Integral of Borcherds type

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Outline

Previous Works

In Arakelov geometry, especially in the Gross-Zagier-Zhang type of formulas, it is very important to have explicit contruction of Green currents of the cycles involved and to compute their integrals.

Let V be a non-degenerate quadratic space over $\mathbb{Q}$ of signature (n,2),and let D be the associated Hermitian symmetric domain, which can be identify as the set of oriented negative 2-planes in $V(\mathbb{R})$. Using theta correspondance for the dual pair $(SL_2(\mathbb{Z}), O(V))$, we can get modular forms of orthogonal groups from the modular forms of $SL_2(\mathbb{Z})$.

In his 1998 paper, Borcherds constructed certain meromorphic modular forms $\Phi(z, f)$ on D with respect to arithmetic subgroups Γ_M of $G = O(V)$ by regularizing the theta integral of vector valued elliptic modular forms f of weight $1 - \frac{n}{2}$ for $SL_2(\mathbb{Z})$ with poles at the cusp, it can be viewed as meromorphic sections of powers of a certain line bundle $\mathbb{L}$ on $X = \Gamma_M \setminus D$.

Taking the standard metric $||\cdot||$ on $\mathbb{L}$, Kudla computed the integral:

$$\kappa(\Phi(f)) := -vol(X)^{-1}\int_{\Gamma_M\backslash D}\log||\Phi(z,f)||^2 d\mu(z),$$

where $d\mu(z)$ is a $G(\mathbb{R})$-invariant volume form on D.

Bruinier and Yang considered the CM cycles (subvariety of signature (0,2)) on X for harmonic weak Maass form f, and express the integral of the theta lift of f, which is a Green function in the Arakelov sense, in terms of the derivative of a certain Rankin-Selberg L-function, which is the basis for their results on a relation between Faltings height and a certain Rankin-Selberg L-function.

For any compact open subgroup $K \subset H(\mathbb{A}_f)$, we can construct a Shimura variety

$$X_K = H(\mathbb{Q}) \diagdown (\mathbb{D} \times H(\mathbb{A}_f) \diagup K).$$

1. (B, reduced norm), for K an Eichler order of level n, we have
$$X_K = \Gamma(n) \backslash \mathbb{H} \times \Gamma(n) \backslash \mathbb{H}.$$
2. $F = Q(\sqrt{\Delta})$ real quadratic field.
$V = \{A \in M_2(F) : A^{'} = A^t\}$, $Q(A) = detA$;
$K = \{g \in GL_2(\widehat{\mathbb{O}_F}) : detg \in \hat{\mathbb{Z}}\}$. Then
$$X_K = SL_2(\mathbb{O}_F) \backslash \mathbb{H}^2.$$

For a subspace U of signature $(r, 2)$, where $0 < r < n$, its associated Shimura variety is a subvariety of X_K of codimension $n-r$ and thus gives a cycle of codimension $n-r$, and we will give a similar results as Bruinier and Yang do.

Notations

Let $H = Gspin(V)$, $G = SL_2$; $G'_{\mathbb{A}}$ for the two fold metaplectic cover of $G(\mathbb{A}) = G_{\mathbb{A}}$. Then groups $G'_{\mathbb{A}}$ and $H(\mathbb{A})$ act on the space $S(V(\mathbb{A}))$ of Schwartz-Bruhat functions of $V(\mathbb{A})$ via the Weil representation ω.

∞-part: Gaussion function

For any $z \in \mathbb{D} = \{z \subset V(\mathbb{R}); dim(z) = 2 \; and \; Q|_z < 0\}$, we may consider the corresponding majorant

$$(x, x)_z = (x_{z^\perp}, x_{z^\perp}) - (x_z, x_z),$$

which is a positive definite quadratic form on the vector space $V(\mathbb{R})$. The Gaussian $\phi_\infty(x, z) = exp(-\pi(x, x)_z)$ belongs to $S(V(\mathbb{R}))$.

finite part: Lattice function

Let $L \subset V$ be an even lattice and write L' for the dual lattice, then the discriminant group L'/L is finite. We consider the subspace S_L of Schwartz functions in $S(V(\mathbb{A}_f))$ which are supported on $\hat{L}' := L' \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ and which are constant on cosets of $\hat{L} := L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. For any $\mu \in L'/L$, the characteristic function $\phi_\mu = char(\mu + \hat{L})$ belongs to S_L, and we have

$$S_L = \oplus_{\mu \in L'/L} \mathbb{C}\phi_\mu \subset S(V(\mathbb{A}_f)).$$

hence the dimension of S_L equals $|L'/L|$.

Theta kernel

For the above choices of speical Schwart-Bruhat functions, we can construct a S_L-valued theta function:

$$\Theta_L(\tau, z, h_f) = \sum_{\mu \in L'/L} v^{-\frac{n}{4}+\frac{1}{2}} \sum_{x \in V(\mathbb{Q})} (\omega(g'_\tau, (h_z, h_f))\phi_\infty(\cdot, z_0) \otimes \phi_\mu(\cdot))(x)\phi_\mu$$

where $h_z \in H(\mathbb{R})$ so that $h_z z_0 = z$ and $z_0 \in \mathbb{D}$ denotes a fixed base point; $\tau = \mu + vi$ in the upper half plane; $g'_\tau = [g_\tau, 1]$ and $g_\tau i = \tau$.

Harmonic weak Maass forms

A twice continuously differentiable function $f : \mathfrak{H} \longrightarrow S_L$ is called a harmonic weak Maass form of weight k with respect to $\Gamma' = Mp_2(\mathbb{Z})$ and ρ_L if it satisfies:

1. $f|_{k,\rho_L}\gamma' = f$ for all $\gamma' \in \Gamma'$;
2. there is a S_L-valued Fourier polynomial
$$P_f(\tau) = \sum_{\mu \in L'/L} \sum_{n \leq 0} c^+(n,\mu) q^n \phi_\mu,$$
such that,$f(\tau) - P_f(\tau) = O(e^{-\epsilon v})$ as $v \to \infty$ for some $\epsilon > 0$.
3. $\Delta_k f = 0$, where $\Delta_k := -v^2(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}) + ikv(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v})$ is the usual weight k hyperbolic Laplace operator.

Denote this space as H_{K,ρ_L}.

There is an antilinear differential operator

$$\xi = \xi_k : H_{k,\rho_L} \longrightarrow S_{2-k,\bar{\rho}_L}$$

$$f(\tau) \longmapsto \xi(f)(\tau) := v^{k-2}\overline{L_k f(\tau)}.$$

Here $L_k = -2iv^2 \frac{\partial}{\partial \bar{\tau}}$ is the Maass lowering operator. (Note the kernel of this operator is just $M^!_{k,\rho_L}$, the space of weakly holomorphic modular forms.) Hence there is a bilinear pairing between the spaces $M_{2-k,\bar{\rho}_L}$ and H_{k,ρ_L} defined by the Petersson scalar product

$$\{g, f\} = (g, \xi(f))_{Pet}.$$

Theta integral

On shimura variety X_K, where $K \subset H(\mathbb{A}_f)$ acts trivially on S_L, we have Heegner divisors $Z(m, \phi)$. For f a harmonic weak Maass form of weight $1 - \frac{n}{2}$ with representation $\bar{\rho_L}$ for Γ' (the double cover of $SL_2(\mathbb{Z})$), we consider the regularized theta integral

$$\Phi(z, h, f) = \int_{\mathfrak{F}}^{reg} \langle f(\tau), \Theta_L(\tau, z, h) \rangle d\mu(\tau)$$

for $z \in \mathbb{D}, h \in H(\mathbb{A}_f)$.

Green function

Theorem

([Bruinier-Funke])
The function $\Phi(z,h,f)$ *is smooth on* $X_K \setminus Z(f)$, *where* $Z(f) = Z_f = \sum_{\mu \in L'/L} \sum_{m>0} c^+(-m,\mu) Z(m,\mu)$. *It has a logarithmic singularity along the divisor* $-2Z(f)$. *The* $(1,1)$*-form* $dd^c\Phi(z,h,f)$ *can be continued to a smooth form on all of* X_K.
We have the Green current equation

$$dd^c[\Phi(z,h,f)] + \delta_{Z(f)} = [dd^c\Phi(z,h,f)],$$

where δ_Z *denotes the Direc current of a divisor Z.*

So the theorem implies that $\Phi(z, h, f)$ is a Green function for the divisor $Z(f)$ in the sense of Arakelov geometry.

Our goal is to compute the period integral of this Green function.

rational subspace and cycles

Let $U \subset V$ be a rational subspace of signature $(r, 2)$, here $0 < r < n$; and V_+ be the orthogonal complement of U over $\mathbb{Q}$. Let $T = Gspin(U)$, acting trivially on V_+, and put $K_T = K \cap T(\mathbb{A}_f)$, we obtain the cycle

$$X_{K,U} = T(\mathbb{Q}) \diagdown \mathbb{D}_0 \times T(\mathbb{A}_f) \diagup K_T \longrightarrow X_K$$

.

we define lattices $N = L \cap U$, $P = L \cap V_+$, then $N \oplus P \subset L$ is a sublattice of finite index.

For $z \in \mathbb{D}_0, h_2 \in T(\mathbb{A}_F)$, the Siegel theta function $\Theta_L(\tau, z, h_2)$ splits up as a product

$$\Theta_L(\tau, z, h_2) = \Theta_P(\tau, 1) \otimes \Theta_N(\tau, z, h_2),$$

where their respective weights are $\frac{n-r}{2}$ and $\frac{r-2}{2}$.

L-functions

For S_N-valued Eisenstein series of weight ℓ:

$$E_N(\tau, s; \ell) = v^{-\frac{\ell}{2}} \sum_{\mu \in N'/N} E(g'_\tau, s; \Phi^\ell_\infty \otimes \lambda_f(\phi_\mu))\phi_\mu,$$

where λ is an intertwinning operator from $S(V(\mathbb{A}))$ to $I(s_0, \chi)$. For $f \in S_{1+\frac{n}{2}, \rho_L}$, we define an L-function by means of the convolution integral

$$L(f, U, s) = (\Theta_P(\tau, h_1) \otimes E_N(\tau, s; \frac{r}{2} + 1), f(\tau))_{Pet}.$$

Assume

$$E_N(\tau, s; \frac{r}{2}+1) = \sum_{\mu}\sum_{m} A_\mu(s, m, \nu) q^m \phi_\mu,$$

with

$$A_\mu(s, m, \nu) = a_\mu(m, \nu) + b_\mu(m, \nu)(s - s_0) + O(s - s_0)^2.$$

Define:

$$\Sigma_N(\tau) = \sum_{\mu}\sum_{m} K(m, \mu) q^m \phi_\mu,$$

where

$$K(m, \mu) = \begin{cases} \lim_{\nu\to\infty} b_\mu(m, \nu) & m \neq 0 \text{ or } \mu \neq 0 \\ \lim_{\nu\to\infty} b_0(0, \nu) - log\nu & m = 0, \nu = 0. \end{cases}$$

our results

Theorem

Suppose when dim$(U) = 3$, U is anisotropic; when dim$(U) = 4$, U is not split. Then:

$$Y = \frac{1}{Vol(X_{K,U})} \int_{X_{K,U}} \Phi(z, h, f) d\mu(z)$$

$$= 2[CT(< f^{+}(\tau), \Theta_P(\tau) \otimes \Sigma_N(\tau) >) + L^{'}(\xi(f), U, \frac{r}{2})].$$

Remark: $CT[\cdot]$ is often of the form $\sum_p a_p logp$, where a_p is some quantity associated to prime p; this part of the theorem is of arithmetic nature; while the second part is of analytic nature.

Ideas of proof

Fist we use the contraction map to switch the pairing on the whole space V to a pairing on the subspace U; then switch the order of integration on $X_{K,U}$ and the regularization process; by a exact relation between coherent and incoherent Eisenstein series, Maass operators , Siegel-Weil formula and Stokes theorem we get the desired results.

Thank you!