

ON A PARTIAL EULER PRODUCT

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ABSTRACT. In this casual note we study the analytic properties of a certain incomplete Euler product.

1. INTRODUCTION

Let L/K be an extension of numberfields of degree n , which is not necessarily Galois. Let $v = (f_1, \dots, f_r)$ be an r -tuple of positive integers such that

$$f_1 \dots f_r$$

and

$$\sum_i f_i = n.$$

A prime p of K is said to be of type v if

- (1) p is unramified in L ;
- (2) We have

$$p\mathcal{O}_L = \mathfrak{p}_1 \dots \mathfrak{p}_r$$

with $f(\mathfrak{p}_i/p) = f_i$ for each i .

We denote the set of all primes of type v by $\mathcal{P}_v(L/K)$. We note that for a type v , the set $\mathcal{P}_v(L/K)$ maybe empty; for example, if L/K is Galois, $\mathcal{P}_v(L/K)$ willbe empty unless $r|n$ and $f_i = n/r$ for all i .

The purpose of this note is to study the analytic properties of the Euler product

$$F_v(s) = \prod_{p \in \mathcal{P}_v(L/K)} (1 - N(p)^{-s})^{-1}$$

for a fixed type v . Our argument follows the proof of the Chebotarev Density Theroem as described in any textbook on algebraic number theory; here we modify the presentation in [P].

This will eventually be incorporated in a joint work with Nathan Kaplan. In the writing of this note I have benefited from email communications with Mohammad Bardestani, as well as comments from the members of the MathOverflow community.

2. LIGHTNING REVIEW OF CLASSFIELD THEORY

Let K be a number field, and let \mathcal{I}_K be the free group generated by the finite primes of K . There is a natural map $\iota : K^\times \rightarrow \mathcal{I}_K$. A modulus is a formal product of primes of K . If \mathfrak{m} is a modulus, and $x \in K$, we write $x \equiv 1 \pmod{\mathfrak{m}}$ to mean:

- For each finite $\mathfrak{p}|\mathfrak{m}$, $x \equiv 1 \pmod{\mathfrak{p}^k}$ if $\mathfrak{p}^k \parallel \mathfrak{m}$;
- for each real prime $\nu|\mathfrak{m}$, we have $x_\nu > 0$.

If S is a finite set of primes, we let \mathcal{I}_K^S be the subgroup of \mathcal{I} generated by the primes not in S . For a modulus \mathfrak{m} we let $I_K^\mathfrak{m}$ be I_K^S where S is the set of finite primes that divide \mathfrak{m} . Set

$$K^\mathfrak{m} := \iota^{-1}(\mathcal{I}_K^\mathfrak{m})$$

and

$$K_1^\mathfrak{m} := \{x \in K^\mathfrak{m}; x \equiv 1 \pmod{\mathfrak{m}}\}.$$

Let $P_K^\mathfrak{m} = \iota(K_1^\mathfrak{m})$ and define

$$\mathcal{C}_K^\mathfrak{m} = \mathcal{I}_K^\mathfrak{m}/P_K^\mathfrak{m}.$$

This class group is finite. A congruence subgroup modulo \mathfrak{m} is a subgroup $H^\mathfrak{m}$ of $\mathcal{I}_K^\mathfrak{m}$ which contains $P_K^\mathfrak{m}$. We recall the following two main theorems of class field theory:

Theorem 2.1 (Artin Reciprocity Law). *For L/K an Abelian extension of number fields, there is a modulus \mathfrak{m} divisible by all the ramified primes of L/K and a congruence subgroup $H^\mathfrak{m}$ such that the sequence*

$$1 \rightarrow H^\mathfrak{m} \hookrightarrow \mathcal{I}_K^\mathfrak{m} \rightarrow \text{Gal}(L/K) \rightarrow 1$$

is exact. Also, $H^\mathfrak{m} = P_K^\mathfrak{m}.N_{L/K}(\mathcal{I}_L^\mathfrak{m})$.

Theorem 2.2. *For any congruence subgroup $H^\mathfrak{m}$, there is a unique Abelian extension L/K such that L is the class field of K of the congruence class group $\mathcal{I}_K^\mathfrak{m}/H^\mathfrak{m}$.*

3. THE GALOIS CASE

This is the main lemma of this note:

Lemma 3.1. *Let K be a number field, \mathfrak{m} a modulus, and $H^\mathfrak{m}$ a congruence subgroup. If C is coset of $\mathcal{I}_K^\mathfrak{m}/H^\mathfrak{m}$, we set*

$$f_C(s) = \prod_{p \in C} (1 - N(p)^{-s})^{-1}$$

Then $f_C(s)$ is holomorphic for $\Re s > 1$. Furthermore, Then $g_C(s) = f_C(s)^r$, $r = |\mathcal{I}_K^\mathfrak{m}/H^\mathfrak{m}|$, has an analytic continuation to an open set

containing $\Re s = 1$ with a unique pole at $s = 1$. On GRH, $s = 1$ is the only pole for $\Re s > 1/2$.

Proof. Let $G = \mathcal{I}_K^m/H^m$. Then

$$\begin{aligned} \log g_C(s) &= |G| \log f_C(s) \\ &= -|G| \sum_{p \in C} \log(1 - N(p)^{-s}) \\ &= |G| \sum_{p \in C} N(p)^{-s} + |G| \sum_{p \in C} \sum_{m \geq 2} \frac{1}{m} N(p)^{-ms}. \end{aligned}$$

Write

$$h(s) = |G| \sum_{p \in C} \sum_{m \geq 2} \frac{1}{m} N(p)^{-ms}.$$

This is holomorphic for $\Re s > 1/2$. We then write

$$\begin{aligned} \log g(s) - h(s) &= \sum_p \sum_{\chi \in \text{Hom}(G, S^1)} \chi(p) \chi(C^{-1}) N(p)^{-s} \\ &= \sum_{\chi \in \text{Hom}(G, S^1)} \chi(C^{-1}) \sum_p \chi(p) N(p)^{-s} \\ &= \sum_{\chi \in \text{Hom}(G, S^1)} \chi(C^{-1}) \left(\log \prod_p (1 - \chi(p) N(p)^{-s}) - \sum_p \sum_{m \geq 2} \frac{1}{m} \chi(p)^m N(p)^{-ms} \right) \\ &= \log \prod_{\chi} L(s, \chi)^{\chi(C^{-1})} + H(s) \end{aligned}$$

with $H(s)$ a function that is holomorphic for $\Re s > 1/2$. Hence

$$g_C(s) = \prod_{\chi} L(s, \chi)^{\chi(C^{-1})} e^{H(s)+h(s)}.$$

The lemma now follows from results on zero free regions of L -functions. \square

Next we prove the following lemma:

Lemma 3.2. *Let L/K be a Galois extension of number fields with Galois group $G = \text{Gal}(L/K)$. Let $C \subset G$ be a conjugacy class. Let*

$$F_C(s) = \prod_{\substack{p \text{ unramified} \\ (\frac{L/K}{p})=C}} (1 - N(p)^{-s})^{-1}.$$

Then $F_C(s)^{|G|}$ has a pole of order $|C|$ at $s = 1$.

Proof. If L/K is Abelian, this follows from the above lemma and class field theory. In general, let $\sigma \in C$, and let $H = \langle \sigma \rangle$. Let $M = L^H$. Note that L/M is an Abelian Galois extension. Let

$$F_H(s) = \prod_{p \in S} (1 - N_M(p)^{-s})^{-1}$$

where S is the set of primes of L^H satisfying

- $(\frac{L/M}{p}) = \sigma$;
- $f(p/p \cap \mathcal{O}_K) = e(p/p \cap \mathcal{O}_K) = 1$.

We will also consider

$$F'_H(s) = \prod_{p \in S'} (1 - N_M(p)^{-s})^{-1}$$

where S' is the set of primes p of M such that $(L/M/p) = \sigma$. We know from what we proved before that $F'_H(s)^{|H|}$ has a simple pole at $s = 1$. By the proof of Proposition 5.1.2 of [P] we know that $F'_H(s)/F_H(s)$ is holomorphic for $\Re s > 1/2$. Thus $F_H(s)^{|H|}$ has a simple pole at $s = 1$ and otherwise holomorphic in an open set containing $\Re s \geq 1$.

Next, it follows from the reduction step of the proof of the Chebotarev density theorem in [P] that

$$\begin{aligned} F_H(s) &= \left(\prod_{\substack{p \text{ prime of } K \\ (\frac{L/K}{p}) = C}} (1 - N(p)^{-s})^{-1} \right)^{\frac{|G|}{|C| \cdot |H|}} \\ &= (F_C(s))^{\frac{|G|}{|C| \cdot |H|}}. \end{aligned}$$

The lemma is now immediate. \square

Corollary 3.3. *Let C_1, \dots, C_r be the conjugacy classes of $G = \text{Gal}(L/K)$ and suppose $n_1, \dots, n_r \geq 0$ are integers. Consider*

$$Z(s) = \prod_{i=1}^r \prod_{\substack{p \text{ unramified} \\ (\frac{L/K}{p}) = C_i}} (1 - N(p)^{-s})^{-n_i}.$$

Then $Z(s)$ is holomorphic for $\Re s > 1$. Furthermore, $Z(s)^{|G|}$ has an analytic continuation to a domain containing $\Re s \geq 1$ with a unique pole at $s = 1$ of order $\sum_i n_i |C_i|$.

4. THE MAIN RESULT

We now go back to the setup of the introduction. Let E/K be the Galois closure of L/K . Set $G = \text{Gal}(E/K)$ and $H = \text{Gal}(E/L)$. In order to understand the analytic properties of $F_v(s)$ for a type v , we describe the set $\mathcal{P}_v(L/K)$ as the set of those primes p whose associated conjugacy class $(\frac{E/K}{p})$ in G satisfies certain properties. As such $F_v(s)$ will be a product of functions of the form $F_C(s)$, and the desired analytic properties will follow from Lemma 3.2 above.

Let p be an unramified prime of K and write

$$p\mathcal{O}_E = \mathfrak{q}_1 \cdots \mathfrak{q}_s.$$

Since E/K is Galois, $f = f(\mathfrak{q}_i/p)$ is independent of i , and we have

$$sf = [E : K].$$

In particular, $s|[E : K]$. Since p is unramified, the decomposition group $D(\mathfrak{q}_i/p)$ is cyclic and its generator will be $[\frac{E/K}{\mathfrak{q}_i}]$. We note that the conjugacy class $(\frac{E/K}{p})$ will be the set

$$\{[\frac{E/K}{\mathfrak{q}_i}]; 1 \leq i \leq s\}.$$

The orbit of \mathfrak{q}_i under the action of H will be in one-to-one correspondence with

$$H/H \cap \langle [\frac{E/K}{\mathfrak{q}_i}] \rangle.$$

We let r_i be the size of this quotient. Then the orbit of \mathfrak{q}_i under H will have r_i elements. Let $\mathfrak{q}_i^1, \dots, \mathfrak{q}_i^{r_i}$ be the elements of the orbit of \mathfrak{q}_i . Then

$$\mathfrak{p} = \mathfrak{q}_i^1 \cdots \mathfrak{q}_i^{r_i}$$

will be H -invariant, hence of the form $\mathfrak{q}\mathcal{O}_E$ for some prime ideal \mathfrak{q} of M . Since our primes are unramified we have $r_i f(\mathfrak{q}_i^j/\mathfrak{q}) = [E : L]$ and consequently

$$f(\mathfrak{q}_i^j/\mathfrak{q}) = \frac{[E : L]}{r_i}$$

Next, since $f(\mathfrak{q}_i^j/p) = f(\mathfrak{q}_i^j/\mathfrak{q})f(\mathfrak{q}/p)$, we have

$$\frac{[E : k]}{s} = \frac{[E : L]}{r_i} \cdot f(\mathfrak{q}/p),$$

from which it follows

$$f(\mathfrak{q}/p) = \frac{[L : K]r_i}{s},$$

or

$$r_i = \frac{f(\mathfrak{q}/p)s}{[L : K]}.$$

So here is the promised characterization of $\mathcal{P}_v(L/K)$.

Theorem 4.1. *Let $v = (f_1, \dots, f_r)$. Then $\mathcal{P}_v(L/K)$ is the set of those p 's that satisfy the following condition: If we write*

$$\left(\frac{E/K}{p}\right) = \{\sigma_1, \dots, \sigma_s\}$$

and consider the multiset of quotient sizes

$$\{|H/H \cap \langle \sigma \rangle; 1 \leq i \leq r\},$$

then this multiset is the same as

$$\{r_1, \dots, r_1, r_2, \dots, r_2, \dots, r_k, \dots, r_k\},$$

each r_i repeated r_i times, with

$$r_i = \frac{f_i \cdot s}{[L : K]}.$$

REFERENCES

- [P] A. Preygel, *L-functions and the densities of primes*, available at http://modular.math.washington.edu/129-05/final_papers/Anatoly-Preygel.pdf.

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