A remark on a paper of Ahlgren, Berndt, Yee, and Zaharescu

Ramin Takloo-Bighash

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Abstract

A classical theorem of Ramanujan relates an integral of Dedekind eta-function to a special value of a Dirichlet L-function at $s = 2$. Ahlgren, Berndt, Yee and Zaharescu have generalized this result. In this paper, we generalize this result to the context of holomorphic cusp forms on the upper half space.

1 The theorem

Let $\eta(z)$ be the Dedekind eta-function given by

$$\eta(z) = e^{\frac{2\pi iz}{24}} \prod_{n=1}^{\infty}(1 - e^{2\pi inz}) \tag{1.1}$$

for $\Re z > 0$. If we formally set

$$f(-q) = \prod_{n=1}^{\infty}(1 - q^n), \tag{1.2}$$

then we have $\eta(z) = q^{1/24} f(-q)$, whenever $q = e^{2\pi iz}$ with $\Re z > 0$.

If D is a fundamental discriminant, we set

$$\chi_D(n) := \left(\frac{D}{n}\right) \tag{1.3}$$

where the symbol on the right hand side is the usual Kronecker symbol. If χ is a Dirichlet character, we denote by $L(s,\chi)$ the finite part of its L-function defined for $\Re s > 1$ by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{1.4}$$

Recall that $L(s,\chi)$ has an analytic continuation to a meromorphic function on the whole complex plane, and satisfies a functional equation of the usual type. The following is a theorem of Ramanujan

Theorem 1.1 ([3], page 207) *Suppose that* $0 < q < 1$. *Then*

$$q^{\frac{1}{9}} \prod_{n=1}^{\infty} (1-q^n)^{n\chi_{-3}(n)} = \exp\left(-C_3 - \frac{1}{9}\int_q^1 \frac{f^9(-t)}{f^3(-t^3)}\frac{dt}{t}\right), \tag{1.5}$$

where

$$C_3 = \frac{3\sqrt{3}}{4\pi} L(2,\chi_{-3}) = L'(-1,\chi_{-3}). \tag{1.6}$$

For a proof see [2]. For historical remarks and a similar statement involving χ_{-4} see [1]. In fact, the paper [1] contains a vast generalization of Ramanujan's theorem to Eisenstein series of arbitrary weight and level. The main theorem of [1] states that

Theorem 1.2 *Suppose that* α *is real, that* $k \geq 2$ *is an integer, and that* χ *is a nontrivial Dirichlet character which satisfies the condition* $\chi(-1) = (-1)^k$. *Suppose further that* $0 < q < 1$. *Then*

$$q^{\alpha} \prod_{n=1}^{\infty} (1-q^n)^{\chi(n)n^{k-2}} = \exp\left(-C - \int_q^1 \left\{\alpha - \sum_{n=1}^{\infty}\sum_{d|n} \chi(d) d^{k-1} t^n\right\} \frac{d}{dt}\right),$$

where

$$C = L'(2-k,\chi).$$

The purpose of this note is to establish an indentity similar to this identity for holomorphic cusp forms. Namely, we will prove the following theorem:

Theorem 1.3 *Let* $F(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi i n z}$ *be a holomorphic cusp form. Then we have*

$$q^{\alpha}\prod_{n=1}^{\infty}(1-q^n)^{c(n)} = \exp\left(-C - \int_q^1 \left\{\alpha - \sum_{n=1}^{\infty}\sum_{d|n} dc(d)t^n\right\}\frac{dt}{t}\right),$$

where

$$C = \Lambda(0, F).$$

Here $\Lambda(s, F)$ *is the completed L-function of the cusp form* F*.*

2 Proof

The proof of Theorem 1.3 is identical to the proof of the main theorem of [1], except that it is easier. By taking logarithms of both sides of the equation, it is seen that we have

$$C = \lim_{t\to 1^-}\sum_{n=1}^{\infty}\sum_{d|n} dc(d)\frac{t^n}{n}.$$

We set

$$H(q) = \sum_{n=1}^{\infty}\sum_{d|n} dc(d)\frac{q^n}{n}$$

and

$$f(q) = \sum_{d=1}^{\infty} c(d)q^d.$$

It is then easy to see that

$$H(q) = \sum_{n=1}^{\infty}\frac{1}{n}f(q^n).$$

By definition,

$$\begin{aligned} C &= \lim_{q\to 1^-} H(q) \\ &= \lim_{M\to\infty} \frac{1}{M} \sum_{n=1}^{\infty} \frac{1}{n/M} f(e^{-\frac{n}{M}}) \\ &= \int_0^\infty f(e^{-y}) \frac{dy}{y} \\ &= \int_0^\infty F(\frac{iy}{2\pi}) \frac{dy}{y} \\ &= \int_0^\infty F(iy) \frac{dy}{y} \\ &= \Lambda(0, F). \end{aligned}$$

Remark 2.1 As in [1], we can replace $c(n)$ with $n^k c(n)$, and obtain similar identities. The only difference is that instead of the special value of the L-function at $s = 0$, we will have other special values multiplied by explicit constants.

Remark 2.2 If we let $u(n)$ be the coefficient of q^n in the expansion of $H(q)$, then $u(n)$ is given by

$$L(s, F)L(s, \eta) = \sum_{n=1}^{\infty} \frac{u(n)}{n^s} \tag{2.1}$$

where $L(s, F)$ is the finite part of the L-function of F, and $\eta : \mathbb{A}_{\mathbb{Q}}^{\times} \to \mathbb{C}^{\times}$ is the unramified character $\eta(\alpha) = |\alpha|_{\mathbb{A}}^{-1}$. We observe that $L(s, F)L(s, \eta)$ is the L-function of a GL(3) Eisenstein series. For this reason, our formula can be thought of the GL(2) analogue of the GL(1) formula of [1]. There is an obvious generalization to all GL(n).

References

[1] S. Ahlgren, B. C. Berndt, A. J. Yee and A. Zaharescu, *Integrals of Eisenstein series and derivatives of L-functions*, IMRN no. 32 (2002), p. 1723-1738.

[2] B. C. Berndt and A. Zaharescu, An integral of Dedekind eta-functions in Ramanujan's lost notebook, J. Reine Angew. Math. 551 (2002), 33–39.

[3] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.