modular form for $\overline{\Gamma(1)}$.
This theory will be able to address the following type of questions....

- Growth of the coefficients of f.
- Verifying whether f have unbounded denominator (Atkin-Swinnerton-Dyer Conjecture).

Principal Part Map or Mittag-Leffler Map

First we define Principal Part Map or Mittag-Leffler Map for group $\overline{\Gamma(1)}$

$$P_\lambda : \mathcal{M}_0(\overline{\Gamma(1)}, \rho, d) \longrightarrow q^{-1}\mathbb{C}[q^{-1}]^d$$

is defined as

$$P_\lambda(\mathbb{X}) = \sum_{n<0} a_n q^n, \qquad a_n \in \mathbb{C}^d$$

in the Fourier expansion of $q^{-\lambda}\mathbb{X}$.

Examples of Principal Part Map

Consider $d = 1$, trivial multiplier ρ and $w = 0$ then $\mathcal{M}_0(\overline{\Gamma(1)}, \rho, d) = \mathbb{C}[J]$

Consider exponent $\lambda = (0)$.
$P_{(0)} : \mathbb{C}[J] \longrightarrow q^{-1}\mathbb{C}[q^{-1}]$ then

$$P_{(0)}(J) = q^{-1}$$

$$\mathrm{Ker} P_{(0)} = \mathbb{C}$$

So $P_{(0)}$ for exponent $\lambda = (0)$ is not injective.

Examples of Principal Part Map

Consider exponent $\lambda = (1)$.
$P_{(1)} : \mathbb{C}[J] \longrightarrow q^{-1}\mathbb{C}[q^{-1}]$ then

$$P_{(1)}(J) = q^{-2}$$

$$\mathrm{Ker} P_{(1)} = \{0\}$$

$P_{(1)}$ for exponent $\lambda = (1)$ is injective.

Key Lemma - An Important Bound

Lemma (Gannon)

There exists a constant $C = C(\rho, w)$ *such that for every* $\mathbb{X} \in \mathcal{M}_w(\Gamma, \rho, d)$,

$$min_\xi l.p.(\mathbb{X}_\xi) \leq C.$$

Here $1 \leq \xi \leq d$

where $\mathbb{X}(\tau) = \begin{pmatrix} \mathbb{X}_1(\tau) \\ \mathbb{X}_2(\tau) \\ \vdots \\ \mathbb{X}_d(\tau) \end{pmatrix}$

Key Theorem

Theorem

Let $\rho : \Gamma \longrightarrow GL_d(\mathbb{C})$ *be an admissible multiplier of* Γ *then there exists* d *linearly independent vector valued modular forms* $\{\mathbb{Y}^1, \mathbb{Y}^2, \cdots, \mathbb{Y}^d\} \in \mathcal{M}_w(\Gamma, \rho, d)$ *and an exponent* λ *such that* $P_\lambda(\mathbb{Y}^\xi) = q^{-1} e_\xi$.

Remark

This result is a consequence of Rohrl's solution to the Riemann-Hilbert problem.

Γ

- $\mathcal{E}(\mathcal{M}_w)$ is the set of bijective exponent of $\mathcal{M}_w(\Gamma, \rho, d)$.
- i.e. those exponent λ for which P_λ is a vector space isomorphism over $\mathbb{C}$.
- For any weight w and multiplier ρ, $\mathcal{E}(\mathcal{M}_w)$ is nonempty.

$\overline{\Gamma(1)}$

- For example $\mathcal{E}(\mathcal{M}_0(\overline{\Gamma(1)}, 1, 1)) = \{1\}$.
- In general, $\exists$ an exponent λ for any ρ and any w such that $P_\lambda : \mathcal{M}_w(\overline{\Gamma(1)}, \rho, d)) \longrightarrow q^{-1}\mathbb{C}[q^{-1}]^d$ is an isomorphism, i.e.

$$\mathcal{E}(\mathcal{M}_w(\overline{\Gamma(1)}, \rho, d)) \neq \emptyset$$

$\overline{\Gamma(1)}$

- Existence of injective exponent Λ by using the key lemma and form set of all injective exponents.
- Existence of surjective exponent λ by using the key theorem and form set of all surjective exponents.
- Observe the overlap in these two sets.

$\overline{\Gamma(1)}$

- For any exponent λ, $\mathrm{Ker} P_\lambda$ is finite dimensional subspace.
- For any exponent λ, $\mathrm{Coker} P_\lambda$ is finite dimensional subspace.

$\overline{\Gamma(1)}$

Theorem

For any exponent λ, there exists an integer $K_{\mathcal{M}_w}$ such that the principal part map $P_\lambda : \mathcal{M}_w \longrightarrow q^{-1}\mathbb{C}[q^{-1}]^d$ has index

$$dim\mathrm{Ker}P_\lambda - dim\mathrm{Coker}P_\lambda = -Tr\lambda + K_{\mathcal{M}_w}.$$

For any bijective exponent λ,
$K_{\mathcal{M}_w} = Tr\lambda = \frac{(5+w)d}{12} + \frac{e^{\frac{\pi iw}{2}}}{4}TrS + \frac{2}{3\sqrt{3}}Re(e^{\frac{-\pi i}{6}+\frac{-2\pi iw}{3}}TrU)$, where
$S = \rho(s), U = \rho(st^{-1})$.

Thank you all for listening

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Big Thanks to
Ramin