A remark on a paper of Ahlgren, Berndt, Yee, and Zaharescu

Ramin Takloo-Bighash

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Abstract

A classical theorem of Ramanujan relates an integral of Dedekind eta-function to a special value of a Dirichlet L-function at $s = 2$. Ahlgren, Berndt, Yee and Zaharescu have generalized this result. In this paper, we generalize this result to the context of holomorphic cusp forms on the upper half space.

1 The theorem

Let $\eta(z)$ be the Dedekind eta-function given by

$$\eta(z) = e^{\frac{2\pi i z}{24}} \prod_{n=1}^{\infty} (1 - e^{2\pi i nz})$$  \hspace{1cm} (1.1)

for $\Re z > 0$. If we formally set

$$f(-q) = \prod_{n=1}^{\infty} (1 - q^n),$$  \hspace{1cm} (1.2)

then we have $\eta(z) = q^{1/24} f(-q)$, whenever $q = e^{2\pi iz}$ with $\Re z > 0$.

If $D$ is a fundamental discriminant, we set

$$\chi_D(n) := \left( \frac{D}{n} \right)$$  \hspace{1cm} (1.3)
where the symbol on the right hand side is the usual Kronecker symbol. If \( \chi \) is a Dirichlet character, we denote by \( L(s, \chi) \) the finite part of its \( L \)-function defined for \( \Re s > 1 \) by
\[
L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{1.4}
\]
Recall that \( L(s, \chi) \) has an analytic continuation to a meromorphic function on the whole complex plane, and satisfies a functional equation of the usual type. The following is a theorem of Ramanujan

**Theorem 1.1 ([3], page 207)** Suppose that \( 0 < q < 1 \). Then
\[
q^{\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)^{n\chi_3(n)} = \exp \left( -C_3 - \frac{1}{9} \int_{q}^{1} \frac{f_3(-t)}{\sqrt[3]{f_3(-t^3)}} \frac{dt}{t} \right), \tag{1.5}
\]
where
\[
C_3 = \frac{3\sqrt{3}}{4\pi} L(2, \chi_3) = L'(1, \chi_3). \tag{1.6}
\]
For a proof see [2]. For historical remarks and a similar statement involving \( \chi_{-4} \) see [1]. In fact, the paper [1] contains a vast generalization of Ramanujan’s theorem to Eisenstein series of arbitrary weight and level. The main theorem of [1] states that

**Theorem 1.2** Suppose that \( \alpha \) is real, that \( k \geq 2 \) is an integer, and that \( \chi \) is a nontrivial Dirichlet character which satisfies the condition \( \chi(-1) = (-1)^k \). Suppose further that \( 0 < q < 1 \). Then
\[
q^{\alpha} \prod_{n=1}^{\infty} (1 - q^n)^{\chi(n)n^{k-2}} = \exp \left( -C - \int_{q}^{1} \left\{ \alpha - \sum_{n=1}^{\infty} \sum_{d|n} \chi(d)d^{k-1}t^n \right\} \frac{dt}{t} \right),
\]
where
\[
C = L'(2 - k, \chi).
\]
The purpose of this note is to establish an identity similar to this identity for holomorphic cusp forms. Namely, we will prove the following theorem:
Theorem 1.3 Let $F(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi inz}$ be a holomorphic cusp form. Then we have

$$q^\alpha \prod_{n=1}^{\infty} (1 - q^n)^{c(n)} = \exp \left( -C - \int_{q}^{1} \left\{ \alpha - \sum_{n=1}^{\infty} \sum_{d|n} dc(d)t^n \right\} \frac{dt}{t} \right),$$

where

$$C = \Lambda(0, F).$$

Here $\Lambda(s, F)$ is the completed L-function of the cusp form $F$.

2 Proof

The proof of Theorem 1.3 is identical to the proof of the main theorem of [1], except that it is easier. By taking logarithms of both sides of the equation, it is seen that we have

$$C = \lim_{t \to 1} \sum_{n=1}^{\infty} \sum_{d|n} dc(d)t^n.\frac{t^n}{n}.$$

We set

$$H(q) = \sum_{n=1}^{\infty} \sum_{d|n} dc(d)\frac{q^n}{n}$$

and

$$f(q) = \sum_{d=1}^{\infty} c(d)q^d.$$

It is then easy to see that

$$H(q) = \sum_{n=1}^{\infty} \frac{1}{n} f(q^n).$$
By definition,

\[ C = \lim_{q \to 1} H(q) \]

\[ = \lim_{M \to \infty} \frac{1}{M} \sum_{n=1}^{\infty} \frac{1}{n/M} f(e^{-\frac{n}{M}}) \]

\[ = \int_{0}^{\infty} f(e^{-y}) \frac{dy}{y} \]

\[ = \int_{0}^{\infty} F\left(\frac{iy}{2\pi}\right) \frac{dy}{y} \]

\[ = \int_{0}^{\infty} F(iy) \frac{dy}{y} \]

\[ = \Lambda(0, F). \]

**Remark 2.1** As in [1], we can replace \( c(n) \) with \( n^k c(n) \), and obtain similar identities. The only difference is that instead of the special value of the L-function at \( s = 0 \), we will have other special values multiplied by explicit constants.

**Remark 2.2** If we let \( u(n) \) be the coefficient of \( q^n \) in the expansion of \( H(q) \), then \( u(n) \) is given by

\[ L(s, F)L(s, \eta) = \sum_{n=1}^{\infty} \frac{u(n)}{n^s} \quad (2.1) \]

where \( L(s, F) \) is the finite part of the \( L \)-function of \( F \), and \( \eta : \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times \) is the unramified character \( \eta(\alpha) = |\alpha|^{-1}_A \). We observe that \( L(s, F)L(s, \eta) \) is the \( L \)-function of a GL(3) Eisenstein series. For this reason, our formula can be thought of the GL(2) analogue of the GL(1) formula of [1]. There is an obvious generalization to all GL(\( n \)).

**References**
