# AUTOMORPHIC FORMS ON REDUCTIVE GROUPS

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## 1. Introduction

The goal of these notes is the basic theory of automorphic forms and reductive groups, up to and including the analytic continuation of Eisenstein series.

## 2. Notation

2.1. Let $X$ be a set and $f, g$ strictly positive real functions on $X$. We write $f \prec g$ if there exists a constant $c > 0$ such that $f(x) \leq cg(x)$ for all $x \in X$; similarly, $f \succ g$ if $g \prec f$, and $f \asymp g$ if $f \prec g$ and $g \prec f$.

2.2. Let $G$ be a group. The left (resp. right) translation by $g \in G$ is denoted $l_g$ (resp $r_g$); these act on functions via

(1) $$l_g \cdot f(x) = f(g^{-1}x)\,, \qquad r_g \cdot f(x) = f(xg)$$

2.3. Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. The latter may be viewed as the tangent space $T_1(G)$ at the identity, or as the space of left-invariant vector fields on $G$. If $X_1 \in T_1(G)$, the associated vector field is $x \mapsto x \cdot X_1$. The action of $X$ on functions is given by

$$Xf = \frac{d}{dt} f(xe^{tX})|_{t=0}$$

The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is identified with the algebra of left-invariant differential operators; the element $X_1X_2 \ldots X_n$ acts via

$$X_1X_2 \ldots X_n f(x) = \frac{d^n}{dt_1 \ldots dt_n} f(xe^{t_1X_1}e^{t_2X_2} \ldots e^{t_nX_n})|t_i = 0.$$

Let $\mathcal{Z}(\mathfrak{g})$ be the center of $\mathcal{U}(\mathfrak{g})$. If $G$ is connected, it is identified with the left and right invariant differential operators. If $G$ is connected and reductive, it is a polynomial algebra of rank equal to the rank of $G$.

2.4. Let $G$ be unimodular. The convolution $u \star v$ of two functions is defined by

$$u \star v(x) = \int_G u(xy)v(y^{-1})dy = \int_G u(y)v(y^{-1}x)dy \tag{2}$$

whenever the integral converges. It is a smoothing operator: if $u$ is continuous and $v \in C_c^\infty(G)$, then

$$D(u \star v) = u \star Dv, \quad D \in \mathcal{U}(\mathfrak{g}) \tag{3}$$

and, in particular, $u \star v \in C^\infty(G)$. It extends to distributions and is associative. If $\mathfrak{g}$ is identified to distributions with support $\{1\}$, then $Xf = f \star (-X)$; see section 2.2 of [6].

## 3. Notion of automorphic form

Let $G$ be a subgroup of finite index in the group of real points of a connected semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{R}$. Let $K$ be a maximal compact subgroup of $G$. Then $X = G/K$ is the Riemannian symmetric space of noncompact type of $G$. Let $\Gamma \subset G$ be a discrete subgroup.

3.1. A continuous function $f \in C(G; \mathbb{C})$ is an automorphic form for $\Gamma$ if it satisfes the following conditions:

(A1) $f(\gamma x) = f(x)$.
(A2) $f$ is $K$-finite on the right.
(A3) $f$ is $\mathcal{Z}(\mathfrak{g})$-finite.
(A4) $f$ is of moderate growth (or slowly increasing).

*Explanation* 1. $f$ is $K$-finite on the right right means that the set of right translates $r_k f$, $k \in K$ is contained in a finite dimensional space. $f$ is $\mathcal{Z}(\mathfrak{g})$-finite means that there exists an ideal $J$ of finite co-dimension in $\mathcal{Z}(\mathfrak{g})$ which annihilates $f$. If $f$ is not $C^\infty$, this is understood in the sense of distributions, but in any case $f$ will be analytic, cf. below. By definition, $G \subset \mathrm{SL}_N(\mathbb{R})$, and is closed. Let $\|g\|$ be the Hilbert-Schmidt norm of $g \in \mathrm{SL}_N(\mathbb{R})$. Thus $\|g\|^2 = \mathrm{tr}\,({}^t g.g) = \sum_{i,j} g_{ij}^2$. Then $f$ is of <u>moderate growth</u> or <u>slowly increasing</u> if there exists $m \in \mathbb{Z}$ such that

$$|f(x)| \prec \|x\|^m, (x \in G).$$

Let $\nu_m$ be the semi-norm on $C(G, \mathbb{C})$ defined by $\nu_m(f) = \sup f(x).\|x\|^{-m}$. Then $f$ is slowly increasing if and only if $\nu_m(f) < \infty$ for some $m$. We note some elementary properties of $\|\ \|$:

(n1) $\|x.y\| \leq \|x\|.\|y\|$, and there is an $m$ such that $\|x^{-1}\| \prec \|x\|^m$.
(n2) If $C \subset G$ is compact, $\|x.y\| \asymp \|y\|$ for $x \in C$, $y \in G$.

**Remark 3.1.** The notion of moderate growth (but not the exponent $m$) is independent of the embedding. One can also define a canonical Hilbert-Schmidt norm as follows: On $\mathfrak{g}$, let $K(x,y) = \mathrm{tr}\,(\mathrm{ad}\,x \circ \mathrm{ad}\,y)$ be the Killing form, and let $\theta$ be the Cartan involution of $G$ with respect to $K$. Then the form $(x,y) = -K(\theta x, y)$ is positive definite on $\mathfrak{g}$. Then the associated Hilbert-Schmidt norm on the adjoint group is $\|g\|^2 = \mathrm{tr}\,(\mathrm{Ad}\,\theta g^{-1}.\mathrm{Ad}\,g)$ (Exercise).

3.2. **Relation with classical automorphic forms on the upper half plane.** Here $G = \mathrm{SL}_2(\mathbb{R})$, $K = \mathrm{SO}_2$, and $X = \{z \in \mathbb{C} \mid \Im z > 0\}$, the action of $G$ being defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az+b}{cz+d}, (z \in X).$$

Let $(cz+d)^m = \mu(g,z)$. It is an automorphy factor, i.e.

$$\mu(g.g', z) = \mu(g, g'.z)\mu(g', z). \tag{4}$$

$K$ is the isotropy group of $i \in X$. Equation 4 gives for $k, k' \in K$, and $z = i$

$$\mu(kk', i) = \mu(k,i)\mu(k',i), \tag{5}$$

i.e. $k \mapsto \mu(k,i)$ is a character $\chi_m$ of $K$.

Let $\Gamma$ be a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$. An automorphic form $f$ on $X$ of weight $m$ is a function satisfying

(A1') $f(\gamma.z) = \mu(\gamma,z)f(z)$
(A2') $f$ is holomorphic
(A3') $f$ is regular at the cusps.

Let $\tilde{f}$ be the function on $G$ defined by

$$\tilde{f}(g) = \mu(g,i)^{-1}f(g.i).$$

Then (A1') for $f$ implies (A1) and (A2) for $\tilde{f}$ by a simple computation using 5. Note in particular

$$\tilde{f}(g.k) = \tilde{f}(g)\chi_{-m}(k). \tag{6}$$

The condition (A2') implies that $\tilde{f}$ is an eigenfunction of the Casimir operator $C$. As $C$ generates $\mathcal{Z}(\mathfrak{g})$, this yields (A3). Consider the cusp at $\infty$. In the inverse image of the "Siegel set" $|x| \leq c$ and $y > t$, where $c$ and $t$ are positive constants, it is easily seen that $\|g\| \asymp y^{\frac{1}{2}}$, hence moderate growth means $\prec y^m$ for some $m$. On the other hand, $\Gamma$ contains a translation $x \mapsto x + p$, for some non-zero integer $p$. Since $f$ is invariant under this translation in the $x$ direction, $f$ admits a development in a Laurent series $\sum_n a_n \exp(\frac{2\pi i n z}{p})$. For $f$ to be bounded by $y^m$ in the Siegel set, it is necessary and sufficient that $a_n = 0$ for $n < 0$. This is the regularity condition (A3'). (cf. [6], 5.14)

## 4. First properties of automorphic forms

In this section, $G$, $K$, $X$, and $\Gamma$ are as in 1.1, and $f$ is an automorphic form for $\Gamma$.

4.1. $f$ is analytic. For this it suffices, by a regularity theorem, to show that it is annihilated by an analytic elliptic operator.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form. The latter is negative (resp. positive) definite on $\mathfrak{k}$ (resp. $\mathfrak{p}$). Let $\{x_i\}$ (resp. $\{y_j\}$) be an orthonormal basis of $\mathfrak{k}$ (resp. $\mathfrak{p}$). Then a Casimir operator can be written $C_{\mathfrak{g}} = -\sum x_i^2 + \sum y_j^2$, and $C_{\mathfrak{k}} = \sum_i x_i^2$ is a Casimir operator for $\mathfrak{k}$. Let $\Omega = 2C_{\mathfrak{k}} + C_{\mathfrak{g}}$. It is an analytic elliptic operator. We claim that $f$ is annihilated by some non-constant polynomial in $\Omega$, which will prove our assertion. The function $f$ is annihilated by an ideal $J$ of finite codimension in $\mathcal{Z}(\mathfrak{g})$ (by (A3) ) and, since $f$ is $K$-finite on the right, it is annihilated by an ideal $\mathfrak{l}$ of finite codimension of $\mathcal{U}(\mathfrak{k})$. Therefore $f$ is annihilated by an idea of finite codimension of the subalgebra $\mathcal{Z}(\mathfrak{g}).\mathcal{U}(\mathfrak{k})$ of $\mathcal{U}(\mathfrak{g})$. But then there exists a polynomial of strictly positive degree $P(\Omega)$ in $\Omega$ belonging to that ideal.

4.2. A function $\alpha$ on $G$ is said to be $K$-invariant if $\alpha(k.x) = \alpha(x.k)$ for all $x \in G$, $k \in K$. We have the following theorem:

**Theorem.** *Given a neighborhood $U$ of 1 in $G$, there exists a $K$-invariant function $\alpha \in C_c^\infty(U)$ such that $f = f \star \alpha$.*

This follows from the fact that $f$ is $\mathcal{Z}$-finite and $K$-finite on one side, by a theorem of Harish-Chandra. See ([12], theorem 1) or ([1], 3.1), and for $\mathrm{SL}_2(\mathbb{R})$ ([6], 2.13).

4.3. A smooth function $u$ on $G$ is said to be of uniform moderate growth if there exists $m \in \mathbb{Z}$ such that $\nu_m(Df) < \infty$ for all $D \in \mathcal{U}(\mathfrak{g})$.

An elementary computation shows that if $\nu_m(u) < \infty$, then

$$\nu_m(u \star \alpha) < \infty$$

for any $\alpha \in C_c^\infty(G)$. Since, $D(f \star \alpha) = f \star D\alpha$, the previous theorem implies that

**Corollary.** *An automorphic form is of uniform moderate growth. More generally, if $f$ has moderate growth, then $f \star \alpha$ is of uniform moderate growth.*

4.4. We intercalate some facts needed in the sequel. Since $G$ is a closed subgroup of $\mathrm{SL}_N(\mathbb{R})$, it is clear that $\|.\|$ has a strictly positive minimum, say $t_0$ on $G$. Fix a Haar measure on $G$. For $t \geq t_0$, let

$$G_t = \{g \in G \mid \|g\| \leq t\}. \tag{7}$$

There there exists $m \in \mathbb{N}$ such that

$$\mathrm{vol}\,(G_t) \prec t^m\,, (t \geq t_0). \tag{8}$$

(cf Lemma 37 of [13]). The proof will be sketched later, once we have reviewed some structure theory of reductive groups. Also,

$$\#(\Gamma \cap G_t) \prec t^m. \tag{9}$$

To see this, fix a compact neighborhood $C$ of 1 such that

$$\Gamma \cap C.C^{-1} = \{1\}.$$

In view of (n1), there exists a constant $d > 0$ such that

$$\|x\| \leq d\|\gamma\|\,, (\gamma \in \Gamma, x \in C.\gamma).$$

So

$$\bigcup_{\gamma \in G_t \cap \Gamma} C\gamma \subset G_{dt}, \quad \text{(disjoint union)},$$

whence

$$\mathrm{vol}\, C.\#(\Gamma \cap G_t) \prec \mathrm{vol}\, G_{dt} \prec t^m,$$

which proves (9).

If $C$ is a compact subset of $G$, then

$$\#(\Gamma \cap xCy) \prec \|x\|^m.\|y\|^m, \quad (x, y \in G). \tag{10}$$

The property (n1) of $\|.\|$ with respect to product implies the existence of a constant $c > 0$ such that $xCy \subset G_{c\|x\|.\|y\|}$, so that (9) implies (10).

4.5. We have the following lemma:

**Lemma.** *Let $\alpha \in C_c^\infty(G)$. There exists $n \in \mathbb{N}$ such that $|u \star \alpha(x)| \prec \|x\|^n \|u\|_1$.*

([13], Lemma 8 and corollary). We sketch the argument. We have

$$|u \star \alpha(x)| \leq \int_G dy|u(y)||\alpha(y^{-1}x)| = \int_{\Gamma\backslash G} |u(y)| \sum_\gamma |\alpha(y^{-1}\gamma^{-1}x)|dy \tag{11}$$

so that it suffices to show the existence of $N$ such that

$$\sum_\gamma |\alpha(y^{-1}\gamma x)| \prec ||x||^N \tag{12}$$

We may assume that $\alpha$ is the characteristic function of some compact symmetric set $C$. Then the sum on the left of 12 is equal to

$\#(\Gamma \cap yCx^{-1})$. Fix $\delta$ in that set. Then, for any $\gamma$ in it, we have $\delta^{-1}\gamma \in xC^{-1}y^{-1}yCx^{-1} = xC^2x^{-1}$, so that the left hand side of 12 is equal to $\#\Gamma \cap xC^2x^{-1}$, and our assertion follows from 10 and (n1).

**Proposition 4.1.** *Assume that* $\mathrm{vol}(\Gamma\backslash G) < \infty$. *Let* $u$ *be a function on* $G$ *which satisfies (A1), (A2), (A3) and belongs to* $L^p(\Gamma\backslash G)$ *for some* $p \geq 1$. *Then* $f$ *has moderate growth, i.e. is an automorphic form.*

*Proof.* Since $\Gamma\backslash G$ has finite volume, $L^p(\Gamma\backslash G) \subset L^1(\Gamma\backslash G)$, so we may assume $p = 1$. By (A2) and (A3), there exists $\alpha \in C_c^\infty(G)$ so that $u = u \star \alpha$. The proposition now follows from 4.5. □

4.6. We now recall some formalism to describe more precisely the notion of a $K$-finite function. Let $dk$ be the Haar measure on $K$ of mass 1. As usual, let $\hat{K}$ be the set of isomorphism classes of (finite dimensional) continuous irreducible representations of $K$. For $\nu \in \hat{K}$, let $d(\nu)$ be its degree, $\chi_\nu$ its character, and $e_\nu = d(\nu)\chi_\nu dk$, viewed as a measure on $G$ with support on $K$. Let $u \in C(G)$ be $K$-finite on the right, and more precisely, belonging to an irreducible $K$-module, under right translations, of type $\nu$. Then we leave it as an exercise to deduce from the Schur orthogonality relations that:

$$u \star e_\mu = \begin{cases} u & \mu = \nu \\ 0 & \mu \neq \nu \end{cases} \tag{13}$$

and therefore $e_\nu$ is an idempotent, a projector of $C(G)$ on the isotypic subspace of type $\nu$, and we have $e_\mu \star e_\nu = 0$ if $\mu \neq \nu$. Consequently, $u$ is $K$-finite on the right if and only if there exists an idempotent $\xi$ which is a finite sum of $e_\nu$, such that $u \star \xi = u$. The element $\xi$ will be called a standard idempotent.

Assuming (A3) expressed in this way, we let $\mathcal{A}(\Gamma, J, \xi)$ be the space of automorphic forms such that $J.f = 0$ and $f \star \xi = f$, where, as before, $J$ is an ideal of finite codimension of $\mathcal{Z}(\mathfrak{g})$.

We shall see that if $\Gamma$ is arithmetic, then $\mathcal{A}(\Gamma, J, \xi)$ is finite dimensional. This theorem is due to Harish-Chandra.

## 5. Reductive groups (review)

We review here what is needed in this course. The catch-words are split tori, roots and parabolic subgroups. We first start with an example.

5.1. $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{R})$. Let $A$ be the subgroup of diagonal matrices with strictly positive entries and $\mathfrak{a}$ its Lie algebra. The exponential is an isomorphism of $\mathfrak{a}$ onto $A$, with inverse the logarithm. Let $X(A)$ be the smooth homomorphisms of $A$ into $\mathbb{R}^*_+$. IF $\lambda \in X(A)$, we denote by $a^\lambda$ the value of $\lambda$ on $a$. Let $\dot\lambda$ be the differential of $\lambda$ at 1. It is a linear form on $\mathfrak{a}$ and we can also write $a^\lambda = \exp(\dot\lambda(\log(a)))$, a notation which is often used in representation theory. Note that $\lambda \mapsto \dot\lambda$ is an isomorphism of $X(A)$ onto $\mathfrak{a}^*$.

Let $\lambda_i$ be the character which associates to $a$ its $i$th coordinate. The $\lambda_i$ span a lattice in $X(A)$ to be denoted $X(A)_\mathbb{Z}$. Its elements are therefore the characters $a \mapsto a_1^{m_1} a_2^{m_2} \dots a_n^{m_n}$, for $m_i \in \mathbb{Z}$.

[Interpretation: Let $T = (\mathbb{C}^*)^n$, and let $X^\star(T)$ be the group of rational morphisms of $T$ into $\mathbb{C}^*$. It is a free abelian group of rank $n$ and $X(A)_\mathbb{Z}$ is the restriction of $X^\star(T)$ to $A$. In particular, an element of $X(A)_\mathbb{Z}$ extends canonically to $T$.]

5.1.1. *Roots.* For $\beta \in X(A)$, let

$$\mathfrak{g}_\beta = \{x \in \mathfrak{g} : \mathrm{Ad}\,(a).x = a^\beta x\} \tag{14}$$

$\beta$ is called a root if it is nonzero and $\mathfrak{g}_\beta \neq 0$. If so, it is immediately checked that there exist $i, j \leq n$, $i \neq j$, such that $g_\beta$ is one-dimensional, spanned by the matrix $e_{ij}$ with entry 1 at the $(i,j)$ spot and zero elsewhere, and that $a^\beta = a_i/a_j$, i.e. $\beta = \lambda_i - \lambda_j$.

We let $\Phi = \Phi(A, G) = \Phi(\mathfrak{a}, \mathfrak{g})$ be the set of roots of $G$ with respect to $A$ (their differentials are the roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$). They span a lattice of rank $n-1$ and form a root system of type $\mathbf{A_{n-1}}$. We have

$$\mathfrak{g} = \mathfrak{a} \oplus \oplus_{\beta\in\Phi}\mathfrak{g}_\beta \tag{15}$$

Fix the ordering on $X(A)$ defined by $\lambda_1 > \lambda_2 > \dots$ and let $\Delta = \{\alpha_i\}_{1\leq i\leq n-1}$, where $\alpha_i = \lambda_i - \lambda_{i+1}$. The elements of $\Delta$ are the *simple roots* (for the given ordering). Any root is a linear combination of the $\alpha_i$ with integral coefficients of the same sign. The Weyl group $W(\Phi)$ of the root system may be identified to

$$W(\Phi) = \mathcal{N}(A)/\mathcal{Z}(A) \tag{16}$$

Note that:

$$\mathcal{Z}(A) = M \times A, \quad M = (\mathbb{Z}/2\mathbb{Z})^n \tag{17}$$

$$W(\Phi) = S_n \ \text{ where } S_n \text{ is the symmetric group on } n \text{ letters} \tag{18}$$

5.1.2. *Parabolic subgroups.* The general definition of parabolic subgroups will be recalled in 5.2. For $\mathrm{GL}_n$ or $\mathrm{SL}_n$ they are the stability groups of flags, and we take this as a definition. A flag is an increasing sequence $F$

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{s-1} \subset V_s = \mathbb{R}^n \quad \text{of subspaces of } \mathbb{R}^n. \tag{19}$$

It is conjugate to a standard flag in which the $V_i$ are coordinate subspaces. Let $n_i = \dim V_i/V_{i-1}$. Then $n = n_1 + n_2 + \cdots + n_s$, and $\dim V_i = n_1 + \cdots + n_i$. The stabilizer $P$ of $F$ is the group of matrices which are "block triangular"

$$\begin{pmatrix} A_1 & * & * & * \\ 0 & A_2 & * & * \\ 0 & 0 & \dots & * \\ 0 & 0 & 0 & A_s \end{pmatrix}, \tag{20}$$

$A_i \in \mathrm{GL}_{n_i}$. $P$ is a semi-direct product $P = L_P.N_P$ where $N_P$ is the "unipotent radical" and consists of upper triangular matrices with $I$ in the blocks and $L_P$ is reductive, and equal to $\mathrm{GL}_{n_1}(\mathbb{R}) \times \mathrm{GL}_{n_2}(\mathbb{R}) \times \cdots \times \mathrm{GL}_{n_s}(\mathbb{R})$. Let $A_P$ be the intersection of $A$ with the center of $L_P$. It consists of diagonal matrices which are scalar multiples $c_i I_{n_i}$ of the identity in the $i^{\text{th}}$ block. We have

$$L_P = M_P \times A_P \tag{21}$$

where $M_P$ consists of matrices $(g_1, \dots, g_s)$ with $g_i \in \mathrm{GL}_{n_i}(\mathbb{R})$ of determinant $\pm 1$.

Given $J \subset \Delta$, we let $A_J = \cap_{\alpha \in J} \ker \alpha$. Then we have

$$A_P = A_J \quad \text{where } J = \Delta - \{\alpha_{n_1}, \alpha_{n_1+n_2}, \dots, \alpha_{n_1+\cdots+n_{s-1}}\}. \tag{22}$$

We shall also write $P_J$ for the present $P$. Thus $P_\emptyset$ is the group of upper triangular matrices and $P_\Delta = G$. We shall also write $N_J$ and $M_J$ for $N_P$ and $M_P$. Note that $L_J = \mathcal{Z}(A_J)$. Thus we have $P_J = M_J A_J N_J$ where $M_J A_J = \mathcal{Z}(A_J)$. The Lie algebra of $M_J$ (resp. $N_J$) is spanned by the $\mathfrak{g}_\beta$ ($\beta$ linear combination of elements of $J$) (resp. $\beta > 0$, not in the span of $J$).

This was all for $\mathrm{GL}_n(\mathbb{R})$. One gets the similar objects for $\mathrm{SL}_n(\mathbb{R})$ by taking subgroups of elements of determinant 1.

5.2. In this subsection, we review some general facts about linear algebraic groups and in the next one we specialize to the main case of interest in this course. See [1](a) for a more extended survey and [5], [15] for a systematic exposition. $F$ is a field (commutative as usual) and $\tilde{F}$ an algebraically closed extension of infinite transcendence degree over its prime subfield.

5.2.1. The group $\mathbf{G} \subset \mathrm{GL}_n(\tilde{F})$ is algebraic if there exists a set of polynomials $P_i\,(i \in I)$ in $n^2$ variables with coefficients in $\tilde{F}$, such that

$$\mathbf{G} = \{g = (g_{ij}) \in \mathrm{GL}_n(\tilde{F}), P_i(g_{11}, g_{12}, \ldots, g_{nn}) = 0, (i \in I)\}. \tag{23}$$

It is defined over $F$ if the ideal of polynomials in $\tilde{F}[X_{11}, X_{12}, \ldots, X_{nn}]$ vanishing on $\mathbf{G}$ is generated by elements with coefficiens in $F$. Its coordinate ring, or ring of regular functions (resp. defined over $F$) is the ring generated over $\tilde{F}$ (resp. $F$) by the $g_{ij}$ and $(\det g)^{-1}$. We also say that $\mathbf{G}$ is an $F$-group if it defined over $F$. For any extension $F'$ of $F$ in $\tilde{F}$, we let $\mathbf{G}(F')$ be the subgroup of elements in $\mathrm{GL}_n(F')$.

If $\mathbf{G}'$ is another $F$-group, a morphism $q : \mathbf{G} \to \mathbf{G}'$ is a group homomorphism such that $q^\circ : f \mapsto q \circ f$ maps $\tilde{F}[\mathbf{G}']$ into $\tilde{F}[\mathbf{G}]$. It is defined over $F$ if $q^\circ$ maps $F[\mathbf{G}']$ into $F[\mathbf{G}]$. We let $X^\star(\mathbf{G})$ be the group of morphisms of $\mathbf{G}$ into $\tilde{F}^\times$. If $\lambda \in X^\star(\mathbf{G})$ is defined over $F$, it maps $\mathbf{G}(F)$ into $F^\times$.

5.2.2. Recall that any $g \in \mathrm{GL}_n(\tilde{F})$ admits a unique (multiplicative) Jordan decomposition

$$g = g_s g_u, \tag{24}$$

with $g_s$ semi-simple, $g_u$ unipotent (all eigenvalues equal to 1), such that $g_s g_u = g_u g_s$. if $g \in \mathbf{G}$, then so are $g_s$, $g_u$. This decomposition is compatible with morphisms of algebraic groups.

5.2.3. The algebraic group $\mathbf{T}$ is a torus (or an algebraic torus) if it is connected (as an affine variety) and consists of semi-simple elements. It is then commutative, isomorphic to a product of groups $\tilde{F}^\times$ and diagonalizable. The group $X^\star(\mathbf{T})$ is free abelian, of rank equal to the $\dim \mathbf{T} = n$. Any elements of $X^\star(\mathbf{T})$ is of the form $t \mapsto t_1^{m_1} t_2^{m_2} \ldots t_n^{m_n}$ ($m_i \in \mathbb{Z}$). Assume it is defined over $F$. Let $X^\star(\mathbf{T})_{(F)}$ be the subgroup of characters defined over $F$. The group $\mathbf{T}$ is said to split (resp. anisotropic) over $F$, if $X^\star(\mathbf{T})_{(F)} = X^\star(\mathbf{T})$ (resp. $X^\star(\mathbf{T})_{(F)} = \{1\}$). In general, the group $\mathbf{T}$ can be written as an almost direct product

$$\mathbf{T} = \mathbf{T}_{\mathrm{sp}}.\mathbf{T}_{\mathrm{an}} \quad (\mathbf{T}_{\mathrm{sp}} \cap \mathbf{T}_{\mathrm{an}} \text{ finite }),$$

where $\mathbf{T}_{\mathrm{sp}}$ (resp. $\mathbf{T}_{\mathrm{an}}$) is split (resp. anisotropic) over $F$.

If $F = \mathbb{R}$, $\mathbf{T}_{\mathrm{an}}$ is a torus in the usual topological sense (product of circles).

5.2.4. We use the language of Zariski-topology on $\mathbf{G}$. In particular, a subgroup $\mathbf{H}$ of $\mathbf{G}$ is closed if and only if it is algebraic. If so, $\mathbf{G}/\mathbf{H}$ admits a canonical structure of algebraic variety with a universal property: any morphism of $\mathbf{G}$ into an algebraic variety which is constant on the left cosets $x\mathbf{H}$ can be factored through $\mathbf{G}/\mathbf{H}$.

5.2.5. Let $\mathbf{G}$ be connected. A closed subgroup $\mathbf{P}$ is parabolic if $\mathbf{G}/\mathbf{P}$ is a projective variety. The radical $\mathcal{R}\mathbf{G}$ (resp. unipotent radical $\mathcal{R}_u\mathbf{G}$) is the greatest connected normal solvable (resp. unipotent, i.e. consisting of unipotent matrices) subgroup of $\mathbf{G}$. The quotient $\mathbf{G}/\mathcal{R}\mathbf{G}$ (resp. $\mathbf{G}/\mathcal{R}_u\mathbf{G}$) is semisimple (resp. reductive, i.e. almost direct product of a semisimple group and a torus). If $F$ has characteristic zero, the maximal reductive $F$-subgroups of $\mathbf{G}$ are conjugate under $\mathcal{R}_u\mathbf{G}(F)$ and $\mathbf{G}$ is the semi-direct product of $\mathcal{R}_u\mathbf{G}$ by any one of them.

5.3. From now on, $F$ is a subfield of $\mathbb{R}$ (mostly $\mathbb{Q}$ or $\mathbb{R}$) and $\tilde{F} = \mathbb{C}$. Our algebraic groups are then also complex Lie groups. If $\mathbf{G}$ is defined over $\mathbb{R}$, then $\mathbf{G}(\mathbb{R})$ is a real Lie group. We assume familiarity with Lie theory and use both points of view, transcendental (based on the topology inherited from that of $\mathbb{R}$ or $\mathbb{C}$) and algebraic (based on the Zariski topology). The emphasis being here on real Lie groups, we shall deviate on one point from the notation in 5.2.1 by denoting the real points of the $\mathbb{R}$-group $\mathbf{G}$ by the corresponding ordinary capital letter, thus writing $G$ rather than $\mathbf{G}(\mathbb{R})$.

5.3.1. Let $\mathbf{G}$ be a connected reductive $F$-group. Let $\mathbf{S} = \mathcal{Z}(\mathbf{G})^\circ$ be the identity component of its center. It is a torus defined over $F$. We denote by $A_{\mathbf{G}}$ the identity component (ordinary topology) of $\mathbf{S}_{\rm sp}(\mathbb{R})$.

Let $\lambda \in X^\star(\mathbf{G})_{(F)}$. It maps $G$ into $\mathbb{R}^\times$. We let $|\lambda|$ be the composition of $\lambda$ with the absolute value. It maps $G$ into $\mathbb{R}^\times_+$. Let ${}^\circ G$ be the intersection of the kernels of the $|\lambda|$, $\lambda \in X^\star(\mathbf{G})_{(F)}$. Then we have a direct product decomposition

$$G = {}^\circ G \times A_{\mathbf{G}}. \tag{25}$$

This follows from the fact that the restriction to $\mathbf{S}_{\rm sp}$ maps $X^\star(\mathbf{G})_{(F)}$ onto a subgroup of finite index of $X^\star(\mathbf{S}_{\rm sp})$. Note that $A_{\mathbf{G}}$ depends on $F$, which is why we index with $\mathbf{G}$ rather than $G$. The group ${}^\circ G$ contains the derived group of $G$, $S_{\rm an}$ and every compact subgroup of $G$.

5.3.2. The maximal $F$-split tori of $\mathbf{G}$ are conjugate under $\mathbf{G}(F)$. Fix one, say $\mathbf{S}$ and let ${}_FA$, or simply $A$ if $F$ is understood, be the identity component of $\mathbf{S}(\mathbb{R})$. It can be diagonalized over $F$. In the case of $\mathrm{GL}_n(\mathbb{R})$, it may be identified with the group so denoted in 5.1. Let $X(A)$ and $X^\star(A)$ be identified as there. For $\beta \in X^\star(A)$, we define $\mathfrak{g}_\beta$ as in 5.1 and call it a root or $F$-root if it is non-zero and $\mathfrak{g}_\beta \neq 0$. The roots form a root system ${}_F\Phi(A, G)$ (in the sense of Bourbaki) in the subspace of $X(A)$ they generate, which can be identified with $X(A/A_G)$. It is not empty if and only if $A \neq A_G$. Unlike in 5.1, the $\mathfrak{g}_\beta$ need not be one-dimensional and $A$ is not in general of finite index in its normalizer. By (25), we can write

$$\mathcal{Z}(A) = M \times A, \quad \text{where } M = {}^\circ\mathcal{Z}(A). \tag{26}$$

The Weyl group $W = W(A; G)$ of $G$ with respect to $A$ is again $\mathcal{N}(A)/\mathcal{Z}(A)$. The equality (15) is replaced by

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \oplus_{\beta \in {}_F\Phi} \mathfrak{g}_\beta. \tag{27}$$

5.3.3. The maximal tori of $\mathbf{G}$ are its Cartan subgroups. $\mathbf{G}$ is said to be split over $F$ if it has a maximal torus (they always exist) split over $F$. The group $\mathbf{G}$ is anisotropic over $F$ if it does not contain any $F$-split torus of strictly positive dimension and isotropic otherwise. For instance, the group $M$ in (26), or rather its complexification, is anisotropic over $F$. Note that $G$ and $\mathcal{D}G$ have the same $F$-root system, which is empty if and only if $\mathcal{D}G$ is anisotropic over $F$.

5.3.4. The parabolic $F$-subgroup of $G$ are, by definition, the groups of real points of the parabolic $F$-subgroups of $\mathbf{G}$. There exist proper ones if and only if $\mathcal{D}G$ is isotropic. Fix an ordering on ${}_F\Phi$, let ${}_F\Phi^+$ be the set of positive $F$-roots and ${}_F\Delta$ the set of simple $F$-roots. Given $J \subset {}_F\Delta$, let as in 5.1.2

$$A_J = \cap_{\alpha \in J} \ker \alpha. \tag{28}$$

Then the parabolic subgroup $P_J$ or ${}_FP_J$ is generated by $\mathcal{Z}(A_J)$ and the subgroup $N$ with Lie algebra $\oplus_{\beta > 0} \mathfrak{g}_\beta$. It admits the semidirect decomposition

$$P_J = L_J . N_J, \tag{29}$$

where $L_J = \mathcal{Z}(A_J)$. Let $\Phi_J$ be the set of roots which are linear combinations of elements in $J$. The group $N_J$ is the unipotent radical of $P_J$. It has Lie algebra

$$\mathfrak{n}_J = \oplus_{\beta > 0, \beta \notin \Phi_J} \mathfrak{g}_\beta. \tag{30}$$

The Lie algebra $\mathfrak{l}_J$ of $L_J$ is sum of the $\mathfrak{g}_\beta$ $(\beta \in \Phi_J)$ and of $\mathfrak{m} \oplus \mathfrak{a}$. This describes the identity component $L_J^\circ$ of $L_J$, but $L_J$ is not necessarily connected. It is generated by $L_J^\circ$ and $\mathcal{Z}(A)$. By applying (26) we get

$$L_J = M_J \times A_J, \quad \text{where } M_J = {}^\circ L_J. \tag{31}$$

If $P$ is defined over $F$, we denote $L_P$, $A_P$, $M_P$ the corresponding data. We have therefore the decompositions

$$P = N_P.A_P.M_P \ \text{ (semidirect)} \tag{32}$$

$$G = N_P.A_P.M_P.K \tag{33}$$

If we write $g = n.a.m.k$ $(n \in N_P, a \in A_P, m \in M_P, k \in K)$, then $n$ and $a$ are uniquely determined and will be denoted $n(g)$, $a(g)$, whereas $m$ and $k$ are determined up to an element of $M_P \cap K$.

5.3.5. *Langlands decomposition.* In (32), (33), $N_P$ is uniquely determined, but $M_P$ and $A_P$ are determined only up to conjugacy. The group $K$ being fixed once and for all, it is customary to normalize the choice of $L_{\mathbf{P}}$, $A_{\mathbf{P}}$, $M_{\mathbf{P}}$ be requesting them to be invariant under the Cartan involution $\theta = \theta_K$ having $K$ as its fixed point set. This determines them uniquely. Note that by doing so, one usually drops the requirement that they be defined over $F$. However, the projection $\mathbf{P} \to \mathbf{P}/\mathbf{N}_P = \mathbf{L}_P$ is defined over $F$ and induces an isomorphism of any Levi subgroup of $\mathbf{P}$ onto $\mathbf{L}_P$, so that the notions defined over $F$ in $\mathbf{L}_P$ can be transported to any subgroup. The decomposition (32), (33) so normalized are called Langlands decompositions. Note that if $P \subset P'$, then $A_{P'} \subset A_P$, $M_P \subset M_{P'}$ (besides $N_{P'} \subset N_P$ which is true regardless of normalization).

5.3.6. Two parabolic $F$-subgroups are <u>associate</u> if a conjugate of one has a common Levi subgroup with the other and are <u>opposite</u> if their intersection reduces to a Levi subgroup. The transform $\overline{\theta(P)}$ is a parabolic subgroup $P'$ opposite to $P$, the only one such that $P \cap P'$ is $\theta$-stable. It is not necessarily defined over $F$ (unless $K$ is defined over $F$), but it is conjugate to an $F$-subgroup. Indeed $P \cap P'$ is conjugate under $N_P$ to a Levi $F$-subgroup and this conjugation brings $P'$ to an $F$-subgroup $P''$. (If $P = P_J$ is standard, then the Lie algebra of the unipotent radical of $P'' = P_J^-$ is spanned by the spaces $\mathfrak{g}_\beta$ ($\beta < 0$ is not a linear combination of elements in $J$).)

5.3.7. *Caution.* The data constructed above depend on the choice of $F$. It would have been more correct to add a left subscript $F$ to $A_P$, $M_P$, etc. If $F \subset F'$, then

$$ {}_F A_{\mathbf{P}} \subset {}_{F'} A_{\mathbf{P}}, \quad {}_F M_{\mathbf{P}} \supset {}_{F'} M_{\mathbf{P}}, \quad {}_F L_{\mathbf{P}} = {}_{F'} L_{\mathbf{P}}. \tag{34} $$

(but $N_P$ is independent of $F$, of course). If $\mathbf{G}$ splits over $F$, these subgroups are independent of $F$.

5.3.8. We finish up with some notation. We state it over $\mathbb{Q}$, but analogous notions can be defined over $\mathbb{R}$ or any field. A pair $(P, A_P)$ is called a $p$-pair, $\Phi(P, A_P)$ denotes the set of weights of $A_{\mathbf{P}}$ on $\mathfrak{n}_{\mathbf{P}}$. They are the integral linear combinations, with positive coefficients, of the set of simple roots ${}_{\mathbb{Q}}\Delta(P, A_P)$. (If $P = P_J$ is standard, ${}_{\mathbb{Q}}\Delta(P, A_P)$ consists of restrictions of ${}_{\mathbb{Q}}\Delta - J$). The $\mathbb{Q}$-rank $\mathrm{rk}_{\mathbb{Q}}(\mathbf{G})$ of $\mathbf{G}$ is the dimension of its maximal $\mathbb{Q}$-split tori. The parabolic rank $\mathrm{prk}_{\mathbb{Q}}(\mathbf{P})$ over $\mathbb{Q}$ of $\mathbf{P}$ is the common dimension of the maximal $\mathbb{Q}$-split tori of its Levi $\mathbb{Q}$-subgroups. In particular, $\mathrm{prk}_{\mathbb{Q}}\mathbf{G} = \dim A_{\mathbf{G}}$.

The parabolic rank and the sets ${}_{\mathbb{Q}}\Phi(P, A_P)$, ${}_{\mathbb{Q}}\Delta(P, A_P)$ were introduced first over $\mathbb{R}$ in representation theory (by Harish-Chandra).

5.4. **Orthogonal groups.** We provide here a second example giving a concrete interpretation of the objects described in general in 5.3, for $\mathrm{GL}_n$ in 5.1, assuming some familiarity with the theory of quadratic forms. Let $V_{\mathbb{Q}}$ be an $n$-dimensional vector space over $\mathbb{Q}$ and $F$ a non-degenerate quadratic form on $V_{\mathbb{Q}}$. It is said to be isotropic if there exists $v \in V_{\mathbb{Q}} \setminus \{0\}$ such that $F(v) = 0$, anisotropic otherwise. A subspace of $V_{\mathbb{Q}}$ is isotropic if the restriction of $F$ to it is zero. The (common) dimension of the maximal isotropic subspaces of $V_{\mathbb{Q}}$ is the index of $F$.

We let $V = V_{\mathbb{Q}} \otimes \mathbb{C}$ and view $F$ as defined on $V$. Let $\mathrm{O}(F)$ be the subgroup of $\mathrm{GL}(V)$ leaving $F$ invariant and $\mathrm{SO}(F)$ its subgroup of elements of determinant one. Let $G = \mathrm{SO}(F)$. It is defined over $\mathbb{Q}$. We assume the index $p$ of $F$ to be $> 0$. There exists a basis $\{e_i\}$ of $V_{\mathbb{Q}}$ such that

$$ F(x) = \sum_{i=1}^{p} x_i . x_{n-p+i} + F_0, \tag{35} $$

where $F_0$ is a non-degenerate anisotropic quadratic form on the subspace spanned by $e_{p+1}, \ldots, e_{n-p}$. Then $\{e_1, \ldots, e_p\}$ on one hand, and $\{e_{n-p+1}, \ldots, e_n\}$ on the other, span maximal isotropic subspaces. A maximal $\mathbb{Q}$-split torus is the group of diagonal matrices

$$ \mathrm{diag}\,(s_1, \ldots, s_p, 1, \ldots, 1, s_{n-p+1}, \ldots, s_n), \tag{36} $$

where $s_i.s_{n-p+i} = 1$. The corresponding subgroup $A$ is the group of elements of the form (36) with $s_i$ real, strictly positive. Then

$$\mathcal{Z}(A) = A \times \mathrm{SO}(F_0) \times (\mathbb{Z}/2\mathbb{Z})^p \tag{37}$$

The Lie algebra $\mathfrak{g}$ of $G$ is

$$\mathfrak{g} = \{C \in M_n(\mathbb{R}) \,|\, C.F + F.{}^tC = 0\}. \tag{38}$$

Let again $\lambda_i \in X^\star(A)$ be the character with assigns to $a \in A$ its $i^{\text{th}}$ coordinate: $a^{\lambda_i} = a_i$ $(1 \leq i \leq p)$. To find the roots, one has to let $a \in A$ act on $\mathfrak{g}$ by adjoint representation $C \mapsto a.C.a^{-1}$. To determine them, it is convenient to write $C$ as a $3 \times 3$ block $C = (C_{ij})$ $(1 \leq i, j \leq 3)$ corresponding to the subsets $\{e_1, \dots, e_p\}$, $\{e_{p+1}, \dots, e_{n-p}\}$, and $\{e_{n-p+1}, \dots, e_n\}$. We leave it as an exercise to find the conditions on the $C_{ij}$ given by (38) and to see the roots are

$$\lambda_i - \lambda_j \quad (i \neq j) \text{ with multiplicity one}$$

$$\pm(\lambda_i + \lambda_j) \quad (i \neq j) \text{ with multiplicity one}$$

and, if $n \neq 2p$,

$$\pm\lambda_i \quad \text{with multiplicity } n - 2p. \tag{39}$$

If $n \neq 2p$, then a system of simple roots is given by $\alpha_1 = \lambda_1 - \lambda_2$, $\alpha_2 = \lambda_2 - \lambda_3$, $\dots$, $\alpha_{p-1} = \lambda_{p-1} - \lambda_p$, and $\alpha_p = \lambda_p$. Thus ${}_F\Phi$ is of type $B_p$ if $n \neq 2p$, and $D_p$ if $n = 2p$. The group is split over $F$ if (and only if) $n - 2p = 0, 1$.

Again, let $n \neq 2p$. A standard isotropic flag is an increasing sequence of isotropic subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_s = V \tag{40}$$

where $V_i$ is of dimension $d(i)$, spanned by $e_1, \dots, e_{d(i)}$ $(1 \leq d(1) < d(2) < \cdots < d(s) = p)$. Let $n_i = d(i) - d(i-1)$ $(i = 1, \dots, s)$. The standard parabolic subgroups are the stabilizers of standard isotropic flags. The stabilizer of the flag (40) is the group $P_J$, where $J = \{\alpha_{d(1)}, \alpha_{d(2)}, \dots, \alpha_{d(s-1)}\}$. It is represented by matrices which are block upper triangular, with the diagonal blocks consisting of matrices of the form

$$g_1, \dots, g_s, g_0, {}^tg_s^{-1}, \dots, {}^tg_1^{-1},$$

where $g_i \in \mathrm{GL}_{n_i}(\mathbb{R})$ $(1 \leq i \leq s)$ and $g_0 \in \mathrm{SO}(F_0)$. The entries of an elements $g \in P_J$ are zero below the diagonal blocks; those above the blocks are subject only to conditions derived from the fact that $g \in G = \mathrm{SO}(F)$.

If $n = 2p$, there is a slight modification in the description of maximal proper parabolic subgroups (cf. e.g. [7], 7.2.4).

## 6. Arithmetic subgroups. Reduction theory

From now on $F = \mathbb{Q}, \mathbb{R}$ (and $\tilde{F} = \mathbb{C}$ as before).

6.1. Let $\mathbf{G}$ be a $\mathbb{Q}$-group and $G \subset \mathrm{GL}_n$ a $\mathbb{Q}$-embedding. A subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is arithmetic if it is commensurable with $G \cap \mathrm{GL}_n(\mathbb{Z})$ (i.e. $\Gamma \cap (G \cap \mathrm{GL}_n(\mathbb{Z}))$ is of finite index in both groups). This notion is independent of the embedding.

The group ${}^\circ G$ can also be defined as $\cap_{\chi \in X(\mathbf{G})} \ker \chi^2$, hence can be viewed as a $\mathbb{Q}$-group (or rather the group of real points in one).

Let $\mathbf{G}$ be reductive. Then $\Gamma \subset {}^\circ G$ and $\Gamma \backslash {}^\circ G$ has finite invariant measure. It is compact if and only if $\Gamma$ consists of semi-simple elements, or if and only if $\mathcal{D}G$ is anisotropic over $\mathbb{Q}$. In the non-compact case, the purpose of reduction theory is to construct fundamental sets in $G$ with respect to $\Gamma$. It relies on the notion of Siegel set.

6.2. **Siegel sets.** We let $\mathcal{P}_{\mathbb{Q}}$ be the set of parabolic $\mathbb{Q}$-subgroups of $\mathbf{G}$. It is operated upon by conjugation by $\mathbf{G}(\mathbb{Q})$, in particular by $\Gamma$. It is known that $\Gamma \backslash \mathcal{P}_Q$ is finite. Let $P \in \mathcal{P}_{\mathbb{Q}}$. To remind us that we are dealing with the Langlands decomposition of $P$ relative to $\mathbb{Q}$ we shall write in bold face the subscript $P$. Fix $t > 0$. We let

(41) $$A_{\mathbf{P},t} = \{a \in A_P,\ a^\alpha \geq t\ (\alpha \in {}_{\mathbb{Q}}\Delta)\}$$

(it is equivalent to require $a^\alpha \geq t$ for all $\alpha \in {}_{\mathbb{Q}}\Phi^+$).

The Langlands decomposition of $P$ can also be written as $P = N_{\mathbf{P}}.M_{\mathbf{P}}.A_{\mathbf{P}}$. A Siegel set $\mathfrak{S}_{\mathbf{P},t,\omega}$, where $\omega \subset N_P.M_P$ is relatively compact, is that set

$$\omega.A_{\mathbf{P},t}.K.$$

A simple computation ([5], 12.5) shows that if $A_{\mathbf{G}} = \{1\}$, then $\mathfrak{S}_{\mathbf{P},t,\omega}$ has finite volume with respect to Haar measure.

**Remark.** The Siegel sets are defined here as subsets of $G$. Traditionally, they were introduced as subsets of $X$, namely, the orbits of the origin by our Siegel set, and that mostly with respect to a minimal parabolic subgroup, in the split case. For instance, if $\mathbf{G} = \mathrm{GL}_n(\mathbb{R})$, $K = \mathrm{O}_n$, $X$ is the space of positive quadratic forms and $P$ is the upper triangular matrices, then a Siegel set in $X$ is the set of quadratic forms of the form $na.{}^t(na)$, where $a$ is diagonal with entries $a_i$ satisfying $a_i \geq t a_{i+1}$ $(1 \leq i \leq n)$ and $n$ is upper triangular, with coefficients $n_{ij}$ $(i < j)$ bounded in absolute value by some constant. For $\mathrm{SL}_n(\mathbb{R})$, one requires moreover $\prod a_i = 1$. If $n = 2$ and $X$ is the upper half plane $X = \{z = x+iy \in \mathbb{C}, y > 0\}$, then a Siegel set is given by the conditions $|x| \leq c$ and $y > t$.

6.3. We keep the previous notation.

**Theorem.** *Let* $\mathbf{P}$ *be a minimal parabolic* $\mathbb{Q}$*-subgroup and* $\mathfrak{S} = \mathfrak{S}_{\mathbf{P},t,\omega}$ *a Siegel set with respect to* $\mathbf{P}$ *and* $C$ *a finite subset of* $\mathbf{G}(\mathbb{Q})$*. Then the set*

$$\{\gamma \in \Gamma, \gamma(C.\mathfrak{S}) \cap C.\mathfrak{S} \neq \emptyset\} \tag{42}$$

*is finite (Siegel property). There exists such a set* $C$ *and* $\mathfrak{S}$ *such that* $G = \Gamma.C.\mathfrak{S}$.

A set $C.\mathfrak{S}$ satisfying the last condition will be called a fundamental set. It is easily seen that if $c \in \mathbf{G}(\mathbb{Q})$, then ${}^c\mathfrak{S}$ is contained in a Siegel set for ${}^c\mathbf{P}$.

The theorem can also be expressed by stating that if $\mathbf{P}_1, \dots, \mathbf{P}_s$ are representatives of the $\Gamma$-conjugacy classes of minimal parabolic $\mathbb{Q}$-subgroups, then there exists Siegel set $\mathfrak{S}_{\mathbf{P}_i,t_i,\omega_i}$, the union of which covers $\Gamma\backslash G$. $G$ cannot be covered by less than $s$ such subsets. Each is said to represent a cusp. The one-cusp case occurs for some classical arithmetic subgroups such as $\mathrm{SL}_n(\mathbb{Z})$ or $\mathrm{Sp}_{2n}(\mathbb{Z})$. More generally for arithmetic subgroups of $\mathbb{Q}$-split groups associated to admissible lattices in the sense of Chevalley. In the adelic case, where the role of $\Gamma$ is played by $G(\mathbb{Q})$, there is only one cusp.

6.4. **Moderate growth condition.** Fix now a maximal (not necessarily split) torus $T \supset A$. There is a corresponding root system $\Phi(T^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$; the restriction of any $\alpha \in \Phi(T^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$ to any $A_P$ is either zero or an element of $\Phi(G, A_P)$. We choose a positive system in $\Phi(T^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$ so that for each $\alpha \in \Phi(T^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})^+$, the restriction of $\alpha$ to $A_P$ is either zero or an element of $\Phi(P, A_P)$.

Identity $\mathbf{G}$ with a subgroup of some $\mathrm{GL}_N$ via a $\mathbb{Q}$-embedding $\rho$. The representation $\rho$ is the a finite sum of irreducible ones, with highest weights $\lambda_1, \dots, \lambda_k$, say; these are characters of $T^{\mathbb{C}}$ that are real-valued and positive on $A_P$.

Let $\mathfrak{S}$ be a Siegel set with respect to the parabolic subgroup $P$. We claim that

$$\|g\|^2 \asymp \sum_i a_P(g)^{2\lambda_i} \quad (g \in \mathfrak{S}). \tag{43}$$

We can assume $A_{\mathbf{P}}$ to be in diagonal form. The entries are then weights of $\rho$. Given such a weight $\mu$, there exists $i$ such that $\mu = \lambda_i - \sum m_\alpha.\alpha$, where $\alpha \in {}_{\mathbb{Q}}\Delta(P, A_P)$ and $m_i \in \mathbb{N}$. Whence $\|g\|^2 \prec \sum_i a_P(g)^{2\lambda_i}$. The other direction is obvious.

Let $f$ be a function on $\mathfrak{S}$. As a consequence, we see that $f$ is of moderate growth on $\mathfrak{S}$ (in the sense of section 3), if and only if there exists $\lambda \in X(A_{\mathbf{P}})$ such that

$$(44) \qquad |f(g)| \prec a(g)^{\lambda} \quad (g \in \mathfrak{S}).$$

We define $f$ to be fast decreasing on $\mathfrak{S}$ if and only if 44 is true for any $\lambda \in X(A_{\mathbf{P}})$. Equivalently, $|f(g)| \prec \|g\|^{-N}$ for all $N$.

Consider the following three conditions on a continuous function $f$ on $\Gamma\backslash G$:

(i) $f$ has moderate growth.
(ii) $f$ has moderate growth in the sense of (44) on each Siegel set of $G$.
(iii) $f$ has moderate growth in the sense of (44) on a fundamental set $\Omega$ (as defined in theorem 6.3).

*Claim.* These conditions are equivalent.

Clearly, (i) implies (ii), by the above. Also (ii) giving (iii) is obvious. Assume now (iii). Then $f$ has moderate growth on a fundamental set $\Omega$. However, $f(\gamma g) = f(g)$ for all $\gamma \in \Gamma, g \in G$, whereas $\|\gamma g\|$ depends of course also on $\gamma$. But we have

$$(45) \qquad \|g\| \asymp \inf_{\gamma\in\Gamma} \|\gamma.c.g\| \quad (\gamma \in \Gamma, c \in C, g \in \mathfrak{S}),$$

a result of Harish-Chandra which clearly shows that (iii) implies (i). Harish-Chandra's proof of (45) is given in ([2], II, §1, Prop. 5).

6.5. In §3, we assumed, to avoid preliminaries, that $G$ was semi-simple. However, the definition is of course valid if $G$ is reductive. In the present case, we shall have to know more precisely the dependence on $A_G$. To express this, we need the following definition, in which $A$ is a closed connected subgroup of $A_G$.

A function $f$ on $A$ is an exponential polynomial if there exist elements $\lambda_i \in X(A)$ and polynomials $P_i$ on $\mathfrak{a}$ $(1 \le i \le s)$ such that

$$(46) \qquad f(a) = \sum_i a^{\lambda_i}.P_i(\log a) \quad (a \in A).$$

Then one has the following result:

6.6. **Proposition.** Let $f$ be a smooth $\mathcal{Z}(\mathfrak{g})$-finite function on $\Gamma\backslash G$. Then there exist exponential polynomials $Q_i$ on $A$ and polynomials $P_i \in \mathcal{Z}(\mathfrak{g})$ such that

$$f(x.a) = \sum_i Q_i(a) P_i f(x) \quad (a \in A_G, x \in {}^\circ G).$$

More generally, this is valid with respect to a decomposition $G = G' \times A$, with $A \subset A_G$ and ${}^\circ G \subset G'$, and the proof reduces to the case where $A$

is one-dimensional, in which case it is an exercise in ODE. A proof is given in ([2], II, §3, lemma 3) or ([13], p. 20).

6.7. For later reference, we end this section with an elementary fact about the function $a(x)^\lambda$. Namely, if $C$ is compact in $G$, then

$$a(x.c)^\lambda \asymp a(x)^\lambda \quad (x \in G, c \in C) \tag{47}$$

for any $\lambda \in X(A)$.

Write $x = n.m.a.k$ as usual. Then $a(x.c) = a(x).a(k.c)$. But $k$ varies in a compact set, hence $a(k.c) \asymp 1$ as $c$ varies in $C$ and $k \in K$. This proves (47).

## 7. Constant terms. Fundamental estimate

7.1. Let $P \in \mathcal{P}_\mathbb{Q}$ and $P = N_\mathbf{P}.A_\mathbf{P}.M_\mathbf{P}$ be the Langlands decomposition of $P$. We let

$$\Gamma_\mathbf{P} = \Gamma \cap \mathbf{P}, \;\; \Gamma_{N_\mathbf{P}} = \Gamma \cap N_\mathbf{P}, \;\; \Gamma_{M_\mathbf{P}} = \Gamma/\Gamma_{N_\mathbf{P}}, \tag{48}$$

and denote by $\pi_P$ the canonical projection $P \to P/N_P = \bar{L}_P$.

The group $\bar{L}_P$ is defined over $\mathbb{Q}$ and has a decomposition $\bar{L}_P = \bar{M}_P.\bar{A}_P$. Then $\Gamma_M$ is an arithmetic subgroup of $\bar{L}_P$, hence is contained in $\bar{M}_P$. Note that in general, $M_\mathbf{P}$ is not defined over $\mathbb{Q}$, so that it would not make good sense to define $\Gamma_M$ as $\Gamma \cap M_P$. Even if $M_P$ is defined over $\mathbb{Q}$, the group $\Gamma \cap M_P$ might map under $\pi_P$ only to a subgroup of finite index of $\Gamma_M$. We have however, obviously

$$\pi_P^{-1}(\Gamma_M) = \Gamma_P.N_P. \tag{49}$$

The projection $\pi_P$ induces an isomorphism of $M_\mathbf{P}$ onto $\bar{M}_\mathbf{P}$ which allows one to give $M_P$ a $\mathbb{Q}$-structure.

7.2. **Constant term.** It is well-known that $\Gamma_{N_\mathbf{P}}\backslash N_\mathbf{P}$ is compact. Let $dn$ be the Haar measure on $N_\mathbf{P}$ which gives volume 1 to $\Gamma_{N_\mathbf{P}}\backslash N_\mathbf{P}$. Let $f$ be a continuous function on $\Gamma\backslash G$. Its constant term $f_P$, or if $\Gamma$ needs to be mentioned $f_{P,\Gamma}$, is defined by

$$f_P(g) = \int_{\Gamma_{N_\mathbf{P}}\backslash N_\mathbf{P}} f(ng)\, dn. \tag{50}$$

The function $f$ is <u>cuspidal</u> if its constant terms with respect to all proper parabolic $\mathbb{Q}$-subgroups are zero.

7.3. **Elementary properties of the constant term.** Those stated without explanation are left as exercises.

(a) Let $\gamma \in \Gamma$ and $P' = {}^\gamma P$. Then $f_{P'}(g) = f_P(\gamma g)$ for $g \in G$.
(b) If $\Gamma'$ is a subgroup of finite index of $\Gamma$, then $f_{P,\Gamma'} = f_{P,\Gamma}$.
(c) As a function on $G$, $f_P$ is left-invariant under $\Gamma_{\mathbf{P}}.N_{\mathbf{P}}$.
(d) Let $u \in C_c(G)$. Then $(f \star u)_P = f_P \star u$.

7.4. The function $f_P$ is left-invariant under $N_{\mathbf{P}}$, its restriction to $P$ will be viewed as a function on $N_{\mathbf{P}}\backslash P = L_{\mathbf{P}}$. It is left-invariant under $\Gamma_{L_{\mathbf{P}}}$.

We have to see that if $f$ is an automorphic form for $\Gamma$, then $f_P$ (so viewed) is an automorphic form for $\Gamma_M$.

(a) $K_P = K \cap M_P$ is a maximal compact subgroup of $M_P$ (and of $L_P$ and $P$ as well). We choose $K_P$ as maximal compact subgroup of $L_P$. It is then clear that if $f$ is $K$-finite on the right, then $f_P$ is $K_P$-finite on the right, of a type determined by that of $f$.
(b) We may take for H.S. on $L_P$ the restriction of the H.S. norm on $G$ for a given embedding. Then, clearly, if $f$ is of moderate growth (resp. fast decrease) on $G$, then so is $f_P$ on $\bar{L}_P$.
(c) There remains to see that if $f$ is $\mathcal{Z}(\mathfrak{g})$-finite, then $f_P$ is $\mathcal{Z}(\mathfrak{l}_P)$-finite. This relies essentially on a theorem of Harish-Chandra. There is a natural homomorphism $\nu : \mathcal{Z}(\mathfrak{g}) \to \mathcal{Z}(\mathfrak{l}_P)$ such that $z \in \nu(z) + \mathcal{U}(\mathfrak{g}).\mathfrak{n}_P$. Now let $J$ be an idea of $\mathcal{Z}(\mathfrak{g})$ annihilating $f$. Let $z \in J$, and write $z = \nu(z)+w(z)$, where $w(z) \in \mathcal{U}(\mathfrak{g}).\mathfrak{n}_P$. Then
$$(zf)_P = z.f_P = \nu(z)f_P + w(z)f_P.$$
But since $f_P$ is left-$N_P$-invariant, we have $w(z).f_P = 0$, whence
$$(zf)_P = \nu(z).f_P$$
which shows that if the ideal $J$ annihilates $f$, then $\nu(J)$ annihilates $f_P$. By a theorem of Harish-Chandra, $\mathcal{Z}(\mathfrak{l}_P)$ is a finitely generated module over $\mathcal{Z}(\mathfrak{g})$, hence if $J$ has finite codimension in $\mathcal{Z}(\mathfrak{g})$, then $\nu(J)$ has finite codimension in $\mathcal{Z}(\mathfrak{l}_P)$ and condition (A3) is fulfilled by $f_P$.

7.5. **Transitivity of the constant term.** Let $\mathbf{Q}$ be a proper parabolic $\mathbb{Q}$-subgroup contained in $\mathbf{P}$, and let ${}^\star\mathbf{Q} = Q \cap L_P$. Then its image ${}^\star\bar{Q}$ in $L_{\mathbf{P}}$ is a parabolic $\mathbb{Q}$-subgroup, so that $(f_{\mathbf{P}})_{^\star\mathbf{Q}}$ makes sense. We have

$$f_{\mathbf{Q}} = (f_{\mathbf{P}})_{^\star\mathbf{Q}}. \tag{51}$$

We leave this as an exercise. In particular, $f_{\mathbf{P}} = 0$ implies $f_{\mathbf{Q}} = 0$.

As a consequence of this and 7.3(a), $f$ is cuspidal if and only if $f_{\mathbf{P}} = 0$ when $\mathbf{P}$ runs through a set of representatives modulo $\Gamma$ of proper maximal parabolic subgroups.

7.6. Fix $P \in \mathcal{P}_{\mathbb{Q}}$. We shall write $P = N.A.M$ as its Langlands decomposition. Let $f \in C^\infty(\Gamma_N \backslash G)$. We are interested in the asymptotic behavior of $f - f_P$ on a Siegel set $\mathfrak{S} = \mathfrak{S}_{P,t,\omega}$ with respect to $P$.

We can find a decreasing sequence of $\mathbb{Q}$-subgroups of $N$, normal in $N.A$,

$$N = N_1 \supset N_2 \supset \cdots \supset N_q \supset N_{q+1} = \{1\}$$

with dimensions decreasing by one. Thus, $q = \dim N$. Let $\Gamma_j = \Gamma \cap N_j$. It is compact in $N_j$ and $\Gamma_j/\Gamma_{j+1}$ is infinite cyclic. Let $\beta_j$ be the weight of $A$ on $\mathfrak{n}_j/\mathfrak{n}_{j+1}$. (Thus $\beta_j$ runs through the elements of $\Phi(P, A)$, each occurring $\dim \mathfrak{g}_\beta$ times.) Fix $X_j \in \mathfrak{n}_j$ such that $\exp X_j$ generates $\Gamma_j/\Gamma_{j+1}$. Let $f_j$ be the "partial constant term"

$$f_j(x) = \int_{\Gamma_j \backslash N_j} f(nx)\, dn \tag{52}$$

where $dn$ gives volume 1 to $\Gamma_j \backslash N_j$. Thus $f_1 = f_P$ and $f_q = f$, and

$$f - f_P = \sum_j (f_{j+1} - f_j) \tag{53}$$

Choose a basis $\{Y_i\}$ of $\mathfrak{g}$.

7.7. **Proposition.** (the basic estimate) Let $f \in C^1(\Gamma_N \backslash G)$ be of moderate growth, bounded by $\lambda \in X(A)$ on Siegel sets w.r.t to $P$. Let $\mathfrak{S}$ be one and $\mathfrak{S}'$ be the union of the translates $e^{tX_j}\mathfrak{S}$ for $0 \leq t \leq 1$. There exists a constant $c > 0$, independent of $f$, such that

$$|(f_j - f_{j+1})(x)| \leq c a(x)^{\lambda - \beta_j} \sum_i \sup_{y \in \mathfrak{S}'} |Y_i f_{j+1}(y)| a(y)^{-\lambda} \tag{54}$$

In particular, if $f \in C^\infty(\Gamma_N \backslash G)$ is of uniform bounded growth, bounded by $\lambda$, then

$$|(f_j - f_{j+1})(x)| \prec a(x)^{\lambda - \beta_j} \quad (x \in \mathfrak{S}). \tag{55}$$

*Proof.* It is quite similar to that of 7.4 in [6]. Since $\Gamma_j \backslash N_j$ is fibered by $\Gamma_{j+1} \backslash N_{j+1}$ over the quotient of $N_j \backslash N_{j+1}$ by the cyclic group generated by the element $\exp X_j$ (which leaves $f$ invariant), we can write

$$f_j(x) = \int_0^1 dt\, f_{j+1}(e^{tX_j}.x) \tag{56}$$

$$f_j(x) - f_{j+1}(x) = \int_0^1 dt (f_{j+1}(e^{tX_j} x) - f_{j+1}(x)) \tag{57}$$

hence

$$f_j(x) - f_{j+1}(x) = \int_0^1 dt \int_0^t ds \frac{d}{ds} f_{j+1}(e^{sX_j}.x) \tag{58}$$

However, the derivative is along $e^{sX_j}$ viewed as a right-invariant vector field, whereas we consider usually derivatives with respect to left-invariant fields (see §3). Set $y_s = e^{sX_j}x$; it lies in $\mathfrak{S}'$.

$$\frac{d}{ds} f_{j+1}(e^{sX_j}x) = \frac{d}{dr} f_{j+1}(e^{rX_j}y_s)|_{r=0} = \frac{d}{dr} f_{j+1}(y_s.y_s^{-1}e^{rX_j}y_s)|_{r=0} \tag{59}$$

Since the exponential commutes with $G$, we have:

$$y_s^{-1}.e^{rX_j}.y_s = e^{r\,{}^{y_s^{-1}}X_j} \quad ({}^{y_s^{-1}}X_j = \mathrm{Ad}(y_s^{-1}).X_j)$$

we get

$$\frac{d}{dr} f_{j+1}(y_s e^{r\,{}^{y_s^{-1}}X_j})|_{r=0} = {}^{y_s^{-1}}X_j f_{j+1}(y_s) \quad y_s = e^{sX_j}x \tag{60}$$

Write $x = n.a(x).m.k$ using the Langlands decomposition. Note that $y_s = e^{sX_j}x$ has a Langlands decomposition differing from that of $x$ only by the $N$-factor. In particular, $a(y_s) = a(x)$.

For any $u \in N$, we have

$${}^{a(x)^{-1}u}X_j = a(x)^{-\beta_j}X_j + X' \quad X' \in \mathfrak{n}_{j+1}$$

Since $f_{j+1}$ is invariant under $N_{j+1}$, we have $X'f_{j+1} = 0$, hence

$${}^{a(x)^{-1}n^{-1}}X_j f_{j+1} = a(x)^{-\beta_j}X_j f_{j+1}, \quad {}^{y_s^{-1}}X_j f_{j+1} = a(x)^{-\beta_j}.\,{}^{(mk)^{-1}}X_j f_{j+1}.$$

There exist smooth functions $c_{ji}$ on $G$ such that for any $g \in G$, we have

$${}^gX_j = \sum_i c_{ji}(g).Y_j.$$

Now

$${}^{y_s^{-1}}X_j f_{j+1} = a(x)^{-\beta_j} \sum_i c_{ji}((mk)^{-1})Y_i f_{j+1} \tag{61}$$

$mk$ varies in a relatively compact set as $x \in \mathfrak{S}$. There exists therefore $c > 0$ such that $|c_{ji}((mk)^{-1}| \le c$, and we get:

$$|{}^{y_s^{-1}}X_j f_{j+1}(y)| \le c \sum_i a(x)^{-\beta_j} \sum_i |Y_i f_{j+1}(y)|$$

Using 60, 61, we get from 58, multiplying both sides by $a(x)^{-\lambda}$:

(62)

$$|f_j(x) - f_{j+1}(x)|a(x)^{-\lambda} = ca(x)^{-\beta_j} \int_0^1 dt \int_0^t ds \sum_i |Y_i f_{j+1}(y)|a(y)^{-\lambda}$$

and (i) follows. Under the assumption of (ii), the last sum on the right-hand side is a constant, whence (ii). [It would be enough to assume that all first derivatives of $f$ be bounded by $\lambda$.] □

7.8. **Corollary.** Let $f$ be as in the second part of Proposition 7.7. Let $N \in \mathbb{N}$. Then

$$|(f_j - f_{j+1})(x)| \prec a(x)^{\lambda - N\beta_j} \quad (x \in \mathfrak{S}) \tag{63}$$

$$|(f - f_P)(x)| \prec \sum_j a(x)^{\lambda - N\beta_j} \quad (x \in \mathfrak{S}). \tag{64}$$

Clearly, $(f_j - f_{j+1})_{j+1} = 0$ and $(f_j)_j = f_j$. We can apply (55) to $f_j - f_{j+1}$ which yields $|(f_j - f_{j+1})(x)| \prec a(x)^{\lambda - 2\beta_j}$. Now Assertion (i) follows by iteration, and then (ii) is obvious, in view of (53).

Assume now that $P = P^\alpha$ is proper maximal ($\alpha \in {}_{\mathbb{Q}}\Delta$). Then $\beta_j = m_j.\alpha$ with $m_j \in \mathbb{N}$, $m_j > 0$. So we get

$$|(f - f_P)(x)| \prec a(x)^{-N\alpha} \quad (x \in \mathfrak{S}_{P^\alpha}) \tag{65}$$

for any $N \in \mathbb{N}$.

Let now $\mathbf{P}$ not be maximal. Assume that $f_{\mathbf{Q}} = 0$ for any proper maximal parabolic $\mathbf{Q}$ containing $\mathbf{P}$ (there are only finitely many, as follows from 3.3d.) Then it is easily seen that $f$ is fast decreasing on any Siegel set with respect to $P$.

7.9. **Theorem.** Any cusp form is fast decreasing if $A_G = \{1\}$.

7.10. Let $Y_P = \Gamma_P.N_P\backslash G$ and $F(Y_P)$ be a $G$-invariant space of functions on $Y_P$, which are at least measurable, locally $L^1$. By definition, its cuspidal part ${}^\circ F(Y_P)$ is the space of function all of whose constant terms with respect to the proper parabolic $\mathbb{Q}$-subgroups containing $P$ are zero. Let $\pi_{\mathbf{Q}} f = f_{\mathbf{Q}}$. This is an idempotent. The $\pi_{\mathbf{Q}}$ ($\mathbf{Q} \supset \mathbf{P}$) commute with one another, and ${}^\circ_P F(Y_P)$ is the kernel of

$$\mathrm{pr}_{\mathrm{cusp}} = \prod_{G \supset \mathbf{Q} \supset \mathbf{P}} (1 - \pi_{\mathbf{Q}}). \tag{66}$$

If $F(Y_P) = C^\infty_{\mathrm{umg}}(Y_P)$ then, as we saw, the elements of ${}^\circ_P F(Y_P)$ are uniformly fast decreasing on any Siegel set with respect to $P$. If $P$ is minimal, and there is only one cusp, these are cuspidal functions on $\Gamma\backslash G$, but not so otherwise. §11 will give a projection of $C^\infty_{\mathrm{umg}}(\Gamma\backslash G)$ onto ${}^\circ C_{\mathrm{umg}}(\Gamma\backslash G)$ in the general case.

## 8. Finite dimensionality of $A(\Gamma, J, \xi)$

The proof that $\mathcal{A}(\Gamma, J, \xi)$ is finite dimensional is essentially the same as that given in [13] for semisimple groups and in [6] for $\mathrm{SL}_2(\mathbb{R})$. We shall be brief and refer to both for details.

8.1. **Definition.** A locally $L^1$-function of $\Gamma\backslash G$ is cuspidal if its constant term with respect to any proper parabolic subgroup is zero.

If $V$ is the space of locally $L^1$ functions, then ${}^\circ V$ denote the subspace of cuspidal elements in $V$.

We recall two known lemmas.

8.2. **Lemma.** Assume $\mathrm{prk}\,_{\mathbb{Q}}\mathbf{G} = 0$ and let $p \in [1, \infty]$. Then ${}^\circ L^p(\Gamma\backslash G)$ is closed in $L^p(\Gamma\backslash G)$.

cf. 8.2 in [6] or, for $p = 2$, the proof of lemma 18 in [13].

8.3. **Lemma.** Let $Z$ be a locally compact space with a positive measure $\mu$ such that $\mu(Z)$ is finite. Let $V$ be a closed subspace of $L^2(Z, \mu)$ that consists of essentially bounded functions. Then $V$ is finite dimensional.

This lemma is due to R. Godement. For Hörmander's proof of it, see [13] p. 17-18 or [6], 8.3.

8.4. **Theorem (Harish-Chandra).** Let $J$ be an idea of finite codimension in $\mathcal{Z}(\mathfrak{g})$ and $\xi$ a standard idempotent for $K$ (4.6). Then the space $\mathcal{A}(\Gamma, J, \xi)$ of automorphic forms on $\Gamma\backslash G$ of type $(J, \xi)$ is finite dimensional.

cf. [13], Theorem 1 and, for $\mathrm{SL}_2(\mathbb{R})$, [6], §8. The proof is by induction on $\mathrm{rk}\,_{\mathbb{Q}}(G)$. Assume that $\mathrm{prk}\ (G) > 0$. We use 6.6 to reduce the proof to $\Gamma\backslash {}^\circ G$. The functions $P_i f$ have $K$-types defined by $\xi$. Moreover, they are annihilated by $J' = J \cap \mathcal{Z}({}^\circ\mathfrak{g})$. Therefore, the $P_i f$ belong to the space of automorphic forms on $\Gamma\backslash {}^\circ G$ of type $(J', \xi)$, which is finite dimensional by induction. 6.6 then shows that $\mathcal{A}(\Gamma, J, \xi)$ is finite dimensional.

From now on, $\mathrm{prk}_{\mathbb{Q}}(\mathbf{G}) = 0$. If $\mathrm{rk}_{\mathbb{Q}}(G) = 0$, then $\Gamma\backslash G$ is compact (6.1), all automorphic forms are $L^2$ and bounded. The theorem follows in this case from Godement's lemma.

Let now $\mathrm{rk}\,_{\mathbb{Q}}(G) > 0$. Let $\mathcal{Q}$ be a set of representatives of $\Gamma$-conjugacy classes of proper maximal parabolic $\mathbb{Q}$-subgroups. It is finite ([3], 15.6). Let $\mu$ be the collection of the maps $f \mapsto f_{\mathbf{P}}$ ($\mathbf{P} \in \mathcal{Q}$). We have seen in 7.4(c), for any $P$ in fact, that $f_P$ belongs to a space of automorphic forms on $L_P$ of a fixed type, determined by $J$ and $\xi$. By induction and our original argument, they form a finite dimensional space, hence image of $\mu$ is finite dimensional. This reduces us to prove that ker$\mu$ is finite dimensional. In view of 7.4, ker $\mu = {}^\circ\mathcal{A}(\Gamma, J, \xi)$. By 7.9, it consists of bounded functions. To deduce from 8.3 that it is finite dimensional, it suffices to show that it is a closed subspace of $L^2(\Gamma\backslash G)$. The argument is the same as the one given in [13], lemma 18, and in [6], proof of 8.3 for $\mathrm{SL}_2(\mathbb{R})$.

## 9. Convolution operators on cuspidal functions

The proofs of the main results here rely mostly on 2.5 and on the basic estimates and its consequences. To formulate them, it will be useful to introduce some function spaces.

9.1. We let $C_{\rm mg}(\Gamma\backslash G)$ (respectively $C^\infty_{\rm umg}(\Gamma\backslash G)$) be the space of functions of moderate growth (respectively smooth functions of uniform moderate growth). They are endowed with the seminorms $\nu_n(f) = \sup|f(x)|||x||^{-n}$ (respectively $\nu_{D,n}(f) = \sup|Df(x)|||x^{-n}||$, for $D \in \mathcal{U}(\mathfrak{g})$.

For $n \in \mathcal{Z}$, let $C_{\rm mg}(n, \Gamma\backslash G)$ (respectively $C^\infty_{\rm umg}(\Gamma\backslash G, n)$) be the subspace of $C_{\rm mg}(\Gamma\backslash G)$ (respectively $C^\infty_{\rm umg}(\Gamma\backslash G)$) on which the $\nu_n$ (respectively all $\nu_{D,n}$, $D \in \mathcal{U}(\mathfrak{g})$) are finite.

Then 2.5 says that:

(*) Let $\alpha \in C^\infty_c(G)$. Then there exists $n \in \mathbb{N}$ such that the map $f \mapsto f \star \alpha$ induces a continuous map of $L^1(\Gamma\backslash G)$ into $C^\infty_{\rm umg}(\Gamma\backslash G, n)$.

9.2. Let $C_{\rm fd}(\Gamma\backslash G)$ (respectively $C^\infty_{\rm ufd}(\Gamma\backslash G)$) be the space of continuous functions on $G$ which are fast decreasing (respectively, of smooth functions such that $Df$ is fast decreasing for every $D \in \mathcal{U}(\mathfrak{g})$.) It is endowed with the seminorms $\nu_n$ (respectively $\nu_{D,n}$). Then Section 7 proves:

(*) For any $n \in \mathbb{Z}$, the inclusion ${}^\circ C^\infty_{\rm ufd}(\Gamma\backslash G) \to^\circ C^\infty_{\rm umg}(\Gamma\backslash G, n)$ is an isomorphism.

In fact, everything in Section 6 is expressed in terms of the growth condition 4.4 (2) on Siegel sets, so one has to trasnalte this in terms of HS norms, using the equivalence discussed in 4.4

9.3. Assume now $\mathrm{prk}_{\mathbb{Q}}(G) = 0$. Then $\Gamma\backslash G$ has finite volume, and therefore $L^2(\Gamma\backslash G) \subset L^1(\Gamma\backslash G)$, and the inclusion is continuous. Using 9.1 (*), 9.2 (*), remembering [??] that ${}^\circ L^p(\Gamma\backslash G)$ is a closed subspace of $L^p(\Gamma\backslash G)$, for $p \geq 1$, and that $(f \star \alpha)_P = f_P \star \alpha$ (5.4), we see that the composition:

$$(67) \qquad {}^\circ L^2(\Gamma\backslash G) \to^\circ L^1(\Gamma\backslash G) \to^\circ C^\infty_{\rm umg}(\Gamma\backslash G, n) \to^\circ C^\infty_{\rm ufd}(\Gamma\backslash G)$$

where the last arrow is the inverse of the isomorphism (*) of 9.2, is *continuous*, i.e. given $n \in \mathbb{Z}$ there exists a constant $c(D, n)$ so that

$$\nu_{D,n}(f \star \alpha) \leq c(D, n)||f||_2, \quad f \in^\circ L^2(\Gamma\backslash G)$$

**Theorem.** *(Gelfand-Piateteski-Shapiro) Let $\alpha \in C^\infty_c(G)$. Then $\star\alpha$ is a Hilbert-Schmidt operator on $L^2(\Gamma\backslash G)$. In particular, it is compact.*

*Proof.* If $D$ is the identity and $n = 0$, $\nu_{D,n}(f \star \alpha) = \sup_{x \in G} |(f \star \alpha)(x)|$. Therefore, 67 above implies the existence of a constant $c > 0$ such that

$$|(f \star \alpha)(x)| \leq c||f||_2, \quad f \in^\circ L^2(\Gamma\backslash G) \tag{68}$$

This implies the theorem by a standard argument (see, for example, [13], proof of Theorem 2, page 14, or [6], 9.5). In fact, as pointed out in [6], 9.5, this result, in combination with a theorem of Dixmier-Malliavin ([10]), shows that the operator $\star\alpha$ is of trace class.

The compactness of $\star\alpha$ on ${}^\circ L^p(\Gamma\backslash G)$ for $\alpha \in C^1_c(G)$ has been proved by R. Langlands, [14]. See [6],9.3 for $\mathrm{SL}_2(\mathbb{R})$.

**Corollary.** *AS a $G$-module, ${}^\circ L^2(\Gamma\backslash G)$ is a Hilbert direct sum of countably many irreducible $G$-invariant closed subspaces with finite multiplicities.*

Finite multiplicities means that at most finitely many summands are isomorphic to a given unitary irreducible $G$-module. The proof can be found in many places, e.g. [6], 16.1

## 10. Automorphic forms and the regular representation on $\Gamma\backslash G$

In this section, I assume some familiarity with generalities on infinite dimensional representations, all to be found in [4] and many other places. We review a few notions and facts mainly to fix notation. We let $G, K, \Gamma$ be as before, though some of the definitions and results recalled here are valid in much greater generality. We assume $\mathrm{prk} G_{\mathbb{Q}} = 0$.

*8.1.* Let $(\pi, V)$ be a continuous representation of $G$ in some locally complete topological vector space $V$. It extends to the convolution algebra of compactly supported functions by the rule:

$$\pi(\alpha v = \int_G \alpha(x)\pi(x)v dx \quad v \in V \tag{69}$$

The subspace spanned by the $\pi(\alpha)v$, where $v$ runs through $V$ and $\alpha$ through $C^\infty_c(G)$ is dense, as is easily seen by using a Dirac sequence ([4], 3.4). Call it $V^\infty$. It consists of $C^\infty$ vectors (i.e. such that $g \mapsto \pi(g)v$ is a smooth map), and it used to be called the Garding space. However, a special terminology has become superfluous because a theorem of Dixmier-Malliavain ([10]) implies that these are all differentiable vectors, but we shall not need this fact. $V^\infty$ is a $G$ module upon which $\mathfrak{g}$ operates naturally ([4], 3.8) hence $V^\infty$ is also a $\mathcal{U}(\mathfrak{g})$ module.

*8.2.* The operation defined by 69 above extends to compactly supported measures. In particular, $V$ can be viewed as a $K$-module, and as such is completely reducible and for $\lambda \in \hat{K}$, the transformation defined by

$$\pi(e_\lambda)v = \int_K \chi_\lambda(k)kvdk \tag{70}$$

(see 2.6 for the notation) is a projector of $V$ onto the isotypic subspace $V_{\lambda^*}$ spanned by the irreducible $K$-modules isomorphic to the module $\lambda^*$ contragredient to $\lambda$. Let $V_K$ be the algebraic direct sum of the $V_\lambda$. It consists of the $K$-finite vectors and is dense in $V$. Any $\alpha \in C_c^\infty(G)$ which is $K$-invariant leaves the $V_\lambda$ stable; using a Dirac sequence of such functions, one sees that $V_K^\infty = V^\infty \cap V_K$ is dense in $V$ and that $V^\infty \cap V_\lambda$ is dense in $V_\lambda$.

The actions of $\mathcal{U}(\mathfrak{g})$ and $K$ on $V_K^\infty$ satisfy the conditions

$$\pi(k)\pi(X)v = \pi(\mathrm{Ad}(k)X)\pi(k)v \quad k \in K, X \in \mathcal{U}(\mathfrak{g}), v \in V \tag{71}$$

which implies that

(1) $\mathcal{U}(\mathfrak{g})$ leaves $V_K^\infty$ stable.
(2) $W$ is a $K$-stable finite dimensional subspace of $V$ then the representation of $K$ on $W$ is differentiable, and has $\pi|\mathfrak{k}$ as its differential.

Conditions 1 and 2 define the notion of a $(\mathfrak{g}, K)$-module. It is admissible, or a Harish-Chandra module, if each $V_\lambda$ is finite dimensional, in which case $V_K \subset V^\infty$.

$V_K^\infty$ can be viewed as a module over a so called Hecke algebra $\mathcal{H} = \mathcal{H}(G, K)$, which is the convolution algebra of distributions on $G$ with support in $K$, and is isomorphic to $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} A_K$, where $A_K$ is the algebra of finite measures on $K$, but this interpretation plays no role in this course. (However, it will in the adelic framework.)

**8.4.** An element $v \in V^\infty$ is $\mathcal{Z}(\mathfrak{g})$ finite if $\mathcal{Z}(\mathfrak{g}).v$ is finite dimensional, hence if $v$ is annihilated by an ideal of finite codimension of $\mathcal{Z}(\mathfrak{g})$.

**8.5.** Assume now $(\pi, V)$ to be unitary and irreducible. By a theorem of Harish-Chandra ([4],5.25) it is admissible, hence $V_K = V_K^\infty$, and $\mathcal{Z}(\mathfrak{g})$ operates by homotheties. Therefore, every element of $V_K^\infty$ is $\mathcal{Z}(\mathfrak{g})$-finite.

**8.6.** *Discrete spectrum and $L^2$-automorphic forms.* Assume now that $V \subset L^2(\Gamma\backslash G)$ is irreducible. Then $V_K^\infty$ consists of elements which are $K$-finite, $\mathcal{Z}$-finite. By 2.6, they are automorphic forms.

Let $L^2(\Gamma\backslash G)_{dis}$ be the smallest closed subspace of $L^2(\Gamma\backslash G)$ containing all irreducible $G$-invariant subspaces. It is stable under $G$ and is a Hilbert direct sum of irreducible subspaces. It follows from 6.3 that it has finite multiplicities, in the sense of 6.5 Indeed, let $W$ be an irreducible summand of $L^2(\Gamma\backslash G)_{dis}$ and let $\lambda \in K$ be such that $W_\lambda \neq 0$. Then $W_\lambda$ consists of automorphic forms of a prescribed type $(J, \lambda)$, for some ideal $J$ of $\mathcal{Z}(\mathfrak{g})$. Since $\mathcal{A}(\Gamma, J, \lambda)$ is finite dimensional, $L^2(\Gamma\backslash G)_{dis}$ can have only finite many summands isomorphic to $W$ as $G$-modules.

Let $f \in L^2(\Gamma\backslash G)$ be $\mathcal{Z}(\mathfrak{g})$-finite and $K$-finite. Then, the smallest invariant closed subspace of $L^2(\Gamma\backslash G)$ containing $f$ is a finite sum of closed irreducible subspaces ([4], 5.26) therefore $f$ belongs to $L^2(\Gamma\backslash G)_{dis}$. As a consequence, the $L^2$-automorphic forms are the elements of $L^2(\Gamma\backslash G)_{dis}$ which are $\mathcal{Z}(\mathfrak{g})$- and $K$-finite, i.e. $(L^2(\Gamma\backslash G)_{dis})^\infty_K$.

*Remark* 1. We have referred to statements in [**?**] in which $G$ is assumed to be connected. However, our $G$ has at most finitely many connected components. It is standard and elementary that the restriction to a normal subgroup of finite index of an irreducible representation is a finite sum of irreducible representations, so the extension to our case is immediate.

**8.6.** If $\Gamma\backslash G$ is compact, then ${}^\circ L^2(\Gamma\backslash G) = L^2(\Gamma\backslash G)_{dis} = L^2(\Gamma\backslash G)$. In general ${}^\circ L^2(\Gamma\backslash G) \subset L^2(\Gamma\backslash G)_{dis}$, by 6.5. Its complement there is called the residual spectrum $L^2(\Gamma\backslash G)_{res}$. The complement of $L^2(\Gamma\backslash G)_{dis}$ in $L^2(\Gamma\backslash G)$ is the continuous spectrum $L^2(\Gamma\backslash G)_{ct}$. It is analyzed by means of the analytic continuation of Eisenstein series. The elements of $L^2(\Gamma\backslash G)_{res}$ occur as residues or iterated residues at poles of analytically continued Eisenstein series.

As a generalization of the statements made in the proof of 7.4, it is also true that the restriction of $\star\alpha$ ($\alpha \in C^\infty_c(G)$) to $L^2(\Gamma\backslash G)_{dis}$ is of trace class. This was proved by W. Müller first for a $K$-finite $\alpha$, and then in general, independently, by W. Müller and L. Ji.

## 11. A decomposition of the space of automorphic forms.

The main purpose of this section is to define a projector of the space $\mathcal{A}(\Gamma\backslash G)$ of automorphic forms on $\Gamma\backslash G$ onto the space ${}^\circ\mathcal{A}(\Gamma\backslash G)$ of cusp forms.

11.1. The parabolic subgroups $P, Q \in \mathcal{P}_\mathbb{Q}$ are said to be associate if a $G(\mathbb{Q})$-conjugate of one has a common Levi $\mathbb{Q}$-subgroup with the other. Let Ass($\mathbf{G}$) be the set of $\mathbb{Q}$-conjugacy classes of associated parabolic $\mathbb{Q}$-subgroups. It is finite, since $\Gamma\backslash\mathcal{P}_\mathbb{Q}$ is finite.

Before giving the following definition, let us remark that if $f \in C_{mg}(\Gamma\backslash G)$ and $g \in C_{fd}(\Gamma\backslash G)$, then $f.\overline{g}$ is fast decreasing and *a fortiori* in $L^1(\Gamma\backslash G)$, so that the scalar product

$$(f,g) = \int_{\Gamma\backslash G} f(x)\overline{g(x)}dx$$

is well-defined. $f$ and $g$ are said to be orthogonal if $(f,g) = 0$.

11.2. Let $f \in C^{\infty}_{umg}$ and $P \in \mathcal{P}_{\mathbb{Q}}$. The function $f$ is said to be negligible along $P$, in sign $f \sim_P 0$, if $f_P$ is orthogonal to the cusp forms on $\Gamma_{M_P}\backslash M_P = 0$. More generally, given $P \in \mathcal{P}_{\mathbb{Q}}$, if $f$ is negligible along all parabolic $\mathbb{Q}$-subgroups properly contained in $\mathbf{P}$, then $f_P$ is a cusp form. If, in addition, it is negligible along $P$, then $f_P$ is zero. This applies in particular to $G$. This notion and these results are due to Langlands ([14], lemma 37, Cor, p.58, [13] Theorem 6).

11.3. Let us write $\mathcal{V}_\Gamma$ for $C^{\infty}_{umg}(\Gamma\backslash G)$. Given $\mathcal{P} \in \mathrm{Ass}(\mathbf{G})$, let $\mathcal{V}_{\Gamma,\mathcal{P}}$ be the set of elements in $\mathcal{V}_\Gamma$ which are negligible along $\mathbf{Q}$ for all $\mathbf{Q} \notin \mathcal{P}$. The remarks just made show that the sum of the $\mathcal{V}_{\Gamma,\mathcal{P}}$ is direct. It is more difficult to prove:

**Theorem.** *(Langlands)*

$$\mathcal{V}_\Gamma = \oplus_{\mathcal{P}\in \mathrm{Ass}(G)} V_{\Gamma,\mathcal{P}}$$

Let $\mathcal{A}(\Gamma\backslash G)_{\mathcal{P}} = \mathcal{A}(\Gamma\backslash G) \cap V_{\Gamma,\mathcal{P}}$. Among the elements of $\mathrm{Ass}(G)$ there is the class consisting of $\mathbf{G}$ itself. It is clear from the definitions that

$$V_{\Gamma,\{G\}} =^{\circ} V_{\Gamma,G} \quad \mathcal{A}(\Gamma\backslash G)_G =^{\circ} \mathcal{A}(\Gamma\backslash G) \tag{72}$$

These decompositions show the existence of a canonical projector of $V_\Gamma$ (resp. $\mathcal{A}(\Gamma\backslash G)$ onto $^{\circ}V_\Gamma$ (resp. $^{\circ}\mathcal{A}(\Gamma\backslash G)$), with kernel the sum of the terms corresponding to associated classes of proper parabolic $\mathbb{Q}$-subgroups.

*Remark* 2. The theorem was proved by Langlands in a letter to me (1982). A proof, in the more general case of $S$-arithmetic subgroups, is given in [9], §4.

## 12. Some estimates of growth functions

These estimates pertain to the functions $a_P(x)^\lambda$ which measure growth rate on Seigel sets. They will be used to prove the convergence of certain Eisenstein series (11.) The technique to establish them is also basic in reduction theory, but will not be used elsewhere in this course.

12.1. Let $P_0 = N_0A_0M_0$ be the group of real points of a minimal parabolic $\mathbb{Q}$-subgroup $P_0$ and $\Delta = \Delta(A_0, G)$ the associated set of simple $\mathbb{Q}$-roots. We fix a Weyl group invariant scalar product on $X(A)$ (or $\mathfrak{a}^*$). Let $\{\beta_\alpha\}$ be the dual basis of $\{\alpha\}$ so that

$$(\beta_\alpha, \gamma) = \delta_{\alpha\gamma} \tag{73}$$

The open positive Weyl chamber $C$ in $X(A)$ is the set of linear combinations of the $\beta_\alpha$ with strictly positive coefficients. Its closure is the cone of dominant weights $\lambda$, characterized by the conditions $(\alpha, \lambda) \geq 0$ for $\alpha \in \Delta$.

Let $\kappa \in \Delta$. It is known that there exists an irreducible representation of $G$, defined over $\mathbb{Q}$, having a highest weight line defined over $\mathbb{Q}$, with highest weight a rational positive multiple $\omega_\alpha$ of $\beta_\alpha$. The highest weight line is then stable under the maximal parabolic $\mathbb{Q}$-subgroup $P^\alpha$. Any positive integral linear combination $\omega = \sum m_\alpha \omega_\alpha$ of the $\omega_\alpha$ is then the highest weight of an irreducible rational representation $(\sigma_\omega, V_\omega)$ with a similar property : the highest weight line is stable under the parabolic subgroup $P_J$, where $J$ is the set of $\alpha$ for which $m_\alpha \neq 0$. We fix a scalar product on $V$ which is invariant under $K$, and let $|| \circ ||$ be the corresponding Eucldean norm, and write $\sigma$ for $\sigma_\omega$. Let $e_\omega$ be a basis vector of the highest weight line.

Let $x \in G$. It has the decomposition $x = n_x m_x a(x) k$. The elements $n_x m_x$ leave the highest weight line invariant and $k$ is isometric.Therefore

$$||\sigma(x^{-1})e_\omega|| = a(x)^{-\omega} \tag{74}$$

12.2. **Proposition.** We keep the previous ntoation and fix a Siegel set $\mathfrak{S}$ with respect to $P_0$. Then

$$a(y.x)^\omega \prec a(y)^\omega a(x)^\omega \quad (y \in G, x \in \mathfrak{S}) \tag{75}$$

$$a(\gamma) \prec 1 \quad (\gamma \in \Gamma) \tag{76}$$

*Proof.* The element $e_1$ is rational, so some multiple may be assumed to belong to a lattice in $V_\omega(\mathbb{Q})$ that is stable under $\Gamma$. Therefore the set of $\gamma e_\omega$ is discrete in $V_\omega$, and does not contain zero, so that there exists $c > 0$ so that $||\gamma e|| \geq c$ for all $\gamma \in \Gamma$. In view of 10.1(2), this proves 78.

Fix an orthonormal basis $e_i$ of $V_\omega(\mathbb{Q})$ consisting of eigenvectors of $A_0$. We have

$$\sigma(y)^{-1}e_\omega = \sum_i c_i(y)e_i$$

and hence, by (1) and (2),

$$a(y)^{-2\omega} = \sum_i c_i(y)^2 \tag{77}$$

On the other hand,

$$||\sigma(x)^{-1}v|| \asymp ||\sigma(a(x)^{-1})v||, v \in V, x \in \mathfrak{S} \tag{78}$$

We have

$$\sigma(a(x)^{-1}y^{-1})e_\omega = \sum_i c_i(y)a(x)^{-\beta_i}e_i$$

where $\beta_i$ is the weight of $e_i$. It is of the form

$$\beta_i = \omega - \sum_{\alpha\in\Delta} m_\alpha(\beta_i)\alpha \quad m_\alpha(\beta_i) \in \mathbb{N}, m_\alpha(\beta_i) \geq 0$$

therefore

$$a(x)^{-\beta_i} \succ a(x)^{-\omega} \tag{79}$$

from 78 and **??** we get

$$||\sigma(x^{-1}y^{-1})e_\omega|| \asymp \sum_i c_i(y)^2 a(x)^{-2\beta_i} \succ a(x)^{-2\omega}(\sum_i c_i(y)^2)$$

Together with 75, this yields the first assertion. □

*Remark:* Let $D$ be a compact set containing the identity. Then

$$a(y.x)^\omega \prec a(y)^\omega a(x)^\omega \quad (y \in G, x \in D\mathfrak{S}) \tag{80}$$

Indeed, we have first $a(dx)^\omega \prec a(d)^\omega a(x)^\omega$, by the proposition, but since $D$ contains 1, $a(x)^\omega \prec a(dx)^\omega$ for $d \in D, x \in S$, and therefore

$$a(dx)^\omega \asymp a(x)^\omega \tag{81}$$

In particular, we can let $x$ run through a fundamental set $\omega = D.\mathfrak{S}$ for $\Gamma$, with $D \in G(\mathbb{Q})$ finite and containing 1.

This was for $P_0$. The extension of (i) to a general $P$ is easy. Assume that $P_0 \subset P$ so $P = P_J$ for some $J \subset \Delta$. The restrictions of the $\beta_\alpha$, for $\alpha \notin J$, to $X(A_P)$ form a basis dual to $\Delta(P, A_P)$. The rational representations $\sigma_\omega$, where $\omega$ is an integral linear combintation of the $\alpha$ (for $\alpha \notin J$) are highest weights of rational irreducible representtations whose highest weight lines are stable under $P$. Let $\mathfrak{S}_P$ be a Siegel set with respect to $P$. The previous argument yields

12.3. **Corollary.** We have

$$a(yx)^\omega \prec a(y)^\omega a(x)^\omega \quad (y \in G, x \in \mathfrak{S}_P)$$

where $\omega$ is a positive linear combination of the $\omega_\alpha$ for $\alpha \notin J$.

12.4. IT will be useful to express some of the previous results in terms of the elements of $A_0$ or $A_P$. As usual, $A^+$ is the positive Weyl chamber in $A$, the set of elements on which the $\alpha \in \Delta(P, A_P)$ are $\geq 1$. The dual cone ${}^+A$ is the set of elements on which the $\beta_\alpha$ are $\geq 1$. We let ${}^-A = \{a \in A, a^{-1} \in^+ A\}$. Then 12.2 gives

12.5. **Proposition.** There exists $a_0 \in A_P$ such that

$$a(\gamma) \in a_0\, {}^-A_P \quad (\gamma \in \Gamma) \tag{82}$$

$$a(\gamma x) \in a_0\, a(x)^- A_P \quad (\gamma \in \Gamma, x \in \mathfrak{S}_P) \tag{83}$$

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