THE SPINOR L-FUNCTION

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Abstract. In this paper we survey some recent results regarding automorphic forms on the similitude symplectic group of order four. We will also explain recent progress on analytic properties of L-functions associated to such automorphic forms.

Contents

Introduction

The purpose of this note is to put together in a suggestive way some of the recent result in the theory of automorphic forms on the similitude symplectic group of order four. Our emphasis will be on spinor L-function as the title indicates. This is by no means to suggest that the standard L-function is not equally as interesting. A quick search in `mathscinet` reveals however that the standard L-function is better understood, and it may be time to devote some energy to the spinor L-function. There is nothing new in these notes. In fact, this paper is what a pirate's closet might have looked like.

The paper is roughly organized as follows. In the first section we review the basic structure of the group under study. We also define

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Siegel modular forms and explain how one might associate adèlic automorphic forms to them. In the next section, we describe the theory of the spinor L-function. Here we will define two integral representations for the L-function; one due to Novodvorsky and one due to Piatetski-Shapiro. The latter integral works for all automorphic representations, whereas the former works only for generic representation. We include a resume of the results of the author on the determination of the local non-archimedean Euler factors of the Novodvorsky's integral. We will also review Moriyama's result on the archimedean representations. Piatetski-Shapiro's integral representation involve the Bessel functionals. For this reason, we have devoted the third section to the study of Bessel functionals and their existence. As an application of the spinor L-function, we have consider CAP representations in the fourth section where a theorem of Piatetski-Shapiro is discussed. In the last paragraph we have listed a number of recent developments which have a bearing on our understanding of the spinor L-function, e.g. Asgari-Shahidi's transfer of generic cusp forms on GSp(4) to GL(4) which among other things implies the holomorphy of the spinor L-function of such forms.

There are many interesting and important topics that are not mentioned in these notes; in particular no connection to the arithmetic of special values of L-functions is discussed. This is the topic that has fueled this author's interest in the subject. These notes are based on the talks given by the author at the Summer School on Algebraic Groups at the Mathematisches Institut at Göttingen during June and July of 2005. The author wishes to thank the hospitality of the Mathematisches Institut. He also wishes to thank Yuri Tschinkel for making the visit possible. The author was partially supported by the NSA.

1. Classical Siegel forms and automorphic representations

1.1. **Preliminaries on** GSp(4)**.** In this paper, the group GSp(4) over an arbitrary field K is the group of all matrices $g \in \mathrm{GL}_4(K)$ that satisfy the following equation for some scalar $\nu(g) \in K$:

$$ {}^t g J g = \nu(g) J, $$

where $J = \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}$. It is a standard fact that $G = \mathrm{GSp}(4)$ is a reductive group. The map $(F^\times)^3 \longrightarrow G$, given by

$$(a, b, \lambda) \mapsto \mathrm{diag}(a, b, \lambda a^{-1}, \lambda b^{-1})$$

gives a parameterization of a maximal torus T in G. The Weyl group is a dihedral group of order eight. We have three standard parabolic subgroups: The Borel subgroup B, The Siegel subgroup P, and the Klingen subgroup Q with the following Levi decompositions:

$$B = \left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & a^{-1}\lambda & \\ & & & b^{-1}\lambda \end{pmatrix} \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{pmatrix} \begin{pmatrix} 1 & & s & r \\ & 1 & r & t \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\},$$

$$P = \left\{ \begin{pmatrix} g & \\ & \alpha\, {}^t g^{-1} \end{pmatrix} \begin{pmatrix} 1 & & s & r \\ & 1 & r & t \\ & & 1 & \\ & & & 1 \end{pmatrix} ; g \in \mathrm{GL}(2) \right\},$$

and finally Q is the maximal parabolic subgroup with non-abelian unipotent radical associated to the long simple root. If ψ is an additive character of the field K, we define a character θ_ψ of the unipotent radical $N(B)$ of the Borel subgroup by the following:

$$\theta(\begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{pmatrix} \begin{pmatrix} 1 & & s & r \\ & 1 & r & t \\ & & 1 & \\ & & & 1 \end{pmatrix}) = \psi(x + t).$$

When K is a local field, we take always take ψ to be an unramified Tate character.

We define various subgroups of the group $G = \mathrm{Sp}(4)$ over the real numbers. We have

$$G(\mathbb{R}) = \{ g \in \mathrm{GL}_4(\mathbb{R}) \mid {}^t g J g = J \},$$

where as before $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$. Then the Lie algebra $\mathfrak{g}$ of G will be the set of matrices $X \in \mathfrak{sl}_4(\mathbb{R})$ such that ${}^t XJ + JX = 0$. The Cartan involution is given by $\theta(X) = -{}^tX$. Then we let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and -1 eigen-spaces of θ, respectively. We have

$$\mathfrak{k} = \{\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A + iB \in U(2)\},$$

and

$$\mathfrak{p} = \{\begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mid A = {}^tA, B = {}^tB\}.$$

Let K be the analytic subgroup defined by $\mathfrak{k}$.

1.2. **From classical Siegel forms to** $\mathrm{GSp}(4)$**.** In this section we review the relation between classical Siegel modular forms and automorphic forms on the group $\mathrm{GSp}(4)$. We will follows the exposition of [2]. Let $\mathcal{H}_n$ be the complex manifold consisting of complex symmetric $n \times n$ matrices with positive definite imaginary part. Let $\Gamma_2 = \mathrm{Sp}(4, \mathbb{Z})$, and let f be a Siegel modular form with respect to Γ_2 of weight k. By definition, f is a holormophic function on $\mathcal{H}_2$ satisfying

$$f|_k\gamma = f \tag{1.1}$$

for each $\gamma \in \Gamma_2$. Here

$$f|_k h = \mu(h)^{nk/2} \jmath(h, Z)^{-k} f(h < Z >) \tag{1.2}$$

for $h \in \mathrm{GSp}_4(\mathbb{R})^+$ and $Z \in \mathcal{H}_n$. In this equation, μ is the multiplier, $\jmath(h, Z) = \det(CZ + D)$ for $h = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, and $h(Z) = (AZ + B)(CZ + D)^{-1}$. We now associate to f an adelic function Φ_f as follows. By strong approximation

$$\mathrm{GSp}(4, \mathbb{A}) = \mathrm{GSp}(4, \mathbb{Q})\mathrm{GSp}(4, \mathbb{R})^+ \prod_{v<\infty} \mathrm{GSp}(4, \mathbb{Z}_p). \tag{1.3}$$

According to this decomposition, write an element $g \in \mathrm{GSp}(4, \mathbb{A})$ as $g = g_{\mathbb{Q}} g_\infty k_0$ with the obvious notation. Define

$$\Phi_f(g) = (f|_k g_\infty)(I), \tag{1.4}$$

where $I = \mathrm{diag}(i, i, \ldots, i) \in \mathcal{H}_n$. The function $\Phi = \Phi_f$ has the following properties:

(1) $\Phi(\gamma g) = \Phi(g)$ for $\gamma \in \mathrm{GSp}(4, \mathbb{Q})$;
(2) $\Phi(gk_0) = \Phi(g)$ for $k_0 \in K_0$;
(3) $\Phi(gk_\infty = \Phi(g) j(k_\infty, I)^{-k}$ for $k_\infty \in K_\infty$;
(4) $\Phi(zg) = \Phi(g)$ for $g \in Z(\mathbb{A})$.

The map $f \to \Phi_f$ send $S_k(\Gamma_n)$ to the space of cusp forms. This map is also a Hecke-equivariant isometry between the L^2-spaces for appropriately normalized invariant measures.

Let f be a cuspidal Hecke eigenform for the full Hecke algebra. Denote by V_f the subspace of $L^2_0(\mathrm{GSp}(4,\mathbb{Q})Z(\mathbb{A})\backslash\mathrm{GSp}(4,\mathbb{A}))$ spanned by all the right translates of Φ_f. We let π_f be the irreducible automorphic cuspidal representation obtained from the right action of $\mathrm{GSp}(4,\mathbb{A})$ on V_f. Notice that here we are ignoring the issue of L-indistinguishability. As f is modular for Γ_2, the representation π_f will be unramified at all finite places. Fix a finite place v, then by general theory the v-component of π_f will be a representation of the form $\chi_1 \times \chi_2 \rtimes \chi_3$. As usual we call the complex numbers $\chi_0(\varpi)$, $\chi_1(\varpi)$, and $\chi_2(\varpi)$ the v-Satake parameters of the local v-component of the representation π_f. There is a simple relation relating these Satake parameters to the classical Satake parameters which will then induce a shift in the functional equation of the L-functions. In fact, the new Satake parameters are equal to $p^{\frac{3}{2}-k}a_0$, a_1, and a_2 with a_0, a_1, a_2 the classical ones.

Let us also say a word about the archimedean component of the representation π_f. Let

(1.5)
$$K_\infty = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{GL}_{2n}(\mathbb{R}) | A\,{}^tA + B + {}^tB = I, A\,{}^tB = B\,{}^tA \right\};$$

the Lie algebra of K_∞, denoted by $\mathfrak{k}$, is the collection of matrices of the same shape with A anti-symmetric and B symmetric. This is the $+1$-eigen-space of the Cartan involution $\theta X = -{}^tX$. We let $\mathfrak{p}$ be the (-1)-eigenspace of the Cartan involution. Clearly $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. One can see that $\mathfrak{p}_\mathbb{C} = \mathfrak{p}^+_\mathbb{C} \oplus \mathfrak{p}^-_\mathbb{C}$ with

$$\mathfrak{p}^\pm_\mathbb{C} = \left\{ \begin{pmatrix} A & \pm iA \\ \pm iA & -A \end{pmatrix} \in \mathsf{M}_{2n}(\mathbb{R}) | A = {}^tA \right\}. \tag{1.6}$$

Let

$$T_i = -i \begin{pmatrix} 0 & D_i \\ -D_i & 0 \end{pmatrix} \tag{1.7}$$

where D_i is the diagonal matrix with entry 1 at position (i,i) and zero everywhere else. Now Lemma 11 of [2] states that the representation π_∞ contains a smooth vector v_∞ with the following properties

(1) $\pi(k_\infty)v_\infty = j(k_\infty, I)^{-k} v_\infty$ for all $k_\infty \in K_\infty$.
(2) $T_i v_\infty = k v_\infty$ for $i = 1, 2$.

(3) $\mathfrak{p}_\infty^{\pm} v_\infty = 0$.

This last condition is equivalent to the holomorphy of the starting function f. Because of this π_∞ is a holomorphic discrete series representation if $k > 2$ and it is a limit of discrete series when $k = 2$. There is a similar description of the automorphic form associated to a vector-valued Siegel cusp form which is nicely explained in paragraph 4.5 of [2].

In the classical setting, if for each prime p, a_0, a_1, a_2 are the Satake parameters of the form f, we set

$$L_1(s,f) = \prod_p \left((1-p^{-s}) \prod_{i=1}^{2} (1-a_i p^{-s})(1-a_i p^{-s}) \right)^{-1}, \tag{1.8}$$

and

$$L_2(s,f) = \prod_p \left((1-a_0 p^{-s})(1-a_0 a_1 p^{-s})(1-a_0 a_2 p^{-s})(1-a_0 a_1 a_2 p^{-s}) \right)^{-1}. \tag{1.9}$$

We now observe that

$$L_1(s,f) = L(s, \pi_f, \text{Standard}), \tag{1.10}$$

and

$$L_2(s,f) = L(s, \pi_f, \text{Spinor}) \tag{1.11}$$

with the right hand side of the equations being Langlands L-functions of the associated representation π_f. In this survey we will concentrate on the spinor L-functions.

2. The Spinor L-function for GSp(4)

In this section, we review the integral representation given by Novodvorsky [28] for $G = \mathrm{GSp}(4)$. The details of the material in the following paragraphs appear in [5], [46]. In the last paragraph, we will review a different construction due to Piatetski-Shapiro that also works for non-generic representations.

2.1. **Whittaker models.** As we will primarily be dealing with representations which have Whittaker models, we take a moment to review basic definition and properties of such models.

Let π be an automorphic cuspidal representation of the group G. For each $\phi \in \pi$, we set

$$W_\phi(g) = \int_{(\mathbb{Q}\backslash\mathbb{A})^4} \phi\left(\begin{pmatrix} 1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & & x_4 & x_3 \\ & 1 & x_3 & x_1 \\ & & 1 & \\ & & & 1 \end{pmatrix} g\right) \times \psi^{-1}(x_1 + x_2)\, dx_1\, dx_2\, dx_3\, dx_4$$

Let N be the unipotent radical of the Borel subgroup. For each place v of $\mathbb{Q}$, the restriction of θ to $N(\mathbb{Q}_v)$ is denoted by θ_v. Consider the representation of G induced from the character θ_v of $N(\mathbb{Q}_v)$:

$$C^\infty_{\theta_v}(N(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v)) := \left\{ W : G(\mathbb{Q}_v) \to \mathbb{C} \mid \begin{array}{c} \text{smooth,} \\ {\scriptstyle W(ng)=\theta_v(n)W(g),} \\ {\scriptstyle n\in N(\mathbb{Q}_v), g\in G(\mathbb{Q}_v)} \end{array} \right\}. \tag{2.1}$$

The action of $G(\mathbb{Q}_v)$ on $C^\infty_{\theta_v}(N(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v))$ is by right translation.

If v is a finite place of $\mathbb{Q}$, then for any irreducible admissible representation π_v of $G(\mathbb{Q}_v)$, the intertwining space

$$\mathrm{Hom}_{G(\mathbb{Q}_v)}(\pi_v, C^\infty_{\theta_v}(N(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v)))$$

is at most one dimensional ([37], Theorem 3). If there is a non-zero intertwining operator

$$\Psi \in \mathrm{Hom}_{G(\mathbb{Q}_v)}(\pi_v, C^\infty_{\theta_v}(N(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v))) \tag{2.2}$$

then we say that π_v is generic, and call the image $W_u := \Psi(u)$ of $u \in \pi_v$ the *local Whittaker function corresponding to* $u \in \pi_v$. The space of all W_u ($u \in \pi_v$) is called the *Whittaker model of* π_v *with respect to* θ_v.

Now let $v = \infty$ be the archimedean place. We say that a $\mathbb{C}$-valued function W on $G(\mathbb{R})$ is of moderate growth if there exists $C > 0$ and $M > 0$ such that $|W(g)| \le C\|g\|^M$ for all $g \in G(\mathbb{R})$. The form $\|g\|$ of $g = (g_{ij})$ is defined by $\|g\| := \max\{|g_{ij}|, |(g^{-1})_{ij}\}$. The space of functions $W \in C^\infty_{\theta_v}(N(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v))$ which is of moderate growth is denoted by $\mathcal{A}_{\theta_\infty}(N(\mathbb{R})\backslash G(\mathbb{R}))$. Improving Shalika's local multiplicity one theorem ([44], Theorem 3.1), Wallach ([51], Theorem 8.8 (1)) showed that for an arbitrary $(\mathfrak{g}, K)$-module π_∞ the intertwining space

$$\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\infty, \mathcal{A}_{\theta_\infty}(N(\mathbb{R})\backslash G(\mathbb{R})))$$

is at most one-dimensional. Again, if there is a non-zero intertwining operator

$$\Psi \in \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\infty, \mathcal{A}_{\theta_\infty}(N(\mathbb{R})\backslash G(\mathbb{R}))), \tag{2.3}$$

then we say π_∞ is generic and call the image $W_u := \Psi(u)$ of $u \in \pi_\infty$ the *local Whittaker function corresponding to u.*

2.2. Let φ be a cusp form on $\mathrm{GSp}(4, \mathbb{A})$, belonging to the space of an irreducible cuspidal automorphic representation π. Consider the integral

$$Z_N(s,\phi,\mu) = \int_{\mathbb{A}^\times/\mathbb{Q}^\times}\int_{(\mathbb{A}/\mathbb{Q})^3} \phi\left(\begin{pmatrix}1 & x_2 & x_4 & \\ & 1 & & \\ & & 1 & \\ & z & -x_2 & 1\end{pmatrix}\begin{pmatrix}y & & & \\ & y & & \\ & & 1 & \\ & & & 1\end{pmatrix}\right) \times \psi(-x_2)\mu(y)|y|^{s-\frac{1}{2}}\,dz\,dx_2\,dx_4\,d^\times y.$$

Since ϕ is left invariant under the matrix

$$\begin{pmatrix} & & & 1\\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix},$$

this integral has a functional equation $s \to 1-s$. A usual unfolding process as sketched in [5] then shows that

$$Z_N(s,\phi,\mu) = \int_{\mathbb{A}^\times}\int_{\mathbb{A}} W_\phi\begin{pmatrix}y & & & \\ & y & & \\ & & 1 & \\ & x & & 1\end{pmatrix}\mu(y)|y|^{s-\frac{3}{2}}\,dx\,d^\times y. \tag{2.4}$$

Here the Whittaker function W_φ is given by

$$W_\phi(g) = \int_{(\mathbb{A}/\mathbb{Q})^4}\phi\left(\begin{pmatrix}1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1\end{pmatrix}\begin{pmatrix}1 & & x_4 & x_3\\ & 1 & x_3 & x_1\\ & & 1 & \\ & & & 1\end{pmatrix} g\right) \times \psi^{-1}(x_1+x_2)\,dx_1\,dx_2\,dx_3\,dx_4$$

The basic idea in establishing the above identity is to show that

(2.5)
$$\int_{(F\backslash\mathbb{A})^3} \phi\left(\begin{pmatrix}1 & x_2 & & x_4\\ & 1 & & \\ & & 1 & -x_2\\ & & & 1\end{pmatrix} g\right)\psi(-x_2)\,dx_2\,dx_4 = \sum_{\substack{\alpha\in F^\times\\ \beta\in F}} W\left(\begin{pmatrix}\alpha & & & \\ & \alpha & & \\ & \alpha\beta & 1 & \\ & & & 1\end{pmatrix} g\right)$$

For this one uses the following identity from Fourier analysis

$$f(0) = \sum_{\alpha\in F}\int_{F\backslash\mathbb{A}} f(x_1)\psi(-\alpha x_1)\,dx_1, \tag{2.6}$$

for any reasonable F invariant function f.

Equation (2.4) implies that, in order for $Z_N(\varphi, s)$ to be non-zero, we need to assume that W_φ is not identically equal to zero. A representation satisfying this condition is called "generic." Every irreducible cuspidal representation of GL(2) is generic. On other groups, however, there may exist non-generic cuspidal representations. In fact, those representations of GSp(4) which correspond to holomorphic cuspidal Siegel modular forms are not generic.

If φ is chosen correctly, the Whittaker function may be assumed to decompose locally as $W(g) = \prod_v W_v(g_v)$, a product of local Whittaker functions. Hence, for $\Re s$ large, we obtain

$$\mathcal{Z}(\varphi, s) = \prod_v \mathcal{Z}(W_v, s), \tag{2.7}$$

where

$$Z_N(W_v, s) = \int_{F_v^\times}\int_{F_v} W_v(\begin{pmatrix}y & & & \\ & y & & \\ & & 1 & \\ & x & & 1\end{pmatrix})|y|^{s-\frac{3}{2}}\,dx\,d^\times y. \tag{2.8}$$

As usual, we have a functional equation: There exists a meromorphic function $\gamma(\pi_v, \psi_v, s)$ (rational function in $\mathbb{N}v^{-s}$ when $v < \infty$) such that

$$Z_N(W_v, s) = \gamma(\pi_v, \psi_v, s)\tilde{\mathcal{Z}}(W_v^w, 1-s), \tag{2.9}$$

with w as above,

$$\tilde{\mathcal{Z}}(W_v, s) = \int_{F_v^\times} \int_{F_v} W_v(\begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & x & & 1 \end{pmatrix}) \chi_v^{-1}(y) |y|^{s-\frac{3}{2}} \, dx \, d^\times y,$$

and χ_v the central character of π_v.

We also consider the unramified calculations. Suppose v is any nonarchimedean place of F such that W_v is right invariant by $\mathrm{GSp}(4, \mathrm{O}_v)$ and such that the largest fractional ideal on which ψ_v is trivial is O. Then the Casselman-Shalika formula [7] allows us to calculate the last integral (cf. [5]). Let us review the computation. For this, we need to recall what the Casselman-Shalika formula says in this context. Set

$$u(\xi_1, \xi_2, \xi_3, \xi_4) = \begin{pmatrix} \xi_1\xi_2 & & & \\ & \xi_1\xi_3 & & \\ & & \xi_2\xi_4 & \\ & & & \xi_3\xi_4 \end{pmatrix}. \tag{2.10}$$

Then the Weyl group acts on the polynomial ring $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_4^{\pm 1}]$. The action is in such a way that the two generators

$$w_1 = \begin{pmatrix} 1 & & & \\ & & -1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} \tag{2.11}$$

act via $w_1 : x_1 \to x_2 \to x_1, x_3 \to x_4 \to x_3$ and $w_2 : x_1 \to x_1, x_2 \to x_3 \to x_2, x_4 \to x_4$. Now define a group algebra element

$$\mathcal{B} = \sum_{w \in W} (-1)^{l(w)} w, \tag{2.12}$$

where $l(w)$ is the length function on the Weyl group. Let

$$T_{k_1,k_2}(x_1, x_2, x_3, x_4) = \frac{\mathcal{B}(x_1^{k_1+k_2+3} x_2^{k_1+2} x_3 x_4^{-k_2})}{\mathcal{B}(x_1^3 x_2^2 x_3)}. \tag{2.13}$$

The Casselman-Shalika formula states the following: Let $X_\pi = u(\xi_1, \xi_2, \xi_3, \xi_4)$ be the semi-simple conjugacy class in $\mathrm{GSp}_4(\mathbb{C})$ associated to the unramified representation π, and W the normalized Whittaker function for π.

Then if $ord(y_i) = k_i$, then

(2.14)

$$W\left(\begin{pmatrix} y_0y_1y_2 & & & \\ & y_0y_1 & & \\ & & y_0 & \\ & & & y_0y_2^{-1} \end{pmatrix}\right) = q^{-\frac{3}{2}k_1-2k_2}(\xi_1\xi_2\xi_3\xi_4)^{k_0}T_{k_1,k_2}(\xi_1,\xi_2,\xi_3,\xi_4)$$

if $k_1, k_2 \geq 0$, and zero otherwise.

The result is the following:

$$\mathcal{Z}(W_v, s) = L(s, \pi_v, \mathrm{Spin}). \tag{2.15}$$

Let us explain the notation. The connected L-group ${}^LG^0$ is $\mathrm{GSp}_4(\mathbb{C})$. Let ${}^LT$ be the maximal torus of elements of the form

$$t(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \alpha_3 & \\ & & & \alpha_4 \end{pmatrix},$$

where $\alpha_1\alpha_4 = \alpha_2\alpha_3$. The fundamental dominant weights of the torus are λ_1 and λ_2 where

$$\lambda_1 t(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \alpha_1,$$

and

$$\lambda_2 t(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \alpha_1\alpha_3^{-1}.$$

The dimensions of the representation spaces associated with these dominant weights are four and five, respectively. In our notation, Spin is the representation of $\mathrm{GSp}(4,\mathbb{C})$ associated with the dominant weight λ_1, i.e. the standard representation of $\mathrm{GSp}(4,\mathbb{C})$ on $\mathbb{C}^4$. The L-function $L(s,\pi,\mathrm{Spin})$ is called the Spinor, or simply the Spin, L-function of $\mathrm{GSp}(4)$.

Next step is to use the integral introduced above to extend the definition of the Spinor L-function to ramified non-archimedean and archimedean places.

Remark 2.1. There are constructions for the degree 8 (resp. 12) L-function for generic representations of $\mathrm{GSp}(4) \times \mathrm{GL}_2$ (resp. $\mathrm{GSp}(4) \times \mathrm{GL}_3$). For a review see [5].

2.3. **Local Euler factors of the spinor L-function.** We now sketch the computation of the local non-archimedean Euler factors of the Spin L-function of generic representations given by the integral representation introduced above. In order for this to make sense, we need the following lemma:

Lemma 2.2 (Theorem 2.1 of [46])**.** *Suppose* Π *is a generic representation of* $\mathrm{GSp}(4)$ *over a non-archimedean local field* K*,* q *order of the residue field. For each* $W \in \mathcal{W}(\Pi, \psi)$*, the function* $\mathcal{Z}(W, s)$ *is a rational function of* q^{-s}*, and the ideal* $\{\mathcal{Z}(W, s)\}$ *is principal.*

Sketch of proof. For $W \in \mathcal{W}(\Pi, \psi)$, we set

$$Z(W, s) = \int_K W\left(\begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\right) |y|^{s-\frac{3}{2}}\, d^\times y. \tag{2.16}$$

The first step of the proof is to show that the vector space $\{Z(W, s)\}$ is the same as $\{\mathcal{Z}(W, s)\}$ (cf. Proposition 3.2 of [46]). Next, we use the asymptotic expansions of the Whittaker functions along the torus to prove the existence of the g.c.d. for the ideal $\{Z(W, s)\}$. Indeed, Proposition 3.5 of [46] (originally a theorem in [7]) states that there is a finite set of finite functions S_Π, depending only on Π, with the following property: for any $W \in \mathcal{W}(\Pi, \psi)$, and $c \in S_\Pi$, there is a Schwartz-Bruhat function $\Phi_{c,W}$ on K such that

$$W(\begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & & & 1 \end{pmatrix}) = \sum_{c \in S_\Pi} \Phi_{c,W}(y) c(y) |y|^{\frac{3}{2}}.$$

The lemma is now immediate. □

We have the following theorem:

Theorem 2.3. *Suppose* Π *is a generic representation of the group* $\mathrm{GSp}(4)$ *over a non-archimedean local field* K*. Then*

1. *If* Π *is supercuspidal, or is a sub-quotient of a representation induced from a supercuspidal representation of the Klingen parabolic subgroup, then* $L(s, \pi, \mathrm{Spin}) = 1$*.*
2. *If* π *is a supercuspidal representation of* $\mathrm{GL}(2)$ *and* χ *a quasi-character of* $K^\times$*, and* $\Pi = \pi \rtimes \chi$ *is irreducible, we have*
$$L(s, \Pi, \mathrm{Spin}) = L(s, \chi).L(s, \chi.\omega_\pi).$$
3. *If* χ_1*,* χ_2*, and* χ_3 *are quasi-characters of* $K^\times$*, and* $\Pi = \chi_1 \times \chi_2 \rtimes \chi_3$ *is irreducible, we have*
$$L(s, \Pi, \mathrm{Spin}) = L(s, \chi_3).L(s, \chi_1\chi_3).L(s, \chi_2\chi_3).L(s, \chi_1\chi_2\chi_3).$$

(4) *When* Π *is not irreducible, one can prove similar statements for the generic subquotients of* $\Pi = \pi \rtimes \chi$ *(resp.* $\Pi = \chi_1 \times \chi_2 \rtimes \chi_3$*) according to the classification theorems of Sally-Tadic* [38] *and Shahidi* [41] *(cf. theorems 4.1 and 5.1 of* [46]*).*

Remark 2.4. Sally and Tadic [38] and Shahidi [41] have completed the classification of representations supported in the Borel and Siegel parabolic subgroups. In particular, they have determined for which representations the parabolic induction is reducible. From their result, one can immediately establish a classification for all the generic representations supported in the Borel or Siegel parabolic subgroups.

Sketch of proof. By the proof of the lemma, we need to determine the asymptotic expansion of the Whittaker functions in each case. The argument consists of several steps:

Step 1. *Bound the size of* S_Π. Fix $c \in S_\Pi$, and define a functional Λ_c on $\mathcal{W}(\Pi, \psi)$ by

$$\Lambda_c(W) = \Phi_{c,W}(0). \tag{2.17}$$

If $c, c' \in S_\Pi$, and $c \neq c'$, the two functionals Λ_c and $\Lambda_{c'}$ are linearly independent. Furthermore, the functionals Λ_c belong to the dual of a certain twisted Jacquet module $\Pi_{N,\bar{\theta}}$ (notation from [46], page 1095). Hence $\#S_\Pi = \dim \Pi_{N,\bar{\theta}}$. Then one uses an argument similar to those of [44], distribution theory on p-adic manifolds, to bound the dimension of the Jacquet module. The result (proposition 3.9 of [46]) is that if Π is supercuspidal or supported in the Klingen parabolic subgroup (resp. Siegel parabolic, resp. Borel parabolic), then $\#S_\Pi = 0$ (resp. ≤ 2, resp. ≤ 4). Note that this already implies the first part of the theorem.

From this point on, we concentrate on the Siegel parabolic subgroup, the Borel subgroup case being similar. We fix some notation. Suppose $\Pi = \pi \rtimes \chi$, with π supercuspidal of GL(2). Let λ_Π (resp. λ_π) be the Whittaker functional of Π (resp. π) from [40]. It follows from the proof of the lemma 2.2 that, for $f \in \Pi$, there is a positive number $\delta(f)$, such that

$$\lambda_\Pi(\Pi(\begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & & & 1 \end{pmatrix})f) = \sum_{c \in S_\Pi} \Lambda_c(f) c(y) |y|^{\frac{3}{2}},$$

for $|y| < \delta(f)$. Here, Λ_c is the obvious functional on the space of Π.

Step 2. *Uniformity.* For $f \in Ind(\pi \times \chi | P \cap K, K)$, and $\tau \in \mathbb{C}$, define f_τ on G by

$$f_\tau(pk) = \delta_P(p)^{\tau+\frac{1}{2}} \pi \otimes \chi(p) f(k).$$

It is clear that f_τ is a well-defined function on G, and that it belongs to the space of a certain induced representation Π_τ. The *Uniformity Theorem* (Proposition 3.9 of [46]) asserts that one can take $\delta(f_\tau) = \delta(f)$.

Step 3. *Regular representations.* This is the case where $\omega_\pi \neq 1$. In this situation, we have

(2.18)

$$\lambda_\Pi(\Pi(\begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & & & 1 \end{pmatrix})f) =$$

$$\lambda_\pi(A(w,\Pi)(f)(e))\chi(y)|y|^{\frac{3}{2}} + C(w\Pi, w^{-1})^{-1}\lambda_\pi(f(e))\chi(y)\omega_\pi(y)|y|^{\frac{3}{2}},$$

for $|y| < \delta(f)$. Here $w = \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}$, $A(w,\Pi)$ is the intertwining integral of [40], and $C(w\Pi, w^{-1})$ is the local coefficient of [40]. The proof of this identity follows from the the above lemma 2.2, and the *Multiplicity One Theorem* [44]. The idea is to find one term of the asymptotic expansion using the open cell; then apply the long intertwining operator to find the other term.

Note that the identity of *Step* 3 also applies to reducible cases. For example, if $f \in \Pi$ is in the kernel of the intertwining operator $A(w,\Pi)$, the first term of the right hand side vanishes.

Step 4. *Irregular Representations.* The idea is the following: we twist everything in *Step* 3 by the complex number τ, so that the resulting representation Π_τ is regular. By *Step* 2, the identity still holds uniformly for all τ. By a theorem of Shahidi [40] (essentially due to Casselman and Shalika [7]), we know that the left hand side of the identity is an entire function of τ. This implies that the poles of the right hand side, coming from the intertwining operator and the local coefficient, must cancel out. Next, we let $\tau \to 0$. An easy argument (l'Hopital's rule!) shows the appearance of $\chi(y)|y|^{\frac{3}{2}}$ and $\chi(y)|y|^{\frac{3}{2}} \log_q |y|$ in the asymptotic expansion.

This finishes the sketch of proof of the theorem. □

Corollary 2.5. *Let π be an irreducible generic representation of* GSp(4) *over a non-archimedean local field K. Let μ be a quasi-character of $K^\times$. If μ is highly ramified, we have*

$$L(s, \pi \otimes \mu) = 1.$$

2.4. **Moriyama's results at the archimedean place.** In this paragraph we review Moriyama's computation of the archimedean Euler factors of Novodvorsky's integral for generic linits of discrete series. In order to state his results, however, we need to set up some notation.

Here let $G = \mathrm{GSp}(4)$ and $G_0 = \mathrm{Sp}(4, \mathbb{R})$. A maximal compact subgroup K (resp. K_0) of $G(\mathbb{R})$ (resp. G_0) is given by $K := G(\mathbb{R}) \cap \mathrm{O}(4)$ (resp. $K_0 := G_0 \cap \mathrm{O}(4)$). The group K_0 is isomorphic to the unitary group $U(2) := \{g \in \mathrm{GL}(2, \mathbb{C}) | \, {}^t\bar{g}g = I_2\}$. Define an isomorphism $\kappa : U(2) \cong K_0$ by

$$A + \sqrt{-1}B \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K_0 \tag{2.19}$$

if $A + \sqrt{-1}B \in U(2)$. We write the Lie algebra of $G(\mathbb{R})$, G_0, and K_0 by $\mathfrak{g}$, $\mathfrak{g}_0$ and $\mathfrak{k}$ respectively. For an arbitrary Lie algebra $\mathfrak{l}$, we denote its complexification by $\mathfrak{l}_\mathbb{C}$. The differential κ_* of κ defines an isomorphism of complex Lie algebras $\mathfrak{gl}(2, \mathbb{C}) \cong \mathfrak{k}_\mathbb{C}$. The simple Lie algebra $\mathfrak{g}_0$ has a compact Cartan subalgebra $\mathfrak{h} := \mathbb{R}T_1 \oplus \mathbb{R}T_2$, where

$$T_1 := \kappa_* \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad T_2 = \kappa_* \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{-1} \end{pmatrix}. \tag{2.20}$$

Define a $\mathbb{C}$-basis $\{\beta_1, \beta_2\}$ of $\mathfrak{h}_\mathbb{C}^*$ by $\beta_i(T_j) = \sqrt{-1}\delta_{ij}$, and fix an inner product $<,>$ on $\mathfrak{h}_\mathbb{C}^*$ by $< \beta_i, \beta_j >= \delta_{ij}$. Then the root system $\Delta = \Delta((\mathfrak{g}_0)_\mathbb{C}, \mathfrak{h}_C)$ is given by $\{\pm 2\beta_1, \pm 2\beta_2, \pm\beta_1 \pm \beta_2\}$. Denote by Δ_c the set of compact root $\{\pm(\beta_1 - \beta_2\}$ and take as a positive system of compact roots the set $\Delta_c^+ = \{\beta_1 - \beta_2\}$.

The irreducible finite dimensional representations of K_0 are parametrized by the set of their highest weights relative to Δ_c^+:

$$\{q = q_1\beta_1 + q_2\beta_2 = (q_1, q_2) \in \mathfrak{h}_C^*, q_i \in \mathbb{Z}, q_1 \geq q_2\}. \tag{2.21}$$

For each dominant weight $q = (q_1, q_2)$, we set $d_q = q_1 - q_2 \geq 0$. Then the degree of the representation $(\tau_{(q_1,q_2)}, V_{(q_1,q_2)})$ with highest weight (q_1, q_2) is $d_q + 1$. We can take a basis $\{v_k | 0 \leq k \leq d\}$ of $V_{(q_1,q_2)}$ so that the representation of $\mathfrak{k}_{\mathbb{C}}$ associated to (q_1, q_2) is given by

$$\tau(\kappa_* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})v_k = (q_2 + k)v_k;$$
$$\tau(\kappa_* \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})v_k = (q_1 - k)v_k;$$
$$\tau(\kappa_* \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})v_k = (k+1)v_{k+1};$$
$$\tau(\kappa_* \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})v_k = (d+1-k)v_{k-1}. \tag{2.22}$$

We call $\{v_k\}$ the standard basis of $V_{(q_1,q_2)}$.

There are four positive systems of Δ containing Δ_c:

$$\Delta_I^+ = \{(1,-1), (2,0), (1,1), (0,2)\};$$
$$\Delta_{II}^+ = \{(1,-1), (0,-2), (2,0), (1,1)\};$$
$$\Delta_{III}^+ = \{(1,-1), (-1,-1), (0,-2), (2,0)\};$$
$$\Delta_{IV}^+ = \{(1,-1), (-2,0), (-1,-1), (0,-2)\}. \tag{2.23}$$

Let $J \in \{I, II, III, IV\}$. Set $\Delta_{J,n}^+ = \Delta_J^+ \backslash \Delta_c^+$. For each index J, we denote by Ξ_J the set of integral weight $\Lambda = (\Lambda_1, \Lambda_2) \in \mathfrak{h}_C^*$, $\Lambda_i \in \mathbb{Z}$, satisfying (i) $< \Lambda, \beta >\geq 0$ for all $\beta \in \Delta_{J,n}^+$ and (ii) $< \Lambda, \beta >> 0$ for all $\beta \in \Delta_c^+$ that is a simple root in Δ_J^+. Then the set $\{(J, \Lambda) | \Lambda \in \Xi_J\}$ gives the collection of Harish-Chandra parameters of the (limits of) discrete series representations for G_0. We denote by $\pi(\Lambda, \Delta_J^+)$ the representation of G_0 associated to the parameter. If $< \Lambda, \beta >> 0$ for all $\beta \in \Delta_J^+$, then $\pi = \pi(\Lambda, \Delta_J^+)$ is a discrete series representations; otherwise a limit of discrete series. The Blattner parameter $\lambda_{min} \in \mathfrak{h}_{\mathbb{C}}^*$ of π is given by

$$\lambda_{min} := \Lambda + \frac{1}{2} \sum_{\alpha \in \Delta_J^+} \alpha - \sum_{\beta \in \Delta_c^+} \beta. \tag{2.24}$$

The highest weights of the K_0-types of π are of the form $\lambda_{min} + \sum_{\alpha \in \Delta_J^+} m_\alpha \alpha$ with m_α integral and non-negative. Furthermore, $\tau_{\lambda_{min}}$ occurs in π with multiplcity one, and we call it the minimal K_0 type of π. We denote by $D_{(q_1,q_2)}$ the representation of this form with minimal K_0-type equal to $\tau_{(q_1,q_2)}$. A (limit of) discrete series representation $\pi(\Lambda, \Delta_J^+)$ is called *large* if $J = II$ or III. Observe that $\pi(\Lambda, \Delta_{II}^+ = \Delta_{\Lambda+(1,0)}$ and $\pi(\Lambda, \Delta_{II}^+ = \Delta_{\Lambda+(0,-1)}$.

Now suppose $\Pi_{\mathbb{R}}$ is a representation of $\mathrm{GSp}(4,\mathbb{R})$ whose restriction to $\mathrm{Sp}(4,\mathbb{R})$ is the direct sum of two (limits of) discrete series representation $D_{(\lambda_1,\lambda_2)}$ and $D_{(-\lambda_2,-\lambda_1)}$. Let $\omega_{\Pi_{\mathbb{R}}}$ be the central character of $\Pi_{\mathbb{R}}$ and define a complex number ω_∞ by $\omega_{\Pi_{\mathbb{R}}}(t) = t^{\omega_\infty}$ $(t \in \mathbb{R}, t > 0)$. Assume that the representation $\Pi_{\mathbb{R}}$ has a local Whittaker model. Hence by a theorem of Kostant on the existence of local Whittaker models, the representation of G_0 occurring in Π_∞ must be large. This means that we must have $1-\lambda_1 \leq \lambda_2 \leq 0$ or $1+\lambda_2 \leq -\lambda_1 \leq 0$. Without loss of generality we will assume that $1-\lambda_1 \leq \lambda_2 \leq 0$. Set

$$L(s,\Pi_{\mathbb{R}}) = \Gamma_{\mathbb{C}}(s + \frac{\omega_\infty+\lambda_1+\lambda_2-1}{2})\Gamma_{\mathbb{C}}(s+\frac{\omega_\infty+\lambda_1-\lambda_2-1}{2}). \tag{2.25}$$

Define the ϵ-factor by $\epsilon(s,\Pi_{\mathbb{R}},\psi_\infty) := (-1)^{\lambda_1}$.

Let $\{v_k | 0 \leq k \leq d = \lambda_1 - \lambda_2\}$ be the standard basis of the minimal K_0-type $\tau_{(-\lambda_2,-\lambda_1)}$ of $D_{(-\lambda_2,-\lambda_1)}$. We denote by $W_k \in \mathsf{W}(\Pi_{\mathbb{R}},\psi_\infty)$ the local Whittaker function corresponding to $v_k \in \Pi_{\mathbb{R}}$. Define a vector subgroup A of $\mathrm{GSp}(4,\mathbb{R})$ by

$$A := \{\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_1^{-1} & \\ & & & a_2^{-1}\end{pmatrix} | a_i > 0\}. \tag{2.26}$$

Moriyama [27] proves the following explicit formula for the value of the function W_k on A:

Theorem 2.6. *For each $0 \leq k \leq d$, the support of W_k is contained in the identity component of* $\mathrm{GSp}(4,\mathbb{R})$*. Furthermore, if (σ_1,σ_2) is a pair of real numbers satisfying*

$$\sigma_1+\sigma_2+1 > 0, \quad \sigma_1 > 0 > \sigma_2, \tag{2.27}$$

then there is a non-zero constant $C \in \mathbb{C}^\times$ *independent of* $0 \le k \le d$ *such that*

(2.28)

$$W_k \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_1^{-1} & \\ & & & a_2^{-1} \end{pmatrix} = C \binom{d}{k} (s\sqrt{-1})^{-k} \exp(-2\pi a_2^2)$$
$$\times \int_{L(\sigma_1)} \frac{ds_1}{2\pi\sqrt{-1}} \frac{\Gamma(s_1+k)}{\Gamma(s_1)} (4\pi^3 a_1^2)^{\frac{-s_1+\lambda_1+1-k}{2}} \int_{L(\sigma_2)} \frac{ds_2}{2\pi\sqrt{-1}} (4\pi a_2^2)^{\frac{-s_2+\lambda_2+k}{2}}$$
$$\Gamma(\frac{s_1+s_2-2\lambda_2+1}{2})\Gamma(\frac{s_1+s_2+1}{2})\Gamma(\frac{s_1}{2})\Gamma(\frac{-s_2}{2}).$$

Here for a real number σ, $L(\sigma)$ *denotes the path* $\sigma - \sqrt{-1}\infty$ *to* $\sigma + \sqrt{-1}\infty$.

For an outline of the proof, which involves solving partial differential equations, see 3.2 in [26].

Moriyama then introduces the following slightly more general integral

$$Z_N^\infty(s, y_1, W) = \int_{\mathbb{R}^\times} \int_{\mathbb{R}} W \begin{pmatrix} yy_1 & & & \\ & y & & \\ & & y_1^{-1} & \\ & x & & 1 \end{pmatrix} |y|^{s-\frac{3}{2}} dx\, dy^\times. \tag{2.29}$$

Clearly we are interested in the value of this integral when $y_1 = 1$.

Theorem 2.7 (Proposition 8 of [26])**.** *The integrals* $Z_N^\infty(s, y_1, W_k)$, $0 \le k \le d$, *converge absolutely for* $\Re s > (-\lambda_1 - \lambda_2 + 1)/2$ *and*

(2.30)

$$\frac{Z_N^\infty(s, y_1, W_k)}{L(s, \Pi_{\mathbb{R}})} = C_1 \binom{d}{k} (\sqrt{-1})^{-k} (4\pi)^s$$
$$\int_{L(\sigma_1)} \frac{ds_1}{2\pi\sqrt{-1}} (4\pi y_1)^{-s_1-s+\lambda_1+1} \frac{\Gamma(s_1 + \frac{-\lambda_1-\lambda_2+1}{2})\Gamma(s_1 + \frac{-\lambda_1+\lambda_2+1}{2})}{\Gamma(\frac{s_1-s-d+k+2}{2})\Gamma(\frac{s_1+s-k+1}{2})}$$

with a non-zero constant $C_1 \in \mathbb{C}$ *independent of* $0 \le k \le d$. *Here* $\sigma \in \mathbb{R}$ *is taken so that* $\sigma_1 > (\lambda_1 - \lambda_2 - 1)/2$. *Moreover, for each fixed* $y_1 > 0$, *the integral in the above expression extends to an entire function of* s.

The proof of this theorem involves a very clever application of Barnes' first lemma. Barnes' first lemma says

(2.31)
$$\frac{1}{2\pi\sqrt{-1}}\int_{L(0)}\Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s)\,ds = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}$$

for all complex numbers a, b, c, d, provided that neither of the numbers $a+c, a+b, b+c, b+d$ is a non-positive integer.

When $y_1 = 1$, one can simplify this expression in the theorem even further. For $\lambda_2 = 0$, the expression turns out to be zero for all k (!). For $\lambda_2 < 0$, we have

(2.32)
$$\frac{Z_N^\infty(s, W_k)}{L(s, \Pi_{\mathbb{R}})} = C_1\binom{d}{k}(\sqrt{-1})^{-k}\sum_{l=1}^{-\lambda_2}(-4\pi)^{-l}\frac{(l-1)!}{(-\lambda_2-l)!} \Gamma(\frac{s-k+l+\frac{\lambda_1+\lambda_2+1}{2}}{2})^{-1}\Gamma(\frac{-s+k+l+\frac{-\lambda_1+3\lambda_2+3}{2}}{2})^{-1}.$$

This is about the best one could have hoped for. Clearly, the expression on the right hand side has a very strong arithmetic scent. It should now be possible to obtain arithmeticity results for generic automorphic forms with archimedean components in the generic limits of discrete series.

[26] contains the treatment of some other representations which are induced from (limits of) discrete series representations of the Levi factor of the Klingen parabolic subgroups. The main result of [26] is the statement that if Π is a generic irreducible automorphic cuspidal representation of GSp(4) over a totally real field, then $L(s, \Pi, spin)$ is entire, and satisfies the correct functional equation. This statement now trivially follows from the general results of Asgari and Shahidi [4]. But what does not follow from [4] is the explicit computation of the integral as in equation (2.32).

2.5. **A different construction.** Here we review the article [32] which contains a construction for L-functions of automorphic representations of GSp(4) which works for representations that are not necessarily generic. This is a construction of the spinor L-function that also works for non-generic representations. This is also very closely related to the Bessel functional. The genesis of this method is an idea of Andrianov for the case of holomorphic Siegel modular forms. Let the base field be $\mathbb{Q}$. Let K be a quadratic extension of $\mathbb{Q}$ such that ϕ, which in our case is an automorphic form coming from a Siegel cusp form, has a non-zero Fourier coefficient parametrized by a symmetric matrix having eigenvalues in K; K is then an imaginary quadratic field. Then $\mathrm{Sp}(4)$ contains a copy of $Res_{K/\mathbb{Q}}\mathrm{SL}(2)$ the restriction of scalars of SL from K to $\mathbb{Q}$. Andrianov then proved that restricting to this subgroup and integrating against an Eisenstein series gives the degree four L-function of ϕ. Piatetski-Shapiro's contribution was to interpret this adelically and apply it to cusp forms that were not necessarily associated to holomorphic Siegel cusp forms.

Let P be the Siegel parabolic subgroup, and let S be its unipotent radical. Let ℓ be a non-degenerate linear form on S, and let D be the connected component of the stabilizer of ℓ in M, the Levi component of P. Then there is a unique semisimple algebra K over the base field k of degree two, such that $D = K^\times$. It is known that either $K = k \oplus k$ or K is a quadratic extension of k. The important subgroup is $R = DS$. Let $N = \{s \in S | \ell(s) = 0\}$. Set $V = K^2$ and consider the group

$$G = \{g \in \mathrm{GL}_2(K) | \det(g) \in k^\times\}. \tag{2.33}$$

We will write vectors in V in row form and let G act on the right. On V we consider the skew symmetric form

$$\rho(x, y) = \mathrm{tr}_{K/k}(x_1 y_2 - x_2 y_1) \tag{2.34}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ as elements of V. Then G preserves ρ up to a factor in $k^\times$. If we consider V as a four dimensional vector space over k then we obtain an embedding

$$G \hookrightarrow \mathrm{GSp}_\rho. \tag{2.35}$$

If we define the k-linear transformation ι on V by

$$(x_1, x_2) \mapsto (\bar{x}_1, \bar{x}_2) \tag{2.36}$$

then ι preserves ρ and gives a well-defined element of GSp_ρ.

Let ψ be a non-degenerate character on $S_\mathbb{A}$, and ν a character on $D_\mathbb{A} \cong I_K$. Let $V_\mathbb{A}$ be the adelic points of the vector space on which $G_\mathbb{A}$ acts. Take $\Phi \in \mathcal{S}(V_\mathbb{A})$, the Schwartz-Bruhat functions on $V_\mathbb{A}$. Let μ be a Hecke character on k. Then we can associate to Φ a function on $G_\mathbb{A}$ defined by

$$f^\Phi(g, \mu, \nu, s) = \mu(\det g)|\det g|^{s+\frac{1}{2}} \int_{I_K} \Phi((0, t)g)|t\bar{t}|^{s+\frac{1}{2}} \nu(t) d^\times t. \tag{2.37}$$

Here $|.|$ is the idele norm on I_K. Note that $f^\Phi(g,\mu,\nu,s) \in Ind_{B'_{\mathbb{A}}}^{G_{\mathbb{A}}}\chi$ where χ is a character on $B'_{\mathbb{A}}$ defined by

$$\chi\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}\begin{pmatrix} \bar{t} & \\ & t \end{pmatrix}\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}\right) = \mu(x)|x|^{s+\frac{1}{2}}\nu^{-1}(t). \tag{2.38}$$

We then form the Eisenstein series

$$E^\Phi(g,\mu,\nu,s) = \sum_{\gamma\in B'_k\backslash G_k} f^\Phi(\gamma g,\mu,\nu,s). \tag{2.39}$$

Then E^Φ is a meromorphic function on $\mathbb{C}$, and satisfies a functional equation of the usual type. Furthermore, if $\mu(t\bar{t})\nu(t) \neq 1$, E^Φ has no poles. If $\mu(t\bar{t})\nu(t) = 1$, poles are only at $s = -\frac{1}{2}$ with residue $\Phi(0)\mu(\det g)$, and at $s = \frac{3}{2}$ with residue $\hat{\Phi}(0)\mu^{-1}(\det g)\nu_k(\det g)$, where ν_k is obtained by restricting ν to I_k.

Now let π be an automorphic cuspidal representation of GSp(4) acting on V_π. Suppose for ψ and ν as above we have

$$\int_{Z_{\mathbb{A}}R_k\backslash R_{\mathbb{A}}} \varphi(r)\alpha_{\nu,\psi}^{-1}(r)\,dr \neq 0 \tag{2.40}$$

for some $\varphi \in V_\pi$. Here $R = SD$, and $\alpha_{\nu,\psi}^{-1}(sd) = \psi(s)\nu(d)$ if $s \in S$ and $d \in D$. If the above integral is non-zero for some choice of φ, we may set

$$W_\varphi(g) = \int_{Z_{\mathbb{A}}R_k\backslash R_{\mathbb{A}}} \varphi(rg)\alpha_{\nu,\psi}^{-1}(r)\,dr. \tag{2.41}$$

These functions satisfy

$$W_\varphi(rg) = \alpha_{\nu,\psi}(r)W_\varphi(g), \tag{2.42}$$

for $r \in R_{\mathbb{A}}$. Denote the space of all such functions by $\mathcal{W}^{\nu,\psi}$. The action of $\mathrm{GSp}_4(\mathbb{A})$ on $\mathcal{W}^{\nu,\psi}$ is equivalent to π. As in the case of the Whittaker model, there will be local models for the the representations π_v, and the uniqueness of the global model follows from the uniqueness of the local models.

For $\phi \in V_\pi$ set

$$Z(s,\phi,\mu) = \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} \phi(g)E^\phi(g,\mu,\nu,s)\,dg. \tag{2.43}$$

Then $Z(s,\phi)$ converges in some right half plane, and has a meromorphic continuation, and furthermore satisfies a functional equation dictated by that of the Eisenstein series. In fact, if $\mu(t\bar{t})\nu(t) \neq 1$, then $Z(\phi, s, \mu)$ is entire; otherwise it has a poles at $s = -\frac{1}{2}$ with residue

$$\Phi(0) \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \mu(\det g) \phi(g)\, dg, \tag{2.44}$$

and at $s = \frac{3}{2}$ with residue

$$\hat{\Phi}(0) \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \mu^{-1}(\det g) \nu_k(\det g) \phi(g)\, dg. \tag{2.45}$$

One can also show that

$$Z(s, \phi, \mu) = \int_{D_{\mathbb{A}} N_{\mathbb{A}} \backslash G_{\mathbb{A}}} W_\phi(g) f^{\Phi}(g, \mu, \nu, s)\, dg. \tag{2.46}$$

If the data is chosen correctly the latter integral is an infinite product, and this gives us a definition of the local zeta integral. One then proceeds to define the local L-factors $L(s, \pi_p, \mu_p)$. It turns out that if π_p is unramified, then the resulting L-factors is the same as the Langlands L-factors obtained from tensoring the natural embedding $\mathrm{GSp}(4, \mathbb{C}) \hookrightarrow \mathrm{GL}_4(\mathbb{C})$ with $\mathbb{C}^\times$. The global L-function $L(s, \pi, \mu)$ defined by this zeta integral then converges in some right half plane, and has a meromorphic continuation to the entire complex plane. It also satisfies the appropriate functional equation. Furthermore, if some local component of π is generic, the L-function $L(s, \pi, \mu)$ is entire.

Remark 2.8. It is likely that the construction reviewed in this paragraph is the one needed in [39].

3. Bessel functionals

3.1. **Bessel functionals.** Here we study the unique models introduced in the previous section in more details. We recall the notion of Bessel model introduced by Novodvorsky and Piatetski-Shapiro [29]. We follow the exposition of [8]. Let $S \in M_2(\mathbb{Q})$ be such that $S = {}^tS$. We define the discriminant $d = d(S)$ of S by $d(S) = -4 \det S$. Let us define a subgroup $T = T_S$ of $\mathrm{GL}(2)$ by

$$T = \{g \in \mathrm{GL}(2) \mid {}^tgSg = \det g.S\}.$$

Then we consider T as a subgroup of $\mathrm{GSp}(4)$ via

$$t \mapsto \begin{pmatrix} t & \\ & \det t.\, {}^tt^{-1} \end{pmatrix},$$

$t \in T$.

Let us denote by U the subgroup of GSp(4) defined by

$$U = \{u(X) = \begin{pmatrix} I_2 & X \\ & I_2 \end{pmatrix} \mid X = {}^tX\}.$$

Finally, we define a subgroup R of GSp(4) by $R = TU$.

Let ψ be a non-trivial character of $\mathbb{Q}\backslash\mathbb{A}$. Then we define a character ψ_S on $U(\mathbb{A})$ by $\psi_S(u(X)) = \psi(\mathrm{tr}(SX))$ for $X = {}^tX \in \mathsf{M}_2(\mathbb{A})$. Usually when there is no danger of confusion, we abbreviate ψ_S to ψ. Let Λ be a character of $T(\mathbb{Q})\backslash T(\mathbb{A})$. Denote by $\Lambda \otimes \psi_S$ the character of $R(\mathbb{A})$ defined by $(\Lambda \otimes \psi)(tu) = \Lambda(t)\psi_S(u)$ for $t \in T(\mathbb{A})$ and $u \in U(\mathbb{A})$.

Let π be an automorphic cuspidal representation of $\mathrm{GSp}_4(\mathbb{A})$ and V_π its space of automorphic functions. We assume that

$$\Lambda|_{\mathbb{A}^\times} = \omega_\pi. \tag{3.1}$$

Then for $\varphi \in V_\pi$, we define a function B_φ on $\mathrm{GSp}_4(\mathbb{A})$ by

$$B_\varphi(g) = \int_{Z_{\mathbb{A}} R_{\mathbb{Q}} \backslash R_{\mathbb{A}}} (\Lambda \otimes \psi_S)(r)^{-1}.\varphi(rh)\, dh. \tag{3.2}$$

We say that π has a global Bessel model of type (S, Λ, ψ) for π if for some $\varphi \in V_\pi$, the function B_φ is non-zero. In this case, the $\mathbb{C}$-vector space of functions on $\mathrm{GSp}_4(\mathbb{A})$ spanned by $\{B_\varphi \mid \varphi \in V_\pi\}$ is called the space of the global Bessel model of π.

Similarly, one can consider local Bessel models. Fix a local field $\mathbb{Q}_v$. Define the algebraic groups T_S, U, and R as above. Also, consider the characters Λ, ψ, ψ_S, and $\Lambda \otimes \psi_S$ of the corresponding local groups. Let (π, V_π) be an irreducible admissible representation of the group GSp(4) over $\mathbb{Q}_v$, when v is finite, or a $(\mathfrak{g}, K)$-module when v is archimedean. Then we say that the representation π has a local Bessel model of type (S, Λ, ψ) if there is a non-zero map in

$$\mathrm{Hom}(\pi_v, Ind(\Lambda \otimes \psi | R, G)). \tag{3.3}$$

Here the Hom space is the collection of $G(\mathbb{Q}_v)$-intertwining maps when v is finite, and the collection of all $(\mathfrak{g}, K)$-maps when v is archimedean. Also in the archimedean case, as in the Whittaker case, we work with that subspace of Ind which consists of functions of moderate growth.

Typically one is interested in two different types of Bessel models corresponding to two choices of the symmetric matrix S. The two choices of S are:

(1) $S = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$,

(2) $S = \begin{pmatrix} 1 & \\ & d \end{pmatrix}$, with d a positive square-free rational number.

Below, we will determine the subgroups T_S, and R, and explicitly write down the corresponding global Bessel functionals. We fix an irreducible automorphic cuspidal representation π of $\mathrm{GSp}_4(\mathbb{A})$ and a unitary character ψ of $\mathbb{A}$ throughout.

(1) $S = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. This is the case of interest for us in this work. In this case, the subgroup T_S is equal to the subgroup consisting of diagonal matrices. A straightforward analysis then shows that for every character Λ of $T_S(\mathbb{Q})\backslash T_S(\mathbb{A})$ subject to (3.1), there is a Hecke character of $\mathbb{A}^\times$ such that the global Bessel functional (3.2) is given by

$$B_\chi^{\text{split}}(g;\varphi) = \int_{F^\times\backslash\mathbb{A}^\times} \varphi^U(\begin{pmatrix} y & & & \\ & 1 & & \\ & & 1 & \\ & & & y \end{pmatrix})\chi(y)\,d^\times y.$$

Here when ϕ is a cusp form on GSp(4), we have set

$$\phi^U(g) = \int_{(F\backslash\mathbb{A})^3} \phi(\begin{pmatrix} 1 & & u & w \\ & 1 & w & v \\ & & 1 & \\ & & & 1 \end{pmatrix} g)\psi^{-1}(w)\,du\,dv\,dw.$$

(2) $S = \begin{pmatrix} 1 & \\ & d \end{pmatrix}$. In this case, the subgroup T_S is equal to a non-split torus. Then there is a Hecke character of the torus T_S, say χ, in such a way that

$$B_\chi(g;\varphi) = \int_{T_S(F)\mathbb{A}^\times\backslash T_S(\mathbb{A})} \varphi^U(\begin{pmatrix} \alpha & \\ & \det\alpha.\,{}^t\alpha^{-1} \end{pmatrix})\chi(\alpha)\,d\alpha,$$

with ϕ^U defined as before. The case of immediate interest is the case where $d = 1$, in which case,

$$\begin{aligned} T_S &= \{g \in \mathrm{GL}_2 \mid {}^t g.g = \det g\,\} \\ &= \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 \in \mathrm{GL}_1\}. \end{aligned}$$

The problems of existence of Bessel functionals for this choice of the matrix S seem to be more delicate.

3.2. **An interesting relevant pair.** In this paragraph, we explain how Bessel functionals are related to the setup of [14]. Here we follow the conventions of [30]. Let k be a field of characteristic not equal to two. The space

$$V = \{T \in \mathsf{M}_4(k) \,|\, TJ_2 \text{ is skew-symmeric and } \mathrm{tr}T = 0\}$$

is a five dimensional vector space over k. Here $J_n = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}$. The group $\mathrm{GSp}(4,k)$ acts on V by

$$g : T \mapsto g^{-1}Tg.$$

A symmetric non-degenerate form on V is given by

$$(T_1, T_2) = \frac{1}{4}\mathrm{tr}T_1T_2.$$

If we set $Q(T) = (T,T)$, then we have $Q(g.T) = Q(T)$ for all $g \in \mathrm{GSp}(4)$ and $T \in V$. In fact, we have an isomorphism $\mathrm{PGSp}(4) \simeq \mathrm{SO}(5)$.

More explicitly, the vector space V can be given in the following way:

$$V = \left\{ \begin{pmatrix} M & xJ_1 \\ yJ_1 & {}^TM \end{pmatrix} \mid M \in \mathsf{M}_2(k), \mathrm{tr}M = 0, x, y \in k \right\}.$$

Also, the quadratic form Q is given by

$$Q(T) = -\det M - xy,$$

for $T = \begin{pmatrix} M & xJ_1 \\ yJ_1 & {}^TM \end{pmatrix}$. We will denote the element T by $[M,x,y]$. The action of the group $\mathrm{GSp}(4)$ on the space V is explicitly given by the following relations:

$$\begin{pmatrix} A & \\ & \lambda\, {}^TA^{-1} \end{pmatrix}.T = [A^{-1}MA, x\lambda(\det A)^{-1}, y\lambda^{-1}\det A],$$

$$\begin{pmatrix} I_2 & S \\ & I_2 \end{pmatrix}.T = [M - ySJ_1, (x - y\det S)J_1 + MS - {}^T(MS), y]$$

where $S = {}^TS$, and

$$\begin{pmatrix} & -I_2 \\ I_2 & \end{pmatrix}.T = [{}^TM, -y, -x].$$

We will also need the following even more explicit realization. The space is given by

$$V = \left\{ T = \begin{pmatrix} t & v+w & 0 & x \\ v-w & -t & -x & 0 \\ 0 & y & t & v-w \\ -y & 0 & v+w & -t \end{pmatrix} \right\}$$

equipped with the quadratic form

$$Q(T) = t^2 + v^2 - w^2 - xy.$$

Then V has a two dimensional quadratic subspace

$$W = \left\{ T' = \begin{pmatrix} t & v & & \\ v & -t & & \\ & & t & v \\ & & v & -t \end{pmatrix} \right\}$$

equipped with

$$Q'(T') = t^2 + v^2.$$

The pair (V, W) is relevant, as $\dim V - \dim W = 3$ is odd, and

$$W^{\perp} = \left\{ T'' = \begin{pmatrix} 0 & w & 0 & x \\ -w & 0 & -x & 0 \\ 0 & y & 0 & -w \\ -y & 0 & w & 0 \end{pmatrix} \right\}$$

equipped with $Q''(T'') = -w^2 - xy$ is split. We set

$$X = \{[0_2, x, 0], x \in k\},$$
$$X' = \{[0_2, 0, y], y \in k\}.$$

Then X and X' are isotropic dual spaces. The stabilizer of X is the Siegel parabolic subgroup given by

$$P = \left\{ \begin{pmatrix} A & \\ & \lambda\, {}^T A^{-1} \end{pmatrix} \begin{pmatrix} I_2 & S \\ & I_2 \end{pmatrix} \right\}$$

The group $\mathrm{SO}(W)$ has the following realization

$$\left\{ \begin{pmatrix} A & \\ & {}^T A^{-1} \end{pmatrix}, A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \text{ with } a^2 + b^2 = 1 \right\}.$$

We note that $\mathrm{SO}(W) \subset P$. Then we consider a character of the unipotent radical N of the parabolic subgroup P. The subgroup N is abelian. The stabilizer of this character will just be a subgroup isomorphic to $\mathrm{SO}(W)$ in M. So in this case, the spherical subgroup is simply $\mathrm{SO}(W) \rtimes N_P$.

It remains to say something about the pure inner forms. We know from the discussion following corollary 8.10 of [14] that the only relevant inner form of the above $G = \mathrm{SO}(V) \times \mathrm{SO}(W)$, which is $\mathrm{SO}(3,2) \times \mathrm{SO}(2,0)$, is $G' = \mathrm{SO}(1,4) \times \mathrm{SO}(0,2)$. Here we give a realization for this group. Here we will copy the first couple of pages of [9].

Let D be the division algebra of Hamiltonian quaternions over $\mathbb{R}$, and let us denote the canonical involution of D by $x \mapsto \bar{x}$. Then we define G_D, the quaternion similitude unitary group of degree two over D, by

$$G_D = \left\{ g \in \mathrm{GL}_2(D) \,|\, g^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mu(g) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mu(g) \in D^\times \right\},$$

where

$$g^* = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \qquad \text{where} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We regard G_D as an algebraic group over $\mathbb{R}$. The group G_D is an inner form of $\mathrm{GSp}(4)$. In fact, if $D = \mathsf{M}_2(\mathbb{R})$, then in $\mathrm{GL}_4(\mathbb{R})$ we have

$$\xi G_D \xi^{-1} = G \quad \text{where} \quad \xi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We refer the reader to lemma 1.2 of [9] for the proof of the last statement.

3.3. **Existence.** The existence of Bessel functionals of various types poses an interesting problem. It is a theorem of Li [25] that any automorphic cuspidal representation of $\mathrm{GSp}(4)$ either has a Whittaker model, or some Bessel model. When $S = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, it follows from [46] that the existence of the the local Bessel functional, at least at the non-archimedean places, is equivalent to the existence of Whittaker models. At the archimedean place, the existence of the Bessel functional certainly implies the genericity. We certainly expect the converse too as suggested by the results of [26, 47]. If this is indeed the case, we will have the following global result:

Theorem 3.1. *Let (π, V_π) be an irreducible generic automorphic cuspidal representation. Let $S = I_2$, and χ a finite order Hecke character with*

χ_∞ *trivial. Then* π *has a non-zero global* (S, χ)*-Bessel model if and only if* $L(\frac{1}{2}, \pi \otimes \chi) \neq 0$.

This is of course not a hard theorem, and follows from simple observations.

It is well-known that automorphic representations associated to holomorphic Siegel modular forms are not generic; that is, they fail to have Whittaker models. It is also known that the genericity of such representations certainly fails at the archimedean place. For this reason it is desirable to determine when holomorphic discrete series representations posses Bessel models which seem to be the next best thing in applications to L-functions [8, 9]. We now turn our attention to the case where $S = \begin{pmatrix} 1 & \\ & d \end{pmatrix}$. For simplicity, assume $d = 1$. The conjecture of Gross and Prasad (Conjecture 6.9 of [14]) predicts that the existence of Bessel models for holomorphic discrete series is intertwined with the existence of such models for other members of the Vogan L-packet of the given discrete series representation, in particular the generic discrete series.

Let Π be a generic discrete series representation of $\mathrm{GSp}(4, \mathbb{R})$, with trivial central character. Then there is a pair (D_k, D_l) of discrete series representations of $\mathrm{GL}(2, \mathbb{R})$ with trivial central character such that Π is obtained by a theta lift from $\mathrm{GO}(2, 2)$ by the representation that the pair (D_k, D_l) defines. In order to land in generic discrete series, we need to assume that $k, l \geq 2$ satisfy $k \neq l$ and they have the same parity. Let n be an integer with $n > \max(k, l)$, and with different parity from k (or l). We set $\chi_n \left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right) = e^{in\theta}$. With these notations, we prove in [47] that

Theorem 3.2. Π *has a* $(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \chi_n, \psi)$*-Bessel model.*

A few remarks are in order. It is clear from the setup that the proof of the theorem uses theta correspondence; in fact, we will use the pull-back of the Bessel functional via the global theta correspondence, along with various substantial local and global results from the theory of automorphic forms [15, 23, 33, 50]. It may be desirable to find a direct local proof of the existence theorem as in [53]. Our attempts in this direction, however, have not been successful. Inspired by [42, 43], one is tempted to write down an integral and try to prove that the integral does not vanish for the correct choice of the data. There are convergence issues that one needs to deal with. In the Whittaker situation, what saves the day is the fact that one can do the analysis of the integrals "one root at a time"; we have not been able to successfully follow such an approach for the Bessel integrals. In order to establish the conjecture of Gross-Prasad for the pair $(\mathrm{SO}(5), \mathrm{SO}(2))$ for discrete series packets, one needs to study generic discrete series representations of PGSp(4), holomorphic discrete series representations of PGSp(4), and related representations of $\mathrm{SO}(4,1)$. The case of $\mathrm{SO}(4,1)$ is simpler as the group in question has rank one. Here we have considered the representations of the group PGSp(4). Thanks to Wallach's recent paper [53], the case of holomorphic representations is much better understood. This is the reason why we can concentrated our efforts on the generic case. Shalika has informed the author that he can prove the converse statement of the theorem using local methods based on [22]. Consequently, the "if" in the theorem may be replaced by "if and only if." Perhaps, it should also be pointed out here that, in light of Theorem 3.4 of [52], our results automatically extend to generic limits of discrete series. In [47] we have used the same idea of pulling back global Bessel functional via the theta correspondence to prove the analytic continuation of the spinor L-function of a large class of automorphic cusp forms. The class of cuspidal representation to which our result applies is restricted by a condition at the archimedean place, and for this reason our theorem is weaker than those obtained by Asgari and Shahidi. At a crucial point in our argument we have to use twisting with highly ramified characters and a density result from [49].

4. Saito-Kurokawa and CAP representations

4.1. **The failure of multiplicity one.** One of the features that distinguishes the symplectic group from the general linear groups is the failure of the *multiplicity one phenomena*. We know by the results of Jacquet and Shalika that if $\Pi_1 = \otimes_v \Pi_{1v}$ and $\Pi_2 = \otimes_v \Pi_{2v}$ are two automorphic cuspidal representations of GL_n such that $\Pi_{1v} \simeq \Pi_{2v}$ for almost all v, then $\Pi_1 = \Pi_2$. This means that first of all $\Pi_{1v} \simeq \Pi_{2v}$ for *all* v, and that the representations Π_1 and Π_2 correspond to the *same* irreducible subspace of L^2. Let us construct two irreducible cuspidal representations of GSp(4) that agree at almost all places, but are not isomorphic globally. Let D be a quaternion algebra over $\mathbb{Q}$ and compact at infinity. Let π_1, π_2 be two cuspidal representations of $D^\times$ with, say, trivial central character. Also assume that π_1 and π_2 are not equal. Then we know that the pair (π_1, π_2) defines a cuspidal representation of GSO(4); extend this representation trivially to GO(4). Then if we consider the dual pair $(\mathrm{GO}(4), \mathrm{GSp}(4))$, we can transfer the resulting representation to GSp(4) and obtain a non-zero automorphic cuspidal representation Π_1 of GSp(4) which is globally non-generic. On the other hand, let π_1^{JL} and π_2^{JL} be the representations of GL(2) which are the respective images of π_1, π_2 under the Jacquet-Langlands correspondence. Then we can construct an automorphic cuspidal representation of $\mathrm{GO}(2,2)$, and by considering the pair $(\mathrm{GO}(2,2), \mathrm{GSp}(4))$, a non-zero automorphic cuspidal representation of GSp(4) which is globally generic. Then it follows that the the representations Π_1 and Π_2 agree at all those places where D is split. For details of these constructions, see [18].

4.2. **CAP representations.** Another feature of sympletic groups that puts them in sharp contrast with the general linear groups is the existence of representations that are *Cuspidal Associated to a Parabolic*. In general, let G be a quasi-split reductive group, and let $P = MU$ be a parabolic subgroup of G. Also let τ be an automorphic representation of M. It is well-known that any irreducible constituent of $Ind_{P(\mathbb{A})}^{G(\mathbb{A})}$ is an automorphic representation, and that if $\Pi_1 = \otimes_v \Pi_{1v}$ and $\Pi_2 = \otimes_v \Pi_{2v}$ are two irreducible constituents then $\Pi_{1v} = \Pi_{2v}$ for almost all v.

Definition 4.1. Let $\pi = \otimes_v \pi_v$ be an irreducible automorphic cuspidal representation of G. We say π is *Cuspidal Associated to a Parabolic*, or simply CAP, if there is a parabolic subgroup $P = MU$ and an automorphic representation τ of M, and an irreducible constituent $\Pi = \otimes_v \Pi_v$ of $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}$ such that $\pi_v \simeq \Pi_v$ for almost all v.

When G is not quasi-split over the number field F, then it has an inner form G' that is quasi-split over F. Note that for almost all v, we have $G(F_v) = G'(F_v)$. In that situation, we call a cuspidal representation π of G *Cuspidal Associated to a Parabolic* if there is a parabolic subgroup $P = MU$ of G' and an automorphic representation τ of M such that π agrees locally at almost all places with an irreducible constituent of $Ind_{P(\mathbb{A})}^{G'(\mathbb{A})}$. This definition is suggested in [10].

In the case of GSp(4) we have three conjugacy classes of parabolic subgroups: the Borel subgroup B, the Siegel parabolic subgroup P, and the Klingen parabolic subgroup Q. So we may have representations that are CAP with respect to either of these parabolics. Note that if a representation is CAP with respect to B then it is CAP with respect to P and Q as $B \subset P$ and $B \subset Q$. For this reason, we introduce the notion of *Strongly*

Cuspidal Associated to a parabolic. This means that the representation τ in the definition is cuspidal. In the sequel, unless otherwise noted, CAP with respect to a parabolic subgroup is always to mean strongly CAP with respect to that parabolic.

4.3. **CAP representations for** $\mathrm{GSp}(4)$ **and its inner forms.** The following theorem is due to Piatetski-Shapiro [30]:

Theorem 4.2. *If π is CAP with respect to B or P with central character ω_π, then there is a representation π' of* $\mathrm{GSp}(4)$*, CAP with respect to B or P and with trivial central character, such that for all $g \in \mathrm{GSp}(4, \mathbb{A})$ we have*

$$\pi(g) = \pi'(g)\omega(\nu(g)). \tag{4.1}$$

A representation of $\mathrm{GSp}(4)$ with trivial central character is nothing but a representation of $\mathrm{PGSp}(4)$. We have identified the latter group with $\mathrm{SO}(3, 2)$. We may then consider the dual reductive pair $(\widetilde{\mathrm{SL}}_2, \mathrm{PGSp}(4))$. The following theorem is one of the main results of [30]:

Theorem 4.3. *Let π be an irreducible cuspidal representation of* $\mathrm{PGSp}(4)$*. Then the following are equivalent:*

(1) *π is CAP with respect to B or P;*
(2) *$L(s, \pi, spin)$ has poles;*
(3) *π is the theta lift of an irreducible cuspidal automorphic representation σ of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$.*

Observe that implicit in the theorem is the statement that the theta lift of any representation σ as in the last part of the theorem is always a non-trivial representation; this is Theorem 5.1 of [30]. This follows from the Rallis inner product formula and the fact that the pair $(\widetilde{\mathrm{SL}}_2, \mathrm{PGSp}(4))$ is in the stable range. For details see Theorem 2.9, 2.16, and 2.17 of [10]. The proof in [30] is more direct.

E. Sayag has proved the analogous statement for rank one inner form of the group $\mathrm{PGSp}(4)$; see [10] for the description of the results. Sayag's argument does not work in the anisotropic situation. [10] contains an interesting result for all inner forms, but Gan's result falls short of the complete characterization of the collection of CAP representations.

5. Further developments

5.1. **Roberts Packets.** Here we have used the lecturenotes of a talk by Roberts at Princeton seminar in February of 2002. Let F be a totally real number field and let E be either a totally real quadratic extension of F or $F \times F$. Let τ be a tempered cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$ such that τ is not Galois invariant, and ω_τ is Galois invariant. Then there must exist a Hecke character χ of F such that $\omega = \chi \circ N_{E/F}$. Then for each place v of F Brooks defined a local L-packet of tempered irreducible representations $\Pi(\chi_v, \tau_v)$ of $\mathrm{GSp}_4(F_v)$ associated conjecturally to an L-parameter $\varphi(\chi_v, \tau_v) : L_{F_v} \to \mathrm{GSp}_4(\mathbb{C})$. It is shown that these packets have many of the properties that one expects of Langlands packets. For example, $|\Pi(\chi_v, \tau_v)| = 1$ or 2, and equal to 1 for almost all v. One can also define global L-packets via

$$\Pi(\chi, \tau) = \otimes_v \Pi(\chi_v, \tau_v). \tag{5.1}$$

Roberts further proves the following multiplicity statement which is compatible with Arthur's conjectures:

Theorem 5.1. (1) *If E is a field, then every element of $\Pi(\chi, \tau)$ occurs with multiplicity one in the cusp forms of* $\mathrm{GSp}(4)$ *of central character χ;*
(2) *Assume $E = F \times F$, and $\Pi = \otimes_v \Pi_v \in \Pi(\chi, \tau)$. Let N be the number of times Π_v is non-generic. Then Π occurs with multiplicity $(1 + (-1)^N)/2$ in the cusp forms on* $\mathrm{GSp}(4)$.

Note that one still does not know that the packets only depend on the equivalent class of $\varphi(\chi_v, \tau_v)$, i.e. we do not know whether $\varphi(\chi_v, \tau_v) \cong \varphi(\chi'_v, \tau'_v)$ implies that $\Pi(\chi_v, \tau_v) = \Pi(\chi'_v, \tau'_v)$. Also, we do not have a purely local construction of the packets i.e. one that does not require global objects τ and χ. It would also be interesting to know exactly which local or global packets are obtained this way. Locally, if v is non-archimedean place not dividing 2, then $\varphi(\chi_v, \tau_v)$ include all but one parameter $\mathrm{sp}(2) \otimes \rho$ with $\dim \rho = 2$ irreducible orthogonal. When $v|2$, there a finite number of parameters not included. So most parameters are covered. The situation is similar to GL(2) in that there are in fact representations that cannot be obtained as theta lifts, namely the special representation.

5.2. **Asgari and Shahidi's results.** As explained earlier the connected component of the identity of the L-group of $\mathrm{GSp}(4)$ is the group $\mathrm{GSp}_4(\mathbb{C})$. This group has an obvious embedding into the group $\mathrm{GL}_4(\mathbb{C})$. Then Langlands' theory of functoriality predicts the existence of a map φ from the collection of automorphic forms on $\mathrm{GSp}(4)$ over a global field F

to those on GL(4) over the same field F in such a way that for any automorphic form π on GSp(4) we have

$$L(s,\pi,Spin) = L(s,\varphi(\pi)). \tag{5.2}$$

The right hand side of this equation is the standard L-function of the automorphic form $\varphi(\pi)$ on GL(4). In a series of two paper [3, 4], Asgari and Shahidi have established this transfer for the case where the representation in question is generic. In [3] they establish the weak transfer for split general spinor groups. This means that they can show that given a globally generic representation π on a general spinor group, there is an automorphic representation on the corresponding general linear group such that the local representation match up via the local Langlands' at almost all places. Currently they cannot show the matching at all places. Observing that GSp(4) is nothing but the split general spinor group of order five, they get weak transfer in the case of GSp(4). Then they use what is know about the theory of L-function a la Piatetski-Shapiro and Piatetski-Shapiro and Soudry to get more precise information about the transfer. Let us describe their result more carefully. By Langlands' theory of Eisenstein series we just need to do the transfer for cuspidal unitary automorphic representations.

Theorem 5.2 (Asgari and Shahidi)**.** *Let π be a unitary cuspidal representation of* $\mathrm{GSp}(4,\mathbb{A})$ *which is globally generic. Then π has a unique transfer to an automorphic representation Π of* $\mathrm{GL}(4,\mathbb{A})$*. The transfer is globally generic (hence locally generic). If ω_π, ω_Π are the central characters of π and Π, respectively, then $\omega_\Pi = \omega_\pi^2$ and $\Pi \simeq \tilde{\Pi} \otimes \omega_\pi$.*

Asgari and Shahidi can also determine when the resulting representation on GL(4) is cuspidal. They show that it is cuspidal unless when the starting representation π is in a Roberts packet, in which case the resulting representation is an isobaric sum of two GL(2) cuspidal representations. In [4] various applications of these important results are listed including bounds towards Ramanujan, and the entireness of the global spinor L-function for generic representations. It should be noted that if the automorphic cuspidal representation π on GSp(4) is not generic, then the L-function may not be entire.

Remark 5.3. The transfer of a general automorphic form on GSp(4) to GL(4) should follow from the trace formula. This is explained in the paper of Arthur in the Shalikafest [1]. There is the new thesis of David Whitehouse under Dinakar Ramakrishnan which establishes the *fundamental lemma* for the situation under consideration. Here we should also point that there is an unpublished manuscript due by Flicker in which the author claims to have established the transfer from GSp(4) to GL(4).

5.3. **Roberts and Schmidt's theory of new forms.** The theory of new forms in the context of classical modular forms is well-understood. There is an interpretation and extension of this theory to the representation theory of the group GL(2) due to Casselman [6] who also found interesting connections to Jacquet-Langlands theory. Casselman's theory of new vectors was then generalized to GL(n) by Jacquet, Piatetski-Shapiro, and Shalika in an amazing beautiful work [21] who also showed the connection to the theory of L-functions. Unfortunately, in general we do not have a theory of new forms for a general reductive group, even conjectural, except for a few isolated cases. One of the rare cases in which there now seems to exist a nice theory of new forms is the case of PGSp(4) where the theory is due to Roberts and Schmidt [36] where a fairly precise conjecture has been formulated; they have recently claimed to have proved their conjecture and are now preparing a manuscript. Let us now explain their conjecture.

We work over a non-archimedean local field F with ring of integers denoted by $\mathcal{O}$, and suppose $\mathfrak{p}$ be the prime idea, and ϖ a local uniformizer. We first define a compact open subgroup of the Klingen parabolic subgroup. For a non-negative integer n, let $Kl(\mathfrak{p}^n)$ be the collection of matrices k in the subgroup

$$\begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p}^n & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathcal{O} & \mathfrak{p}^n \\ \mathfrak{p}^n & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix} \tag{5.3}$$

subject to the extra condition that $\nu(k) \in \mathcal{O}^\times$. Define the Atkin-Lehner element of level $\mathfrak{p}^n$ by

$$u_n = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -\varpi^n & & \\ \varpi^n & & & \end{pmatrix} \tag{5.4}$$

We finally define the paramodular group $K(\mathfrak{p}^n)$ of level $\mathfrak{p}^n$ to the subgroup of $\mathrm{GSp}(4, F)$ generated by $Kl(\mathfrak{p}^n)$ and $u_n Kl(\mathfrak{p}^n) u_n^{-1}$. An equivalent description is to say that $K(\mathfrak{p}^n)$ is the collection of matrices in the subgroup

$$\begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-n} & \mathcal{O} \\ \mathfrak{p}^n & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathcal{O} & \mathfrak{p}^n \\ \mathfrak{p}^n & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix} \tag{5.5}$$

which have their similitude norm in the group $\mathcal{O}^\times$.

Conjecture 5.4 (Roberts, Schmidt)**.** *Let π be a generic irreducible admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character. For each non-negative integer n, let $\pi(\mathfrak{p}^n)$ be the subspace of π of vectors fixed by $K(\mathfrak{p}^n)$. Then*

(1) *For some non-negative integer n, the space $\pi(\mathfrak{p}^n)$ is non-zero.*
(2) *If N_π is the smallest n such that $\pi(\mathfrak{p}^n)$ is non-zero, then*

$$\dim \pi(\mathfrak{p}^{N_\pi}) = 1.$$

(3) *There is a W_π in the Whittaker model of π coming from a non-zero vector in $\pi(\mathfrak{p}^{N_\pi})$ such that*

$$Z(s, W_\pi) = L(s, \pi).$$

As in the $\mathrm{GL}(n)$ theory there is a nice connection to ϵ-factors. For example it is shown in [36] that if the conjecture is true, then

$$\pi(u_{N_\pi}) W_\pi = \epsilon_\pi W_\pi \tag{5.6}$$

for some $\epsilon_\pi \in \{\pm 1\}$. Furthermore,

$$\epsilon(s, \pi, \psi) = \epsilon_\pi q^{-N_\pi(s - \frac{1}{2})}. \tag{5.7}$$

Observe that the subgroups $K(\mathfrak{p}^n)$ do NOT form a descending chain of subgroups. This is an indication of how strange a theory of new forms for a general reductive group might seem at first. Recall that in the case of $\mathrm{GL}(k)$, the role of the subgroups $K(\mathfrak{p}^n)$ is played by the subgroup

$$\left\{ \begin{pmatrix} & & & u \\ & g & & \vdots \\ & & & v \\ r & \dots & s & m \end{pmatrix} ; \begin{array}{c} g \in \mathrm{GL}_{k-1}(\mathcal{O}), u, \dots, v \in \mathcal{O}, \\ r, \dots, s \equiv 0 \mod \mathfrak{p}^n, m \equiv 1 \mod \mathfrak{p}^n \end{array} \right\}. \tag{5.8}$$

and in this case, the subgroups clearly form a descending chain.

In an interesting recent paper, Schmidt considers the problem for the case of non-generic representations and offers a theory for the square-free case. Schmidt considers various congruence subgroups, and studies each case carefully. What is needed to complete the theory is a theory of L functions of degree four for representations with Iwahori-fixed vectors similar to the one proposed by Novodvorsky, and worked out by Bump, Takloo-Bighash, and Moriyama for the generic situation.

References

[1] J. Arthur, *Automorphic representations of* $\mathrm{GSp}(4)$, Contributions to automorphic forms, geometry, and number theory, 65-81, Johns Hopkins University Press, Baltimore, MD 2004.
[2] M. Asgari, and Ralf Schmidt, *Siegel modular forms and representations*, manusctripta math. **104** (2001), 173-200.
[3] M. Asgari, and F. Shahidi, *Generic transfer for general Spin groups*, preprint math.NT/0411035.
[4] M. Asgari, and F. Shahidi, *Generic transfer from* $\mathrm{GSp}(4)$ *to* $\mathrm{GL}(4)$, preprint math.NT/0412505.
[5] D. Bump, *The Rankin-Selberg Method: A survey,* Number Theory, trace formulas, and discrete groups (K. E. Bombieri and D. Goldfeld, eds.), Academic Press, San Diego, 1989.
[6] W. Casselman, *On some results of Atkin and lehner*, Math. Ann. **201** (1973), 301-314.
[7] W. Casselman and J. Shalika, *The unramified principal series of p-adic groups. II. The Whittaker function.* Compositio Math. 41 (1980), no. 2, 207–231.
[8] M. Furusawa, *On L-functions for $GSp(4) \times GL(2)$ and their special values.* J. Reine Angew. Math. 438 (1993), 187–218.
[9] M. Furusawa, and J. A. Shalika, *On central critical values of the degree four L-function for* $\mathrm{GSp}(4)$*: The fundamental lemma*, Mem. of the AMS, vol 782, 2003.
[10] W. T. Gan, *The Saito-Kurokawa space of $PGSp(4)$ and its transfer to inner forms*, preprint.
[11] S. Gelbart and F. Shahidi, *Analytic Properties of Automorphic L-Functions*, Prospectives in Mathematics, Academic Press, 1988.
[12] D. Ginzburg, *L-functions for* $\mathrm{SO}_n \times \mathrm{GL}_k$, J. Reine Agnew. Math, **405**(1990), 156-180.
[13] B. Gross and D. Prasad, *On the decomposition of a representation of* SO_n *when restricted to* SO_{n-1}. Canad. J. Math. 44 (1992), no. 5, 974–1002.
[14] B. Gross and D. Prasad,*On irreducible representations of* $\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2m}$. Canad. J. Math. 46 (1994), no. 5, 930–950.
[15] M. Harris, *Nonvanishing of L-functions of 2×2 unitary groups.* Forum Math. 5 (1993), no. 4, 405–419.
[16] M. Harris and S.S. Kudla, *Arithmetic automorphic forms for non-holomorphic discrete series of* $\mathrm{GSp}(2)$, Duke Math. J. 66 (1992), 59-121.
[17] M. Harris, D. Soudry, and R. Taylor, *l-adic representations associated to modular forms over imaginary quadratic fields I: lifting to* $\mathrm{GSp}_4(\mathbb{Q})$, Invent. Math. 112 (1993), 377-411.

[18] R. Howe and I. Piatetski-Shapiro, *Some examples of automorphic forms on* Spb4*.* Duke Math. J. 50 (1983), no. 1, 55–106.

[19] H. Jacquet and R. Langlands, *Automorphic Forms on $GL(2)$,* Lecture Notes in Mathematics, Vol. 114, Springer, New York, 1970.

[20] H. Jacquet, I. Piatetski-Shapiro, and J. A. Shalika, *Automorphic Forms on $GL(3)$ I,* Ann. of Math., 109 (1979), 169-212.

[21] H. Jacquet, I. Piatetski-Shapiro, and J. A. Shalika, *Conducteur des reprsentations du groupe linaire*, Math. Ann. 256 (1981), no. 2, 199–214..

[22] A. Knapp and N. Wallach, *Szegö kernels associated with discrete series.* Invent. Math. 34 (1976), no. 3, 163–200.

[23] P. Michel, *Appendix*, [47].

[24] R. P. Langlands, *On the classification of irreducible representations of real algebraic groups*, *Representation theory and harmonic analysis on semisimple Lie groups*, 101-170, Math. Surveys Monogr., 31, Amer. Math. Soc., Providence, RI, 1989.

[25] J.-S. Li, *Non-existence of singular cusp forms*, Comp. Math., **83**(1992), 43-51.

[26] T. Moriyama, *Entireness of the spinor L-functions for certain generic cusp forms on* GSp(2)*.* Amer. J. Math. **126** (2004), no. 4, 899–920.

[27] T. Moriyama, *A remark on Whittaker functions on $Sp(2,\mathbb{R})$*, J. Math. Sci. Univ. Tokyo, **9**(2002), 627-635.

[28] M. Novodvorsky, *Automorphic L-Functions for the Symplectic Group $GSp(4)$,* AMS Proc. Symp. Pure Math., vol 33(2), 1979, 87-95.

[29] M. Novodvorsky and I. Piatetski-Shapiro, *Generalized Bessel Models for the Symplectic Group of Rank 2,* Math. Sb., 90(2), 1973, 246-256.

[30] I. I. Piatetski-Shapiro, *On the Saito-Kurokawa lifting*, Invent. Math. **71** (1983), no. 2, 309–338.

[31] I. I. Piatetski-Shapiro, *A remark on my paper "On the Saito-Kurokawa lifting"*, Invent. Math, **76** (1984), 75-76.

[32] I. I. Piatetski-Shapiro, *L-functions for* GSp(4), Pacific J. Math. 1997, Special Issue, 259-275.

[33] B. Roberts, *Global L-packets for* GSp(2) *and theta lifts*, Documenta Mathematica 6 (2001), 247-314.

[34] B. Roberts, *Electronic communication*, Feb 14, 2002.

[35] B. Roberts, *Epsilon factors for some representations of* GSp(2) *and Bessel coefficients*, in preparation.

[36] B. Roberts and R. Schmidt, *New vectors for* GSp(4)*: a conjecture and some evidence*, preprint.

[37] Rodier, F., *Whittaker models for admissible representations of reductive p-adic split groups.* Proc. Sympos. Pure Math. **26** (1973), 425-430.

[38] P. Sally, Jr. and M. Tadic, *Induced representations and classifications for* $\mathrm{GSp}(2,F)$ *and* $\mathrm{Sp}(2,F)$, Mem. Soc. Math. France (N.S.) No. 52 (1993), 75–133.

[39] R. Schmidt, *Iwahori-spherical representations of $GSp(4)$ and Siegel modular forms of degree 2 with square-free level*

[40] F. Shahidi, *On certain L-functions,* Amer. J. Math. 103 (1981), no. 2, 297–355.

[41] F. Shahidi, *A Proof of Langlands' Conjecture on Plancheral Measures; Complementary Series for p-adic Groups,* Ann. of Math., 132 (1990), 273-330.

[42] Shahidi, *Whittaker models for real groups*, Duke Math. J. **47** (1980), no. 1, 99-125.

[43] Shahidi, *Local coefficients as Artin factors for real groups*, Duke Math. J. **52** (1985), no. 4, 973-1007.

[44] J. A. Shalika, *Multiplicity One for* $\mathrm{GL}(n)$, Ann. of Math. (2) 100 (1974), 171–193.

[45] D. Soudry, *Rankin-Selberg Convolutions for* $\mathrm{SO}_{2l+1} \times \mathrm{GL}_n$*: Local theory*, Mem. Amer. Math. Soc., no. 500, vol. **105** (1993).

[46] R. Takloo-Bighash, *L-functions for the p-adic group* $\mathrm{GSp}(4)$, American Journal of Mathematics, 122 (2000), 1085-1120.

[47] R. Takloo-Bighash, *Bessel coefficients, theta correspondence, and spinor L-functions*, preprint.

[48] R. Takloo-Bighash, *Some results on the spinor L-function for the group* $\mathrm{GSp}(4)$, in Shahyad, A volume dedicated to S. Shahshahani's 60th birthday (ed. Lajevardi, P. Safari, and Y. Tabesh).

[49] A. Venkatesh, *The Burger-Sarnak method and operations on the unitary dual of* $\mathrm{GL}(n)$, preprint.

[50] J.-L. Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symtrie*. Compositio Math. 54 (1985), no. 2, 173–242.

[51] N. Wallach, *Asymptotic expansions of generalized matrix entries of representations of real reductive groups.* Lecture Notes in Mathematics **1024**, Springer-Verlag (1983), 287-369.

[52] N. Wallach, *Lie Algebra Coholomology and Holomorphic Continuation of Generalized Jacquet Integrals*, Advanced Studies in Pure Mathematics, **14** (1988), 123-151.

[53] N. Wallach, *Generalized Whittaker vectors for holomorphic and quatertionic representations*, Comment. Math. Helv. **78** (2003), 266-307.

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