APPENDIX: NON-VANISHING OF SPECIAL VALUES OF RANKIN-SELBERG $L$-FUNCTIONS

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In this paper we establish several non-vanishing results for central values of Rankin-Selberg $L$-functions which are used in [1B]. We follow the methods of [KMV] where rather strong non-vanishing results were obtained (but not in the generality needed for the applications to [1B]). Let $g(z)$ be some primitive modular form (either an holomorphic or a Maass form of weight 0 or 1) of level $D$ and nebentypus $\chi_g$, we denote $\lambda_g(n)$, $n \geq 1$ the $n$-th (unitary normalized) Hecke eigenvalue of $g$. For $k \geq 2$, $q$ a prime number coprime with $D$ and $\chi$ a Dirichlet character of modulus $q$ such that $\chi(-1) = (-1)^k$, we denote by $S_k^\chi(q, \chi)$ the set of arithmetically normalized primitive holomorphic modular forms of weight $k$, level $q$ and nebentypus $\chi$: for $f \in S_k^\chi(q, \chi)$ we denote by $\lambda_f(n)$ the $n$-th (unitary normalized) Hecke eigenvalue: we have the fourier expansion

$$f(z) = \sum_{n \geq 1} \lambda_f(n)n^{\frac{k-1}{2}}e(nz).$$

Since $(q, D) = 1$, the Rankin-Selberg $L$-function is the dirichlet series

$$L(f \otimes g, s) = L(\chi, \chi, 2s)\sum_{n \geq 1} \frac{\lambda_f(n)\lambda_g(n)}{n^s}$$

this is an Euler product of degree four which has analytic continuation to $\mathbb{C}$ and which satisfies a functional equation of the form

$$\left(\frac{qD}{4\pi^2}\right)^s \Gamma_{g,k}(s)L(f \otimes g, s) = \varepsilon(f \otimes g)\left(\frac{qD}{4\pi^2}\right)^{1-s}\Gamma_{g,k}(1-s)L(f \otimes g, 1-s)$$

where $\Gamma_{g,k}(s)$ is the local factor at $\infty$, i.e. a product of Gamma functions depending only on the infinity type of $g$ and on $k$ (see [KMV]), and $\varepsilon(f \otimes g)$ is a complex number of modulus 1 (the root number) given by (see [KMV] (4.4) or [Li2] Thm. 2.2 and Example 2.)

$$\varepsilon(f \otimes g) = \begin{cases} 
\chi_D(-q)\chi_q(D)\eta_f(q)^2\eta_g(D)^2 = \varepsilon(g)\eta_f(q)^2 \ (\text{say}) & \text{if } g \text{ is holomorphic and } k' \geq k, \\
\chi_D(q)\chi_q(-D)\eta_f(q)^2\eta_g(D)^2 = \varepsilon(g)\eta_f(q)^2 \ (\text{say}) & \text{else.}
\end{cases}$$

Here $\eta_f(q)$, $\eta_g(D)$ are the pseudo-eigenvalues of $f$, $g$ for the Atkin-Lehner-Li operators $W_q, W_D$. In particular since $q$ is squarefree, one has ([KMV] A.4.)

$$\eta_f(q) = \sqrt{q}G_{\chi},$$

where $G_{\chi}$ is the Gauss sum (with value $-1$ when $\chi$ is trivial). In particular when $\chi$ is trivial, $\varepsilon(f \otimes g) = \varepsilon(g)$ does not depend on $f$ inside $S_k^\chi(q, \chi)$. We prove here following

**Theorem 0.1.** With the notations above, assume that either

- $k \geq 3$ or,
- $k \geq 2$, $\chi$ is the trivial character and $\varepsilon(g) \neq -1$ whenever $g$ has real Hecke eigenvalues (in which case $x_{\chi}$ is real),

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then for \( q \) a sufficiently large prime (large depending on \( g \) and \( k \)) there exists \( f \in S_k^*(q, \chi) \) such that \( L(f \otimes g, 1/2) \neq 0 \).

**Remark.** In the case where \( \chi \) is trivial, \( g \) has real Hecke eigenvalues and \( \varepsilon(g) = -1 \), \( L(f \otimes g, 1/2) = 0 \) but one can prove that there exists an \( f \in S_k^*(q, \chi) \) such that \( L(f \otimes g, 1/2) \neq 0 \).

**Remark.** In fact it is not difficult to show that under the above hypotheses the number of such \( f \) is \( \gg \varepsilon , g, k \) \( q^{1-\varepsilon} \) for every \( \varepsilon > 0 \). With more work, mollifications methods (see [KMV] for instance) should provide a positive proportion of \( f \) satisfying \( L(f \otimes g, 1/2) \neq 0 \).

**Remark.** With more work (ie. using the methods of [KMV] and the refined Voronoi summation formula of [M] Appendix) one should be able to prove the above non-vanishing result when \( k = 2 \) and \( \chi \) non trivial as well.

**Proof.** As usual, the proof uses Petersson summation formula: let \( B_k(q) \) denote the space of cuspforms of weight \( k \) level \( q \) and nebentypus \( \chi \) and let \( B_k(q, \chi) \) be an orthogonal basis of \( S_k(q, \chi) \); we denote by \( \psi_f(n) \) the \( n \)-th Fourier coefficient of \( f \); we have

\[
\sum_{f \in B_k(q)} \psi_f(m) \overline{\psi_f}(n) := \frac{\Gamma(k - 1)}{(4\pi)^{k-1}} \sum_{f \in B_k(q, \chi)} \frac{\psi_f(m) \overline{\psi_f}(n)}{\langle f, f \rangle} = \delta_{m,n} + \Delta(m,n)
\]

with

\[
\Delta(m,n) := 2\pi i^{-k} \sum_{c \equiv 0(q)} S(m,n;c) J_{k-1}(\frac{4\pi \sqrt{mn}}{c})
\]

and \( S(m,n;c) \) the (twisted) Kloostermann sum

\[
S_\chi(m,n;c) := \sum_{x(c) \equiv 1} \chi(x)e(\frac{mx + n\pi}{c}).
\]

**Remark.** When \( \chi \) is primitive or when \( k < 12 \) there are no old forms and an orthogonal basis is given by \( B_k(q) = S_k^*(q, \chi) \). This is the case.

We will also use the following large sieve type inequality (see [KMV] 5.1.1 for comments about the proof)

**Proposition 0.2.** Let \( k \geq 2 \) be an integer. For \( \eta \) a smooth function supported in \([C, 2C]\) such that \( \eta^{(j)} \ll_j C^{-j} \) for all \( j \geq 0 \), set

\[
\Delta_\eta(m,n) := 2\pi i^{-k} \sum_{c \equiv 0(q)} S(m,n;c) J_{k-1}(\frac{4\pi \sqrt{mn}}{c}) \eta(c).
\]

Then for any sequences of complex numbers \( x_m, y_n \),

\[
\sum_{m \leq M} \sum_{n \leq N} x_m y_n \Delta_\eta(m,n) \ll_{\varepsilon, \eta} C^{k}(\frac{\sqrt{MN}}{C})^k \left(1 + \frac{M}{q}\right)^{1/2} \left(1 + \frac{N}{q}\right)^{1/2} ||x||_2 ||y||_2
\]

with any \( \varepsilon > 0 \).

Using contour shifts and the functional equations of \( L(f \otimes g, s) \) one express \( L(f \otimes g, 1/2) \) in terms of rapidly converging series (the ”approximate functional equation” in asymptotic form): for any \( X \geq 1 \)

\[
L(f \otimes g, 1/2) = \sum_{n \geq 1} \frac{\lambda_f(n) \lambda_g(n)}{n^{1/2}} V_{\chi \chi}(\frac{4\pi^2 n}{qD}) + \varepsilon(f \otimes g) \sum_{n \geq 1} \frac{\lambda_f(n) \lambda_g(n)}{n^{1/2}} V_{\chi \chi}(\frac{4\pi^2 n}{qD})
\]

\[
= \sum_{n \geq 1} \frac{\lambda_f(n) \lambda_g(n)}{n^{1/2}} V_{\chi \chi}(\frac{4\pi^2 n}{qD}) + \varepsilon(g) \sqrt{\frac{q}{q'}} \sum_{n \geq 1} \frac{\lambda_f(q^2 n) \lambda_g(n)}{n^{1/2}} V_{\chi \chi}(\frac{4\pi^2 n}{qD})
\]
Collecting the above estimates, we see that since eventually we find that the first term of \( M \) is not trivial and (We apply Petersson’s formula; from the first term we obtain a diagonal contribution given by (since \( \psi > 0 \) for all \( \epsilon > 0 \) for any \( \epsilon > 0 \) for every \( \epsilon > 0 \) the implied constant depending on \( \epsilon \) and \( g \); here \( \hat{q} \) denotes the conductor of \( \chi \). For simplicity we prove the theorem only in the case where, either \( \chi \) is not trivial or \( 2 \leq k < 12 \). In this case we may take \( B_k(q, \chi) = S_k(q, \chi) \) and we have \( \lambda_f(n) = \psi_f(n) \) We refer to the end of the paper for how to the remaining cases.

1. The case \( \chi \) non-trivial

We show that for \( q \) large enough and \( k \geq 3 \) we have

\[
M_k(q, \chi, 1) = \sum_{f \in S_k(q, \chi)} b \cdot L(f \otimes g, 1/2) \gg_g \delta q^{-1/2}
\]

for every \( \delta > 0 \). By (0.3) \( M_k(q, \chi, 1) \) is the sum of two terms:

\[
\sum_{f \in B_k(q, \chi)} b \cdot \psi_f(1) \sum_{n \geq 1} \psi_f(n) \lambda_g(n) n^{1/2} V_{\chi \chi}(\frac{n}{qDX})
\]

(since \( \psi_f(1) = \lambda_f(1) = 1 \)) and

\[
\epsilon(q) \frac{\hat{q}}{C\chi} \sum_{f \in B_k(q, \chi)} b \cdot \psi_f(1) \sum_{n \geq 1} \psi_f(q^n) \lambda_g(n) n^{1/2} V_{\chi \chi}(\frac{n}{qDX}).
\]

We apply Petersson’s formula; from the first term we obtain a diagonal contribution given by (since \( \chi \) is not trivial and \( (q, D) = 1, \chi \chi \) is not trivial)

\[
V_{\chi \chi}(1/qDX) = L(\chi \chi, 1) + O_g, k, \epsilon(q^{-1/4 + 1/2} X^{1/4})
\]

and an error term to which we apply the large sieve inequality (0.2) and the bound

\[
\sum_{n \geq 1} \frac{\mid \lambda_g(n) \mid^2}{n} |V_g(\frac{n}{qDX})|^2 \ll_g, k, \epsilon \left( (qX) \epsilon \right.
\]

eventually we find that the first term of \( M_k(q, \chi, 1) \) is given by

\[
L(\chi \chi, 1) + O_g, k, \epsilon(q^{-1/4 + 1/2} X^{1/4}) + O_g, k, \epsilon(\epsilon(qX)^{(k-3)/2} X^{1/2})
\]

for any \( \epsilon > 0 \). In the second term, there is no diagonal contribution and we apply Weil’s bound for Kloosterman sums and the bound \( J_{k-1}(y) \ll_k y \) to bound the latter by

\[
O_g, k, \epsilon(\epsilon(qX)^{1/2} X^{1/2}).
\]

Collecting the above estimates, we see that since \( k \geq 3 \) we may choose \( X \) of the form \( X = q^\Delta \) for some \( 1/2 < \Delta < 1 \) such that

\[
M_k(q, \chi, 1) = L(\chi \chi, 1) + O_g(q^{-\Delta})
\]
for some absolute $\delta > 0$, and the lower bound \((1.1)\) follows.

**Remark.** The choice of an asymmetric representation in \((0.3)\) was made in order to get a better control on the portion of $M_k(q, \chi)$ containing the root number $\varepsilon(f \otimes g)$.

2. The case $\chi$ trivial

In this case $\varepsilon(f \otimes g) = \varepsilon(g)$. We assume that $2 \leq k < 12$, and this time we evaluate the twisted moment

$$M_k(q, \chi, \ell) = \sum_{f \in S_k^+(q, \chi)} h \lambda_f(\ell)L(f \otimes g, 1/2),$$

for several integers $\ell < q$. We use now \((0.3)\) for $X = 1$, and using Petersson formula and \((0.2)\) we get

\[
M_k(q, \chi, \ell) = \res_{s=0} \frac{\Gamma_s(k)}{\Gamma_s(1/2)} \left( \frac{qD/4\pi^2}{s} L(q, \chi, 1 + 2s) \lambda_g(\ell) + \varepsilon(g)L(q, \chi, 1 + 2s) \lambda_g(\ell) \right) + O_{g,k,\varepsilon}(q^{(k-3)/2}).
\]

2.1. $g$ selfdual. If $g$ is self-dual, $\varepsilon(g) = \pm 1$, $\chi_g$ is real and taking $\ell = 1$ we obtain

\[
M_k(q, \chi, 1) = (1 + \varepsilon(g)) \res_{s=0} \frac{\Gamma_s(k)}{\Gamma_s(1/2)} \left( \frac{qD/4\pi^2}{s} L(q, \chi, 1 + 2s) + O_{g,k,\varepsilon}(q^{-\delta}) \right)
\]

for some absolute $\delta > 0$. Hence, unless $\varepsilon(g) = -1$ (in which case $L(f \otimes g, 1/2) = 0$ for all $f$ by the functionnal equation), we obtain \((1.1)\) in this case too.

2.2. $g$ non-self-dual. In this case $\chi_g$ is not trivial and we will need an extra trick. For $L \geq 1$ and $W$ some given non negative (non zero) smooth compactly supported function on $[0, +\infty]$ we form the average

$$\sum_{l \geq 1} W\left(\frac{l}{L}\right) \lambda_g(\ell) M_k(q, \chi, \ell) = \sum_{l \geq 1} W\left(\frac{l}{L}\right) \left[ L(q, \chi, 1) |\lambda_g(\ell)|^2 + \varepsilon(g)L(q, \chi, 1) \lambda_g(\ell)^2 \right] + O_{g,k,L}(q^{-\delta})$$

for some absolute $\delta > 0$. Since $g$ is not self-dual we have by the Rankin-Selberg theory that

$$\sum_{l \geq 1} W\left(\frac{l}{L}\right) |\lambda_g(\ell)|^2 = c_g \hat{W}(1)L + O_{g,W}(L^{1/2})$$

for some $c_g > 0$ and where $\hat{W}$ is the Mellin transform, while

$$\sum_{l \geq 1} W\left(\frac{l}{L}\right) \lambda_g(\ell)^2 = O_{g,W}(L^{1/2}).$$

In particular taking $L$ large enough depending on $g$ only we obtain that for $q$ large enough (depending on $g, W$ and $k$) we have

\[
\sum_{l \geq 1} W\left(\frac{l}{L}\right) \lambda_g(\ell) M_k(q, \chi, \ell) \geq \frac{1}{2} c_g L(q, \chi, 1) \hat{W}(1)L \neq 0.
\]

**Remark.** To conclude we say a few word in the case where $\chi$ is trivial and $k \geq 12$. In this case the space of old forms is not empty and $S_k^+(q, \chi)$ is not a full basis of $S_k(q, \chi)$. However one can form an orthogonal basis $B_k(q, \chi)$ by adding to $S_k^+(q, \chi)$ an explicit set of old eigenforms (coming from level 1) as in [ILS]. In order to apply Petersson’s formula to compute the $M_k(q, \chi, \ell)$ one needs to add the corresponding contribution from these old forms but these turn out to be negligible (ie. $O_{g,k,q^{-\delta}}$ for some $\delta > 0$) for trivial reasons. In particular the formulae \((2.2)\) and \((2.3)\) still hold.
References


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