

A remark on a paper of Ahlgren, Berndt, Yee, and Zaharescu

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Abstract

A classical theorem of Ramanujan relates an integral of Dedekind eta-function to a special value of a Dirichlet L -function at $s = 2$. Ahlgren, Berndt, Yee and Zaharescu have generalized this result. In this paper, we generalize this result to the context of holomorphic cusp forms on the upper half space.

1 The theorem

Let $\eta(z)$ be the Dedekind eta-function given by

$$\eta(z) = e^{\frac{2\pi iz}{24}} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}) \quad (1.1)$$

for $\Re z > 0$. If we formally set

$$f(-q) = \prod_{n=1}^{\infty} (1 - q^n), \quad (1.2)$$

then we have $\eta(z) = q^{1/24} f(-q)$, whenever $q = e^{2\pi iz}$ with $\Re z > 0$.

If D is a fundamental discriminant, we set

$$\chi_D(n) := \left(\frac{D}{n} \right) \quad (1.3)$$

where the symbol on the right hand side is the usual Kronecker symbol. If χ is a Dirichlet character, we denote by $L(s, \chi)$ the finite part of its L -function defined for $\Re s > 1$ by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \quad (1.4)$$

Recall that $L(s, \chi)$ has an analytic continuation to a meromorphic function on the whole complex plane, and satisfies a functional equation of the usual type. The following is a theorem of Ramanujan

Theorem 1.1 ([3], page 207) *Suppose that $0 < q < 1$. Then*

$$q^{\frac{1}{9}} \prod_{n=1}^{\infty} (1 - q^n)^{n\chi_{-3}(n)} = \exp \left(-C_3 - \frac{1}{9} \int_q^1 \frac{f^9(-t)}{f^3(-t^3)} \frac{dt}{t} \right), \quad (1.5)$$

where

$$C_3 = \frac{3\sqrt{3}}{4\pi} L(2, \chi_{-3}) = L'(-1, \chi_{-3}). \quad (1.6)$$

For a proof see [2]. For historical remarks and a similar statement involving χ_{-4} see [1]. In fact, the paper [1] contains a vast generalization of Ramanujan's theorem to Eisenstein series of arbitrary weight and level. The main theorem of [1] states that

Theorem 1.2 *Suppose that α is real, that $k \geq 2$ is an integer, and that χ is a nontrivial Dirichlet character which satisfies the condition $\chi(-1) = (-1)^k$. Suppose further that $0 < q < 1$. Then*

$$q^\alpha \prod_{n=1}^{\infty} (1 - q^n)^{\chi(n)n^{k-2}} = \exp \left(-C - \int_q^1 \left\{ \alpha - \sum_{n=1}^{\infty} \sum_{d|n} \chi(d) d^{k-1} t^n \right\} \frac{d}{dt} \right),$$

where

$$C = L'(2 - k, \chi).$$

The purpose of this note is to establish an identity similar to this identity for holomorphic cusp forms. Namely, we will prove the following theorem:

Theorem 1.3 *Let $F(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi inz}$ be a holomorphic cusp form. Then we have*

$$q^\alpha \prod_{n=1}^{\infty} (1 - q^n)^{c(n)} = \exp \left(-C - \int_q^1 \left\{ \alpha - \sum_{n=1}^{\infty} \sum_{d|n} dc(d)t^n \right\} \frac{dt}{t} \right),$$

where

$$C = \Lambda(0, F).$$

Here $\Lambda(s, F)$ is the completed L -function of the cusp form F .

2 Proof

The proof of Theorem 1.3 is identical to the proof of the main theorem of [1], except that it is easier. By taking logarithms of both sides of the equation, it is seen that we have

$$C = \lim_{t \rightarrow 1^-} \sum_{n=1}^{\infty} \sum_{d|n} dc(d) \frac{t^n}{n}.$$

We set

$$H(q) = \sum_{n=1}^{\infty} \sum_{d|n} dc(d) \frac{q^n}{n}$$

and

$$f(q) = \sum_{d=1}^{\infty} c(d)q^d.$$

It is then easy to see that

$$H(q) = \sum_{n=1}^{\infty} \frac{1}{n} f(q^n).$$

By definition,

$$\begin{aligned}
C &= \lim_{q \rightarrow 1^-} H(q) \\
&= \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{n=1}^{\infty} \frac{1}{n/M} f(e^{-\frac{n}{M}}) \\
&= \int_0^{\infty} f(e^{-y}) \frac{dy}{y} \\
&= \int_0^{\infty} F\left(\frac{iy}{2\pi}\right) \frac{dy}{y} \\
&= \int_0^{\infty} F(iy) \frac{dy}{y} \\
&= \Lambda(0, F).
\end{aligned}$$

Remark 2.1 As in [1], we can replace $c(n)$ with $n^k c(n)$, and obtain similar identities. The only difference is that instead of the special value of the L-function at $s = 0$, we will have other special values multiplied by explicit constants.

Remark 2.2 If we let $u(n)$ be the coefficient of q^n in the expansion of $H(q)$, then $u(n)$ is given by

$$L(s, F)L(s, \eta) = \sum_{n=1}^{\infty} \frac{u(n)}{n^s} \tag{2.1}$$

where $L(s, F)$ is the finite part of the L -function of F , and $\eta : \mathbb{A}_{\mathbb{Q}}^{\times} \rightarrow \mathbb{C}^{\times}$ is the unramified character $\eta(\alpha) = |\alpha|_{\mathbb{A}}^{-1}$. We observe that $L(s, F)L(s, \eta)$ is the L -function of a $\mathrm{GL}(3)$ Eisenstein series. For this reason, our formula can be thought of the $\mathrm{GL}(2)$ analogue of the $\mathrm{GL}(1)$ formula of [1]. There is an obvious generalization to all $\mathrm{GL}(n)$.

References

- [1] S. Ahlgren, B. C. Berndt, A. J. Yee and A. Zaharescu, *Integrals of Eisenstein series and derivatives of L-functions*, IMRN no. 32 (2002), p. 1723-1738.

- [2] B. C. Berndt and A. Zaharescu, An integral of Dedekind eta-functions in Ramanujan's lost notebook, *J. Reine Angew. Math.* 551 (2002), 33–39.
- [3] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.