The Arithmetic of Noncongruence Modular Forms

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Modular forms

- A modular form is a holomorphic function on the Poincaré upper half-plane $\mathfrak{H}$ with a lot of symmetries w.r.t. a finite-index subgroup Γ of $SL_2(\mathbb{Z})$.
- Γ is called a *congruence* subgroup if it contains the group

$$\Gamma(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ : \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N\}$$

  for some positive integer N.

  Forms for such Γ are called congruence modular forms.
- Otherwise Γ is called a *noncongruence* subgroup, and forms are called noncongruence modular forms.
- Congruence forms well-studied; noncongruence forms much less understood.

Modular curves

- The group Γ acts on $\mathfrak{H}$ by fractional linear transformations. We compactify the orbit space $\Gamma\backslash\mathfrak{H}$ by joining finitely many cusps to get a Riemann surface, called the modular curve X_Γ for Γ. It has a model defined over a number field.
- The modular curves for congruence subgroups are defined over $\mathbb{Q}$ or cyclotomic fields $\mathbb{Q}(\zeta_N)$.
- Belyi: Every smooth projective irreducible curve defined over a number field is isomorphic to a modular curve X_Γ (for infinitely many finite-index subgroups Γ of $SL_2(\mathbb{Z})$).
- $SL_2(\mathbb{Z})$ has far more noncongruence subgroups than congruence subgroups.

Modular forms for congruence subgroups

Let $g = \sum_{n\geq 1} a_n(g)q^n$, where $q = e^{2\pi i z}$, be a normalized ($a_1(g) = 1$) newform of weight $k \geq 2$ level N and character χ.

I. Hecke Theory

- The Fourier coefficients are multiplicative, i.e.,

$$a_{mn}(g) = a_m(g)a_n(g)$$

whenever m and n are coprime.

- It is an eigenfunction of the Hecke operators T_p with eigenvalue $a_p(g)$ for all primes $p \nmid N$, i.e., for all $n \geq 1$,

$$a_{np}(g) - a_p(g)a_n(g) + \chi(p)p^{k-1}a_{n/p}(g) = 0.$$

For primes $p|N$ and all $n \geq 1$,

$$a_{np}(g) = a_n(g)a_p(g).$$

- The Fourier coefficients of a newform are algebraic integers. In the space of weight k cusp forms for a congruence subgroup, there is a basis of forms with integral Fourier coefficients. An algebraic cusp form has bounded denominators.

II. Galois representations

- (Eichler-Shimura, Deligne) There exists a compatible family of l-adic deg. 2 rep'ns $\rho_{g,l}$ of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ such that at primes $p \nmid lN$, the char. poly.

$$H_p(T) = T^2 - A_p T + B_p = T^2 - a_p(g)T + \chi(p)p^{k-1}$$

of $\rho_{g,l}(\mathrm{Frob}_p)$ is indep. of l, and

$$a_{np}(g) - A_p\, a_n(g) + B_p\, a_{n/p}(g) = 0$$

for $n \geq 1$ and primes $p \nmid N$.

- Ramanujan-Petersson conjecture holds for newforms. That is, $|a_p(g)| \leq 2p^{(k-1)/2}$ for all primes $p \nmid N$.

Langlands philosophy

- Artin and Takagi in 1920's developed the class field theory, which describes abelian extensions of local and global fields.
- Langlands' general reciprocity law (1960's), known as Langlands philosophy: There is a correspondence from

  $$\{\text{equiv. classes of nice rep'ns of a reductive group over } F\}$$
  $$\downarrow$$
  $$\{\text{ equiv. classes of nice rep'ns of the Galois group over } F\}$$

  such that the corresponding rep'ns have the same invariants. Here F is a local or global field.

  The class field theory : reductive group is $GL(1)$, and rep'ns are 1-dim'l.

When reductive group is $GL(2)$ over $\mathbb{Q}$, the automorphic rep'ns are on the classical modular forms or Maass forms.

- The representations of Galois groups are over $\mathbb{C}$ or ℓ-adic fields. When a global Galois representation has a corresponding automorphic representation, it is called "modular".
- Most of the progress concerning Langlands philosophy has been from reductive group side to Galois side, eg. Eichler-Shimura, Deligne, Taylor, Henniart, Harris-Taylor, Drinfeld, Lafforgue, Jiang-Soudry, Gan-Takeda.

- Wiles' work on Taniyama-Shimura modularity conjecture is the first major progress from Galois side to automorphic side. Since then, there has been tremendous progress, including:
  - full proof of Taniyama-Shimura modularity conjecture (Breuil, Conrad, Diamond, Taylor)
  - various modularity lifting criteria (Skinner-Wiles, Diamond-Flach-Guo),
  - proof of the Serre's conjecture over $\mathbb{Q}$ (Khare-Wintenberger, Kisin),
  - many cases of degree 2 Fontaine-Mazur conjecture (Kisin),
  - Artin's conjecture on degree 2 irreducible complex rep'ns of Galois group over $\mathbb{Q}$ (Taylor),
  - Sato-Tate conjecture (Clozel, Harris, Sheppard-Barron, Taylor) for elliptic curves over totally real fields.

Modular forms for noncongruence subgroups

Γ : a noncongruence subgroup of $SL_2(\mathbb{Z})$ with finite index

$S_k(\Gamma)$: space of cusp forms of weight $k \geq 2$ for Γ of dim d

A cusp form has an expansion in powers of $q^{1/\mu}$.

Assume the modular curve X_Γ is defined over $\mathbb{Q}$ and the cusp at infinity is $\mathbb{Q}$-rational.

Atkin and Swinnerton-Dyer: there exists a positive integer M such that $S_k(\Gamma)$ has a basis consisting of forms with coeffs. in $\mathbb{Z}[\frac{1}{M}]$ (called M-integral) :

$$f(z) = \sum_{n \geq 1} a_n(f) q^{n/\mu}.$$

No efficient Hecke operators on noncongruence forms

- Let Γ^c be the smallest congruence subgroup containing Γ. Naturally, $S_k(\Gamma^c) \subset S_k(\Gamma)$.
- $Tr_{\Gamma}^{\Gamma^c} : S_k(\Gamma) \to S_k(\Gamma^c)$ such that $S_k(\Gamma) = S_k(\Gamma^c) \oplus ker(Tr_{\Gamma}^{\Gamma^c})$.
- $ker(Tr_{\Gamma}^{\Gamma^c})$ consists of genuinely noncongruence forms in $S_k(\Gamma)$.

Conjecture (Atkin). The Hecke operators on $S_k(\Gamma)$ for $p \nmid M$ defined using double cosets as for congruence forms is zero on genuinely noncongruence forms in $S_k(\Gamma)$.

This was proved by Serre, Berger.

Atkin-Swinnerton-Dyer congruences

Let E be an elliptic curve defined over $\mathbb{Q}$ with conductor M. By Belyi, $E \simeq X_\Gamma$ for a finite index subgroup Γ of $SL_2(\mathbb{Z})$. Eg. $E : x^3 + y^3 = z^3$, Γ is an index-9 noncongruence subgp of $\Gamma(2)$.

Atkin and Swinnerton-Dyer: The normalized holomorphic differential 1-form $f\frac{dq}{q} = \sum_{n\geq 1} a_n q^n \frac{dq}{q}$ on E satisfies the congruence relation

$$a_{np} - [p + 1 - \#E(\mathbb{F}_p)]a_n + p a_{n/p} \equiv 0 \mod p^{1+\mathrm{ord}_p n}$$

for all primes $p \nmid M$ and all $n \geq 1$.

Note that $f \in S_2(\Gamma)$.

Taniyama-Shimura modularity theorem: There is a normalized congruence newform $g = \sum_{n\geq 1} b_n q^n$ with $b_p = p + 1 - \#E(\mathbb{F}_p)$. This gives congruence relations between f and g.

Back to general case where X_Γ has a model over $\mathbb{Q}$, and the d-dim'l space $S_k(\Gamma)$ has a basis of M-integral forms.

ASD congruences: for each prime $p \nmid M$, $S_k(\Gamma, \mathbb{Z}_p)$ has a p-adic basis $\{h_j\}_{1 \le j \le d}$ such that the Fourier coefficients of h_j satisfy a three-term congruence relation

$$a_{np}(h_j) - A_p(j)a_n(h_j) + B_p(j)a_{n/p}(h_j) \equiv 0 \mod p^{(k-1)(1+\mathrm{ord}_p n)}$$

for all $n \ge 1$. Here

- $A_p(j)$ is an algebraic integer with $|A_p(j)| \le 2p^{(k-1)/2}$, and
- $B_p(j)$ is equal to p^{k-1} times a root of unity.

This is proved to hold for $k = 2$ and $d = 1$ by ASD.

Remarks. (1) The basis varies with p.

(2) The three-term congruence relations for noncongruence forms capture the spirit of the Hecke operators in essence.

(3) From where do $A_p(j)$ and $B_p(j)$ arise? Any modularity interpretations?

Belyi's theorem tells us that, viewed simply as algebraic curves, noncongruence modular curves are very general, hence we should not expect them to have any special arithmetic properties. On the other hand, the uniformization of noncongruence curves by the upper half-plane is quite special, and leads to surprising consequences.

Galois representations attached to $S_k(\Gamma)$ and congruences

Theorem[Scholl] *Attached to $S_k(\Gamma)$ is a compatible family of $2d$-dim'l l-adic rep'ns ρ_l of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ unramified outside lM such that for primes $p > k+1$ not dividing Ml, the following hold.*

(i) The char. polynomial

$$H_p(T) = \sum_{0 \leq r \leq 2d} B_r(p) T^{2d-r}$$

of $\rho_l(\mathrm{Frob}_p)$ lies in $\mathbb{Z}[T]$, is indep. of l, and its roots are algebraic integers with complex absolute value $p^{(k-1)/2}$;

(ii) For any form f in $S_k(\Gamma)$ integral outside M, its Fourier coeffs satisfy the $(2d+1)$-term congruence relation

$$\begin{aligned} & a_{np^d}(f) + B_1(p)a_{np^{d-1}}(f) + \cdots + \\ & + B_{2d-1}(p)a_{n/p^{d-1}}(f) + B_{2d}(p)a_{n/p^d}(f) \\ & \equiv 0 \mod p^{(k-1)(1+\mathrm{ord}_p n)} \end{aligned}$$

for $n \geq 1$.

The Scholl rep'ns ρ_l are generalizations of Deligne's construction to the noncongruence case. The congruence in (ii) follows from comparing l-adic theory to an analogous p-adic de Rham/crystalline theory; the action of $Frob_p$ on both sides have the same characteristic polynomials.

Scholl's theorem establishes the ASD congruences if $d = 1$.

In general, to go from Scholl congruences to ASD congruences, ideally one hopes to factorize

$$H_p(T) = \prod_{1 \le j \le d} (T^2 - A_p(j)T + B_p(j))$$

and find a p-adic basis $\{h_j\}_{1 \le j \le d}$, depending on p, for $S_k(\Gamma, \mathbb{Z}_p)$ such that each h_j satisfies the three-term ASD congruence relations given by $A_p(j)$ and $B_p(j)$.

For a congruence subgroup Γ, this is achieved by using Hecke operators to further break the l-adic and p-adic spaces into pieces. For a noncongruence Γ, no such tools are available.

Scholl representations, being motivic, should correspond to automorphic forms for reductive groups according to Langlands philosophy. They are the link between the noncongruence and congruence worlds.

Modularity of Scholl representations when $d = 1$

Scholl: the rep'n attached to $S_4(\Gamma_{7,1,1})$ is modular, coming from a newform of wt 4 for $\Gamma_0(14)$; ditto for $S_4(\Gamma_{4,3})$ and $S_4(\Gamma_{5,2})$.

Li-Long-Yang: True for wt 3 noncongruence forms assoc. with K3 surfaces defined over $\mathbb{Q}$.

In 2006 Kahre-Wintenberger established Serre's conjecture on modular representations. This leads to

Theorem *If* $S_k(\Gamma)$ *is* 1*-dimensional, then the degree two l-adic Scholl representations of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *are modular.*

Therefore for $S_k(\Gamma)$ with dimension one, we have both ASD congruences and modularity. Consequently, every $f \in S_k(\Gamma)$ with algebraic Fourier coefficients satisfies three-term congruence relations with a wt k congruence form.

Application: Characterizing noncongruence modular forms

The following conjecture, supported by all known examples, gives a simple characterization for noncongruence forms. If true, it has wide applications.

Conjecture. A modular form in $S_k(\Gamma)$ with algebraic Fourier coefficients has bounded denominators if and only if it is a congruence modular form, i.e., lies in $S_k(\Gamma^c)$.

Kurth-Long: quantitative confirmation for certain families of noncongruence groups.

Theorem[L-Long 2010] *The conjecture holds when $S_k(\Gamma)$ is 1-dim'l and X_Γ is defined over $\mathbb{Q}$.*

ASD congruences and modularity for $d \geq 2$

For each $n \geq 1$, there is an index-n subgroup Γ_n of $\Gamma^1(5)$ whose modular curve is defined over $\mathbb{Q}$ and $S_3(\Gamma_n)$ is $(n-1)$-dim'l with explicit basis and attached Scholl rep'n $\rho_{n,l}$.

Case $d = 2$.

Theorem[L-Long-Yang, 2005, for Γ_3]

(1) The space $S_3(\Gamma_3)$ has a basis consisting of 3-integral forms

$$f_{\pm}(z) = q^{1/15} \pm iq^{2/15} - \frac{11}{3}q^{4/15} \mp i\frac{16}{3}q^{5/15} - \\ -\frac{4}{9}q^{7/15} \pm i\frac{71}{9}q^{8/15} + \frac{932}{81}q^{10/15} + \cdots .$$

(2) (Modularity) There are two cuspidal newforms of weight 3 level 27 and character χ_{-3} given by

$$
\begin{aligned}
g_\pm(z) = {} & q \mp 3iq^2 - 5q^4 \pm 3iq^5 + 5q^7 \pm 3iq^8 + \\
& +9q^{10} \pm 15iq^{11} - 10q^{13} \mp 15iq^{14} - \\
& -11q^{16} \mp 18iq^{17} - 16q^{19} \mp 15iq^{20} + \\
& 45q^{22} \pm 12iq^{23} + \cdots .
\end{aligned}
$$

such that $\rho_{3,l} = \rho_{g_+,l} \oplus \rho_{g_-,l}$ *over* $\mathbb{Q}_l(\sqrt{-1})$.

(3) $f_\pm$ *satisfy the 3-term ASD congruences with* $A_p = a_p(g_\pm)$ *and* $B_p = \chi_{-3}(p)p^2$ *for all primes* $p \geq 5$.

Here χ_{-3} is the quadratic character attached to $\mathbb{Q}(\sqrt{-3})$.

Basis functions $f_\pm$ indep. of p, best one can hope for.

Hoffman, Verrill and students: an index 3 subgp of $\Gamma_0(8) \cap \Gamma_1(4)$, wt 3 forms, $\rho = \tau \oplus \tau$ and τ modular, one family of A_p and B_p.

Case $d = 3$.

- $S_3(\Gamma_4)$ has an explicit basis h_1, h_2, h_3 of 2-integral forms.
- $\Gamma_4 \subset \Gamma_2 \subset \Gamma^1(5)$ and $S_3(\Gamma_2) =< h_2 >$.

Theorem[L-Long-Yang, 2005, for Γ_2]

The 2-dim'l Scholl representation $\rho_{2,l}$ attached to $S_3(\Gamma_2)$ is modular, isomorphic to $\rho_{g_2,l}$ attached to the cuspidal newform $g_2 = \eta(4z)^6$. Consequently, h_2 satisfies the ASD congruences with $A_p = a_p(g_2)$ and $B_p = p^2$.

It remains to describe the ASD congruence on the space $< h_1, h_3 >$. Let

$$f_1(z) = \frac{\eta(2z)^{12}}{\eta(z)\eta(4z)^5} = q^{1/8}(1 + q - 10q^2 + \cdots) = \sum_{n\geq 1} a_1(n)q^{n/8},$$
$$f_3(z) = \eta(z)^5\eta(4z) = q^{3/8}(1 - 5q + 5q^2 + \cdots) = \sum_{n\geq 1} a_3(n)q^{n/8},$$
$$f_5(z) = \frac{\eta(2z)^{12}}{\eta(z)^5\eta(4z)} = q^{5/8}(1 + 5q + 8q^2 + \cdots) = \sum_{n\geq 1} a_5(n)q^{n/8},$$
$$f_7(z) = \eta(z)\eta(4z)^5 = q^{7/8}(1 - q - q^2 + \cdots) = \sum_{n\geq 1} a_7(n)q^{n/8}.$$

Theorem[Atkin-L-Long, 2008] [ASD congruence for the space $< h_1, h_3 >$]

1. *If* $p \equiv 1 \mod 8$, *then both* h_1 *and* h_3 *satisfy the three-term ASD congruence at* p *with* $A_p = sgn(p) a_1(p)$ *and* $B_p = p^2$, *where* $sgn(p) = \pm 1 \equiv 2^{(p-1)/4} \mod p$ *;*
2. *If* $p \equiv 5 \mod 8$, *then* h_1 *(resp.* h_3*) satisfies the three-term ASD-congruence at* p *with* $A_p = -4ia_5(p)$ *(resp.* $A_p = 4ia_5(p)$*) and* $B_p = -p^2$*;*
3. *If* $p \equiv 3 \mod 8$, *then* $h_1 \pm h_3$ *satisfy the three-term ASD congruence at* p *with* $A_p = \mp 2\sqrt{-2} a_3(p)$ *and* $B_p = -p^2$*;*
4. *If* $p \equiv 7 \mod 8$, *then* $h_1 \pm ih_3$ *satisfy the three-term ASD congruence at* p *given by* $A_p = \pm 8\sqrt{-2} a_7(p)$ *and* $B_p = -p^2$.

Here $a_1(p), a_3(p), a_5(p), a_7(p)$ *are given above.*

To describe the modularity of $\rho_{4,l}$, let

$$f(z) = f_1(z) + 4f_5(z) + 2\sqrt{-2}(f_3(z) - 4f_7(z)) = \sum_{n\geq 1} a(n)q^{n/8}.$$

$f(8z)$ is a newform of level dividing 256, weight 3, and quadratic character χ_{-4} associated to $\mathbb{Q}(i)$.

Let $K = \mathbb{Q}(i, 2^{1/4})$ and χ a character of $Gal(K/\mathbb{Q}(i))$ of order 4. Denote by $h(\chi)$ the associated (weight 1) cusp form.

Theorem[Atkin-L-Long, 2008][Modularity of $\rho_{4,l}$]

The 6-dim'l Scholl rep'n $\rho_{4,l}$ decomposes over $\mathbb{Q}_l$ into the sum of $\rho_{2,l}$ (2-dim'l) and $\rho_{-,l}$ (4-dim'l). Further, $L(s, \rho_{2,l}) = L(s, g_2)$ and $L(s, \rho_{-,l}) = L(s, f \times h(\chi))$ (same local L-factors).

Proof uses Faltings-Serre method.

Case $d = 5$. Similar result for $S_3(\Gamma_6)$ by Long.

Representations with quaternion multiplication

Joint work with A.O.L. Atkin, T. Liu and L. Long

ρ_l : a 4-dim'l Scholl representation of $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ assoc. to a 2-dim'l subspace $S \subset S_k(\Gamma)$.

Suppose ρ_l has *quaternion multiplication* (QM) over $\mathbb{Q}(\sqrt{s}, \sqrt{t})$, i.e., there are two operators J_s and J_t on $\rho_l \otimes_{\mathbb{Q}_l} \bar{\mathbb{Q}}_l$, parametrized by two non-square integers s and t, satisfying

(a) $J_s^2 = J_t^2 = -id, J_{st} := J_s J_t = -J_t J_s$;

(b) For $u \in \{s, t\}$ and $g \in G_{\mathbb{Q}}$, we have $J_u \rho_l(g) = \varepsilon_u(g) \rho_l(g) J_u$, where ε_u is the quadratic character of $\mathrm{Gal}(\mathbb{Q}(\sqrt{u})/\mathbb{Q})$.

For Γ_3, Scholl representations have QM over $\mathbb{Q}(\sqrt{s}, \sqrt{t}) = \mathbb{Q}(\sqrt{-3})$, and for Γ_4, we have QM over $\mathbb{Q}(\sqrt{s}, \sqrt{t}) = \mathbb{Q}(\sqrt{-1}, \sqrt{2}) = \mathbb{Q}(\zeta_8)$.

Theorem [Atkin-L-Liu-Long 2010] (Modularity)

(a) *If* $\mathbb{Q}(\sqrt{s}, \sqrt{t})$ *is a quadratic extension, then over* $\mathbb{Q}_l(\sqrt{-1})$, ρ_l *decomposes as a sum of two degree* 2 *representations assoc. to two congruence forms of weight* k.

(b) *If* $\mathbb{Q}(\sqrt{s}, \sqrt{t})$ *is biquadratic over* $\mathbb{Q}$, *then for each* $u \in \{s, t, st\}$, *there is an automorphic form* g_u *for* GL_2 *over* $\mathbb{Q}(\sqrt{u})$ *such that the* L*-functions attached to* ρ_l *and* g_u *agree locally at all* p. *Consequently,* $L(s, \rho_l)$ *is automorphic.*

$L(s, \rho_l)$ also agrees with the L-function of an automorphic form of $GL_2 \times GL_2$ over $\mathbb{Q}$, and hence also agrees with the L-function of a form on GL_4 over $\mathbb{Q}$ by Ramakrishnan.

The proof uses descent and modern modularity criteria.

Theorem [Atkin-L-Liu-Long 2010] (ASD congruences)

Assume $\mathbb{Q}(\sqrt{s},\sqrt{t})$ *is biquadratic. Suppose that the QM operators* J_s *and* J_t *arise from real algebraic linear combinations of normalizers of* Γ *so that they also act on the noncongruence forms in* S*. For each* $u \in \{s,t,st\}$*, let* $f_{u,j}$*,* $j = 1,2$*, be linearly independent eigenfunctions of* J_u*. For almost all primes* p *split in* $\mathbb{Q}(\sqrt{u})$*,* $f_{u,j}$ *are* p*-adically integral basis of* S *and the ASD congruences at* p *hold for* $f_{u,j}$ *with* $A_{u,p}(j)$ *and* $B_{u,p}(j)$ *coming from the two local factors*

$$(1 - A_{u,p}(j)p^{-s} + B_{u,p}(j)p^{-2s})^{-1}, \qquad j = 1,2,$$

of $L(s,g_u)$ *at the two places of* $\mathbb{Q}(\sqrt{u})$ *above* p*.*

Note that the basis functions for ASD congruences depend on p modulo the conductor of $\mathbb{Q}(\sqrt{s},\sqrt{t})$.

ASD congruences in general

Now suppose $S_k(\Gamma)$ has dimension d. Scholl representations ρ_l are $2d$-dimensional. For almost all p the characteristic polynomial $H_p(T)$ of $\rho_l(\mathrm{Frob}_p)$ has degree $2d$. The representations are called *strongly ordinary* at p if $H_p(T)$ has d roots which are distinct p-adic units (and the remaining d roots are p^{k-1} times units).

Scholl: ASD congruences at p hold if ρ_l is strongly ordinary at p.

But if the representations are not ordinary at p, then the situation is quite different. Then the ASD congruences at p may or may not hold. We exhibit an example computed by J. Kibelbek.

Ex 1. $X \; : \; y^2 = x^5 + 2$, genus 2 curve defined over $\mathbb{Q}$. By Belyi, $X \simeq X_\Gamma$ for a finite index subgroup Γ of $SL_2(\mathbb{Z})$. Put

$$\omega_1 = x\frac{dx}{2y} = f_1 \frac{dq^{1/10}}{q^{1/10}}, \quad \omega_2 = \frac{dx}{2y} = f_2 \frac{dq^{1/10}}{q^{1/10}}.$$

Then $S_2(\Gamma) =< f_1, f_2 >$, where

$$\begin{aligned} f_1 &= \sum_{n \geq 1} a_n(f_1) q^{n/10} \\ &= q^{1/10} - \frac{8}{5} q^{6/10} - \frac{108}{5^2} q^{11/10} + \frac{768}{5^3} q^{16/10} + \frac{3374}{5^4} q^{21/10} + \cdots , \end{aligned}$$

$$\begin{aligned} f_2 &= \sum_{n \geq 1} a_n(f_2) q^{n/10} \\ &= q^{2/10} - \frac{16}{5} q^{7/10} + \frac{48}{5^2} q^{12/10} + \frac{64}{5^3} q^{17/10} + \frac{724}{5^4} q^{22/10} + \cdots . \end{aligned}$$

The l-adic representations for wt 2 forms are the dual of the Tate modules on the Jacobian of X_Γ.

For primes $p \equiv 2, 3 \mod 5$, $H_p(T) = T^4 + p^2$ (not ordinary). The Scholl congruences give

$$a_{mp^{n+2}}(f_i) + p^2 a_{mp^{n-2}}(f_i) \equiv 0 \mod p^{n+1}$$

for all $n \geq 1$ and $m \geq 1$, but no 3-term congruences exist.

In other words, $S_2(\Gamma)$ has no nonzero form satisfying the ASD congruences for $p \equiv 2, 3 \mod 5$.