POINTS OF BOUNDED HEIGHT ON ALGEBRAIC VARIETIES

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Introduction

The purpose of these somewhat casual notes is to explain various conjectures on the distribution of rational points on Fano varieties. The conventional wisdom is that the arithmetic of algebraic varieties, and in particular the distribution of rational points on them, is determined by their geometry. A geometric invariant that seems to play an immensely important role is the canonical class of the variety. In fact, one expects drastically different distribution patterns in the two extreme cases where the canonical class or its negative is ample. In

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one extreme case where the canonical class is ample, one expects a rarity of rational points. For example, the Bombieri-Lang conjecture predicts that most rational points are contained in a fix Zariski closed set independent of the number field over which one is working. In the other direction where the negative of the canonical class is ample, and this is the Fano case, the expectations are quite different. Manin has conjectured and most human beings believe that rational points are potentially dense; that is, there is a finite extension of the ground field over which the rational points are Zariski dense. The intermediate cases are highly unexplored and except for certain special classes such as abelian varieties, where one has the classical Mordell-Weil theorem, a general picture -even conjectural- is missing.

One way to measure the density of rational points on an algebraic variety in a quantifiable fashion is using the *height function*. The so-called *Height Machine* associates to each class in the Picard group of the variety a height function, or a class of such things. This association is defined in such a way that if the divisor class is *ample* then the associated height function is a counting device, in the sense that given any number field, there will be only a finite number of points rational over the number field whose height is bounded by any given constant. A standard approach to understanding the distribution of rational points on varieties is to understand the growth of the number of rational points of bounded height.

The approach we adopt in this article is the study of the distribution of rational points of bounded height on Fano varieties. This requires some geometric and arithmetic preparation. The first section is devoted to the study of height functions on the projective space. The second chapter deals with height functions on algebraic varieties. In order to do this we recall the elementary theory of divisors and linear systems. In the third section we discuss the conjectures of Manin and various examples. The fourth section is devoted to a review of the recent joint work of the author with Shalika and Tschinkel on the distribution of rational points on compactifications of semi-simple algebraic groups. For reasons of space we have chosen not to cover many of the exciting new developments in the theory; *universal torsors* and *dynamical techniques* have been mentioned only in passing. We refer the reader to the bibliography for a very incomplete list of some recent works.

The first three sections of these notes are based on a series of lectures delivered by the author at the Sharif University of Technology and the Institute for Studies in Theoretical Physics in Tehran, during the summer of 2006. These lectures were largely based on the wonderful book of Hindry and Silverman [15]. It would be obvious to the reader that this book has greatly influenced our presentation of the material, although it would also be clear that that author has been influenced by [17] and [24]. Our reference for the algebraic geometry material has been [13] and [25], although here too the book of Hindry and Silverman has been quite useful. The last section, considerably more technical in nature, is based on lectures delivered at IPM, Institute of Basic Sciences in Zanjan, and Ahwaz University all during July of 2006, and CUNY Graduate Center and the New York Number Theory Seminar in the Academic Year 2006-2007. Here I would like to thank these institutions for their hospitality and Professor Mehrdad Shahshahani for making the visit to Iran possible.

1. Heights on the projective space

1.1. **Basic height function.** For a field k, an *absolute value on k* is a real valued function $|.| : k \to [0, \infty)$ with the following properties

(1) $|x| = 0$ if and only if $x = 0$;
(2) $|xy| = |x|.|y|$ for all $x, y \in k$; and
(3) $|x + y| \leq |x| + |y|$ for all x, y.

We call the absolute value *non-archimedean* if instead of the last inequality we have

$$|x + y| \leq \max\{|x|, |y|\}.$$

For $\mathbb{Q}$, and for any other numberfield, there is a host of absolute values. For each prime p, there is a p-adic absolute value given by the following: For a rational number x, we define the integer $ord_p(x)$ by

$$x = p^{ord_p(x)}.\frac{a}{b}$$

with $a, b \in \mathbb{Z}$, $(a, b) = 1$ and $(p, ab) = 1$. Then we set $|x|_p = p^{-ord_p(x)}$. It is easy to see that $|.|_p$ is a non-archimedean absolute value. In contract, we have the so-called archimedean absolute value defined by the ordinary absolute value. We usually denote this by $|.|_\infty$. We have the following product formula: For each non-zero rational number x, we have

$$\prod_v |x|_v = 1. \tag{1}$$

Here either $v = \infty$ or $v = p$ prime. Given any absolute value on a numbefield, we may construct a completion of the number field using Cauchy sequences; this is what we will denote by $\mathbb{Q}_v$ if the absolute value is $|.|_v$. There are natural ways to generalize these notions to arbitrary number fields. We will use such constructs without further comment, and we will refer the reader to any textbook on algebraic number theory for definitions and details. We just mention here that for general number fields archimedean absolute values are parametrized by embeddings of the number field in $\mathbb{R}$ or $\mathbb{C}$, indeed the collection of embeddings of the number field in $\mathbb{R}$ and half the number of collection of embeddings of the number field in $\mathbb{C}$, and non-archimdean absolute values are parametrized by the collection of the prime ideals of the ring of integers of the number field.

The basic idea in defining *the height* of a rational number is the observation that even though the two numbers 1 and $\frac{1000000001}{1000000000}$ are very close, the latter is much more *arithmetically complicated* than the former; the height function quantifies this intuitive statement. Here is one way to do this. If $x = \frac{a}{b}$, with $(a, b) = 1$, we set

$$H(x) = \max\{|a|, |b|\}.$$

It is clear that $H(1) = 1$ and

$$H(\frac{1000000001}{1000000000}) = 1000000001.$$

We could have considered any other Banach norm instead of $\max\{|.|, |.|\}$, e.g. $|.| + |.|$ or $\sqrt{|.|^2 + |.|^2}$. The fundamental observation is that

Theorem 1.1. *For any rational number x we have*

$$H(x) = \prod_v \max\{1, |x|_v\}.$$

Proof is easy and we leave it to the reader. If we had constructed the height function with respect to another Banach norm, we would have to modify the product expression at the archimedean place to recover the modified height function. This theorem makes it possible to give a reasonable definition of height for any number field; we simply define the height function as in the statement of the theorem.

Example 1.2. Given n, we would like to determine the number of elements of the set

$$\{x \in \mathbb{Q}; H(x) = n\}.$$

We will denote this number by N_n. It is clear that if $N_0 = 1$, $N_1 = 2$. Now let $n > 1$. If $x = \frac{a}{b}$, with $(a, b) = 1$, then either $|a| = n$ or $|b| = n$. It is then seen that $N_n = 4\phi(n)$ for $n > 1$. Hence

$$N_n = \begin{cases} 1 & n = 0; \\ 2 & n = 1; \\ 4\phi(n) & n > 1. \end{cases} \tag{2}$$

We can also the following quantity:

$$N(B) = \#\{x \in \mathbb{Q}; H(x) \leq B\}.$$

It is clear that $N(B) = 3 + \sum_{2 \leq n \leq B} N_n$. We leave it to the reader to verify that $N(B)$ grows like CB^2 for an explicitly computable constant C. For this and other elementary examples see [18].

1.2. **Height function on the projective space.** Recall that the n-dimensional projective space over a number field F is the collection of equivalence classes of $(n+1)$-tuples $(x_0, \dots, x_n)$ such that $(x_0, \dots, x_n) \neq (0, \dots, 0)$. Two tuples $(x_0, \dots, x_n)$ and $(y_0, \dots, y_n)$ are defined to be equivalent if there is a non-zero λ such that $x_i = \lambda y_i$ for all i. This is denoted by $\mathsf{P}^n(F)$ and it is indeed the F-rational points of a variety P^n defined over the rational numbers. The equivalence class of a tuple $(x_0, \dots, x_n)$ is usually denoted by $(x_0 : \dots : x_n)$.

Let $(x_0 : \dots : x_n) \in \mathsf{P}^n(\mathbb{Q})$. This means that for all i, j with $x_j \neq 0$ we have $x_i/x_j \in \mathbb{Q}$. It is clear that P will have a representative $(y_0, \dots, y_n)$ such that $y_i \in \mathbb{Z}$ for all i and $\gcd(y_0, \dots, y_n) = 1$. We define

$$H_n(P) = \max(|y_0|, \dots, |y_n|).$$

As before, we may consider more general height function by considering other Banach norms of the vector $(y_0, \dots, y_n)$. There is an obvious embedding $\iota : \mathbb{Q} \to \mathsf{P}^1(\mathbb{Q})$ given by $x \mapsto (x : 1)$. Then it is easily seen that $H(x) = H_1(\iota(x))$. Here too we have a product identity for the height function given by

$$H(P) = \prod_v \max\{|x_0|_v, \dots, |x_n|_v\}$$

for any representative $(x_0, \dots, x_n)$ for the rational point P. This is well-defined. Indeed, let $(y_0, \dots, y_n)$ be another representative for P. Then by definition, $y_i = \lambda x_i$ for some $\lambda \in \mathbb{Q}^\times$. Hence we have

$$\begin{aligned}\prod_v \max\{|y_0|_v, \dots, |y_n|_v\} &= \prod_v |\lambda|_v . \prod_v \max\{|x_0|_v, \dots, |x_n|_v\} \\ &= \prod_v \max\{|x_0|_v, \dots, |x_n|_v\}\end{aligned}$$

by the product formula (1).

As before, the product identity enables us to extend the definition of the height function to arbitrary number fields in the following fashion. Let $P = (x_0 : \dots : x_n) \in \mathsf{P}^n(k)$ for a number field k. We set

$$H_n^k(P) = \prod_v \max\{|x_0|_v, \dots, |x_n|_v\} \tag{3}$$

where the product is taken over the collection of places v. Note that if k' is an extension of k, then $\mathsf{P}^n(k) \subset \mathsf{P}^n(k')$. It is seen then that if $P \in \mathsf{P}^n(k)$, then

$$H_n^{k'}(P) = H_n^k(P)^{[k':k]}.$$

We now define a height function on $\mathsf{P}^n(\overline{\mathbb{Q}})$: For $P \in \mathsf{P}^n(\overline{\mathbb{Q}})$, there is a number field k such that $P \in \mathsf{P}^n(k)$; we set $H_n(P) = K_n^k(P)^{1/[k:\mathbb{Q}]}$. This is independent of the choice of k. This height function has the following two basic properties

- For all P, $H_n(P) \geq 1$; and
- For all $P \in \mathsf{P}^n(\overline{\mathbb{Q}})$ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have $H_n(\sigma P) = H_n(P)$.

When there is no confusion, we drop the subscript n in $H_n(P)$. We have the following important finiteness theorem:

Theorem 1.3 (Northcott)**.** *Given a point $P \in \mathsf{P}^n(\overline{\mathbb{Q}})$, we set $\mathbb{Q}(P) := Q(x_0/x_j, \dots, x_n/x_j)$ for $x_j \neq 0$. Then for all $D, B \geq 0$, the set*

$$\{P \in \mathsf{P}^n(\overline{\mathbb{Q}}); H(P) \leq B, [\mathbb{Q}(P) : \mathbb{Q}] \leq D\}$$

is finite. In particular, for any fixed number field k, the set

$$\{P \in \mathsf{P}^n(k); H(P) \leq B\}$$

is finite.

Sketch of proof. It is sufficient to prove this for $n = 1$, i.e. it suffices to show that the set

$$\{x \in \overline{\mathbb{Q}}; H(x) \leq B, [\mathbb{Q}(x) : \mathbb{Q}] \leq D\} \tag{4}$$

is finite. Let $x \in \overline{\mathbb{Q}}$ be of degree d over $\mathbb{Q}$, and let $x_1, \dots, x_d$ be the conjugates of x. Set

$$F_x(T) = \prod_{j=1}^{d}(T - x_j) = \sum_{r=0}^{d}(-1)^r s_r(x) T^{d-r},$$

for $s_r(x) \in \mathbb{Q}$, and we can estimate its size to be

$$|s_r(x)|_v \leq c(v, d) \max_{1 \leq i \leq d} |x_i|_v^r$$

where $c(v, d) = 2^d$ if v is archimedean, and $c(v, d) = 1$ if v is non-archimedean. This gives

$$\max\{|s_0(x)|_v, \dots, |s_d(x)|_v\} \leq c(v, d) \prod_{i=1}^{d} \max\{|x_i|_v, 1\}^d.$$

Since $H(x_i) = H(x)$ for all i, we get

$$H(s_0(x), \dots, s_d(x)) \le 2^d H(x)^{d^2}.$$

This quickly reduces the problem to $\mathsf{P}^d(\mathbb{Q})$, and there the problem is trivial. □

We will state a quantitative version of this theorem below.

1.3. **Behavior under maps.** The basic problem is to start with a map $\phi : \mathsf{P}^n \to \mathsf{P}^m$ defined over $\overline{\mathbb{Q}}$ and we want to compare $H(\phi(P))$ and $H(P)$ for $P \in \mathsf{P}^n(\overline{\mathbb{Q}})$. Let us start with a couple of examples:

Example 1.4. Let $\phi_{n,m} : \mathsf{P}^n \times \mathsf{P}^m \to \mathsf{P}^{m+n+mn}$ be the Segre embedding given by

$$((x_i)_i, (y_j)_j) \mapsto (x_i y_j)_{i,j}.$$

Then $H(\phi_{n,m}(P, Q)) = H(P)H(Q)$. There is an obvious generalization of this. For $\underline{n} = (n_1, \dots, n_k)$, a k-tuple of positive integers, we define the obvious map $\phi_{\underline{n}} : \mathsf{P}^{n_1} \times \cdots \times \mathsf{P}^{n_k} \to \mathsf{P}^N$, $N = -1 + \prod_i (n_i + 1)$. We then have

$$H(\phi_{\underline{n}}(P_1, \dots, P_k)) = \prod_i H(P_i).$$

Also, there is the diagonal embedding of $\Delta : \mathsf{P}^n \to \mathsf{P}^n \times \dots \mathsf{P}^n$. Then if $\underline{n} = (n, \dots, n)$, k-copies, we can consider $\varphi_k := \phi_{\underline{n}} \circ \Delta$; this is the so-called k-uple embedding $\varphi_k : \mathsf{P}^n \to \mathsf{P}^N$ for the appropriate N. Then we conclude

$$H(\varphi_k(P)) = H(P)^k.$$

The d-uple embedding described in the example is prototypical degree map of degree d between projective spaces. In general, though, one does not expect to get identities like the ones described in the example, but one can still recover a great deal. The following theorem addresses this. Before stating the theorem, let us define the *logarithmic height*. The logarithmic height of a point $P \in \mathsf{P}^n(\overline{\mathbb{Q}})$, $h(P)$, is simply the logarithm of its height $\log H(P)$.

Theorem 1.5. *Let $\phi : \mathsf{P}^n \to \mathsf{P}^m$ be a rational map of degree d defined over $\overline{\mathbb{Q}}$. Let $\mathcal{Z} \subset \mathsf{P}^n$ be the closed subvariety where ϕ is not defined. Then*

1. *$h(\phi(P)) \le dh(P) + O(1)$, for all $P \in \mathsf{P}^n(\overline{\mathbb{Q}})$; and*
2. *Let X be a subvariety contained in $\mathsf{P}^n \backslash \mathcal{Z}$, then $h(\phi(P)) = dh(P) + O(1)$, for all $P \in X(\overline{\mathbb{Q}})$.*

The second statement does not hold in general on the whole of $\mathsf{P}^n \backslash \mathcal{Z}$.

Proof. Note that ϕ is given by polynomials $(f_0, \ldots, f_m)$ all homogeneous of degree d. The locus $\mathcal{Z}$ is the collection of common zeros of the f_i's. Write

$$f_i(X) = \sum_{|e|=d} a_{i,e} X^e.$$

Here e is a $n+1$-tuple of non-negative integers $(e_0, \ldots, e_n)$ and $|e| = \sum e_i$. Also $X^e = X_0^{e_0} \ldots X_n^{e_n}$. We fix a representative $(x_0, \ldots, x_n)$ for the point P. We set $|P|_v = \max\{|x_i|_v\}$, also $|f_i|_v = \max\{|a_{i,e}|_v\}$. We define $\epsilon_v(r) = r$ when v is archimedean, and $= 1$ when v is non-archimedean. This is the constant that comes in the triangle inequality

$$|a_1 + \cdots + a_r|_v \leq \epsilon_v(r) \max\{|a_1|_v, \ldots, |a_r|_v\}.$$

Let $P \in \mathsf{P}^n(k)$ for a number field k. Then it is not hard to see that

$$|f_i(P)|_v \leq \epsilon_v \binom{n+d}{n} |f_i|_v |P|_v^d,$$

which easily gives

$$H(\phi(P)) \leq \binom{n+d}{n} H(\phi) H(P)^d$$

for an appropriate constant $H(\phi)$ depending only on the f_i's. Taking logarithms gives the first part.

The proof of the second part involves the Nullstellensatz. Let $p_1, \ldots, p_r$ be the homogeneous polynomials generating the ideal of X. Then as $X \cap \mathcal{Z} = \varnothing$, we know that $p_1, \ldots, p_r, f_0, \ldots, f_m$ have no common zeros on P^n. By the Nullstellensatz the radical of the ideal generated by these polynomials is the ideal $(X_0, \ldots, X_n)$. This means there is a positive integer $t \geq d$ and polynomials g_{ij} and q_{ij} such that

$$g_{oj} f_0 + \cdots + g_{mj} f_m + q_{1j} p_1 + \cdots + q_{rj} p_r = X_j^t$$

for all j. The g_{ij}'s are homogeneous of degree $t - d$. Without loss of generality, we may assume that $g_{ij}, q_{ij} \in k[X_0, \ldots, X_n]$. Now if $P = (x_0, \ldots, x_n) \in X(k)$, we have

$$g_{0j}(P) f_0(P) + \cdots + g_{mj}(P) f_m(P) = x_j^t.$$

Hence

$$\begin{aligned}
|P|_v^t &= \max_j |x_j^t|_v \\
&= \max_j |g_{0j}(P) f_0(P) + \cdots + g_{mj}(P) f_m(P)|_v \\
&\leq \epsilon_v(m+1) \epsilon_v \binom{t-d+m}{n} (\max_{i,j} |g_{ij}|_v) |P|_v^{t-d} (\max_i |f_i(x)|_v).
\end{aligned}$$

This gives

$$H(P)^t \leq cH(P)^{t-d}H(\phi(P))$$

for a constant c. Taking logarithms gives the second part. $\square$

2. Heights on varieties

In this section, we define height functions on arbitrary projective varieties and explain the *Weil Height Machine*. In order to do this properly, we recall some background from Algebraic Geometry. The material presented here is taken from [15]. The standard reference on this material is [13] espeically II.6 and II.7. Another very useful reference, and a very pleasant read, is [25], Book 1, Chapter 3.

2.1. **Divisors.**

2.1.1. *Weil divisors.* Let X be an algebraic variety over an algebraically closed field k. We let $\mathrm{Div}X$ be the free abelian group generated by the irreducible closed subvarieties of codimension one on X. We call the elements of $\mathrm{Div}X$ *divisors*. Hence a typical element of $\mathrm{Div}X$ will be a formal linear combination $D = \sum_Y n_Y.Y$, where $n_Y \in \mathbb{Z}$ and are zero except for a finite number of exceptions. Given D as above, we define $\mathrm{Supp}D$ to be the union of all Y for which $n_Y \neq 0$. We call D *effective* or *positive* if $n_Y \geq 0$ for all Y. If Y is an irreducible divisor on X, then we denote by $\mathcal{O}_{Y,X}$ the local ring of functions that are regular in a neighborhood of Y. If X is non-singular, then $\mathcal{O}_{Y,X}$ is a discrete valuation ring, and we have a valuation $\mathrm{ord}_Y : \mathcal{O}_{Y,X}\backslash\{0\} \to \mathbb{Z}$. As $k(X)^\times$ is the quotient field of $\mathcal{O}_{Y,X}$, we can extend ord_Y to $k(X)^\times$. The function $\mathrm{ord}_Y : k(X)^\times \to \mathbb{Z}$ enjoys the following properties:

- $\mathrm{ord}_Y(fg) = \mathrm{ord}_Y(f) + \mathrm{ord}_Y(g)$;
- Given $f \in k(X)^\times$, there are only finitely many Y such that $\mathrm{ord}_Y f \neq 0$; and finally
- Suppose X is projective and let $f \in k(X)^\times$. Then the following are equivalent:
  (1) $\mathrm{ord}_Y(f) \geq 0$ for all Y;
  (2) $\mathrm{ord}_Y(f) = 0$ for all Y; and
  (3) $f \in k^\times$.

These properties allow us to define a group homomorphism $\mathrm{div} : k(X)^\times \to \mathrm{Div}X$ given by $\mathrm{div}f = \sum_Y \mathrm{ord}_Y(f).Y$. Divisors of the form $\mathrm{div}f$ are called *principal*. Two divisors $D, D' \in \mathrm{Div}X$ are called *linearly equivalent*, written $D \sim D'$, if $D - D'$ is principal, i.e. there is $f \in k(X)^\times$ such that $D - D' = \mathrm{div}f$. It is easy to check that $\sim$ is indeed an equivalence relation. We set $\mathrm{Cl}(X) = \mathrm{Div}X/\sim$.

Example 2.1. Let f be an irreducible homogeneous polynomial of degree d in $n+1$ variables with coefficients in k. Let $\mathcal{Z} \subset \mathsf{P}^n$ be the irreducible hypersurface defined by f. We define the degree of $\mathcal{Z}$, denoted by $\deg \mathcal{Z}$, to be d. It is a standard fact that irreducible codimension one subvarieties of the projective space are irreducible hypersurfaces ([25], Book 1, Chapter I, 6.1, Theorem 3′). This allows us to extend the deg to $\mathrm{Div}\mathsf{P}^n$. Then $D \in \mathrm{Div}\mathsf{P}^n$ is principal if and only if $\deg D = 0$. The induced map

$$\deg : \mathrm{Cl}(\mathsf{P}^n) \to \mathbb{Z}$$

is an isomorphism. One can also show that $\mathrm{Cl}(\mathsf{P}^{n_1} \times \cdots \times \mathsf{P}^{n_r}) \cong \mathbb{Z}^r$ ([25], Book 1, Chapter III, 1.1, Examples 1-3).

2.1.2. *Cartier divisors.* A Cartier divisor on a variety X is a collection of points $\{(U_i, f_i)\}_{i\in I}$, such that U_i's form an open cover of X and f_i a non-zero rational function on U_i. We further assume that for all i, j we have $f_i f_j^{-1} \in \mathcal{O}(U_i \cap U_j)^\times$, that is no poles or zeros on $U_i \cap U_j$. Two such collections $\{(U_i, f_i)\}_i$ and $\{(V_j, g_j)\}$ are defined to be equal if for all i, j we have $f_i g_j^{-1} \in \mathcal{O}(U_i \cap V_j)^\times$. The set of all such collection subject to the equality just described is denoted by $\mathrm{CaDiv}(X)$, and its elements are called *Cartier Divisors*. This is clearly a group. We call a Cartier divisor $\{(U_i, f_i)\}$ *effective* or *positive* if $f_i \in \mathcal{O}(U_i)$ for all i. For each $f \in k(X)^\times$, we let the Cartier divisor (X, f) be the *principal Cartier divisor defined by* f. The collection of principal Cartier divisors is a subgroup of $\mathrm{CaDiv}(X)$; the quotient group is denoted by $\mathrm{Pic}(X)$.

Let Y be an irreducible subvariety of codimension one on X, and let D be a Cartier divisor given by $\{(U_i, f_i)\}$. Select a U_i such that $U_i \cap Y \neq \varnothing$. Set $\mathrm{ord}_Y D := \mathrm{ord}_Y(f_i)$. It is clear that this is independent of the choice of U_i. We define a map $\mathrm{CaDiv}(X) \to \mathrm{Div}(X)$ by

$$D \mapsto \sum_Y \mathrm{ord}_Y(D).Y.$$

If X is smooth, this map is an isomorphism, descends to an isomorphism $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$.

Example 2.2. Let X be a smooth variety of dimension n, and suppose ω is a non-zero differential n-form on X. We wish to associate a divisor to ω. On any affine open U of X with coordinates $x_1, \ldots, x_n$ we can write $\omega = f_U dx_1 \wedge \cdots \wedge dx_n$ for some rational function $f \in k(X)$. Set $\mathrm{div}\omega := \{(U, f_U)\}$. If ω' is another non-zero form of top degree, we have $\omega' = f\omega$ for some $f \in k(X)^\times$. It can be seen then that $\mathrm{div}\omega' = \mathrm{div}\omega + \mathrm{div} f$. The class in $\mathrm{Pic}(X)$, denoted by K_X, thus defined is called the *Canonical Class of* X. For example, if H is the divisor class of a hyperplane in P^n, then $K_{\mathsf{P}^n} = -(n+1)H$. If X is a hypersurface of degree d in P^n, then $K_X = -(n+1-d)H$ ([25], Book 1, Chapter III, 6.4).

Suppose $g : X \to Y$ is a morphism of varieties and $D = \{(U_i, f_i)\}_i \in \mathrm{CaDiv}Y$. Assume $g(X) \subset |D|$. Then we define a Cartier divisor $g^*D \in \mathrm{CaDiv}X$ by setting

$$g^*D = \{(g^{-1}(U_i), f_i \circ g)\}_i.$$

It is clear that g^* is a group homomorphism and $(f \circ g)^* = g^* \circ f^*$. Also, $g^*(\mathrm{div} f) = \mathrm{div}(f \circ g)$. The following lemma is immensely important:

Lemma 2.3. *Let $f : X \to Y$ be a morphism of varieties. Then*

(1) *Let $D, D' \in \mathrm{CaDiv}Y$ be linearly equivalent divisors. If $f(X)$ is not contained in $\mathrm{Supp}D \cup \mathrm{Supp}D'$, then $f^*D \sim f^*D'$.*
(2) *(Moving Lemma) For every divisor $D \in \mathrm{CaDiv}Y$, there exists a Cartier divisor D' satisfying $D \sim D'$ and $f(X) \nsubseteq \mathrm{Supp}D'$.*

Proof. The first part is easy. For the second part, let $D = \{(U_i, f_i)\}_{i\in I}$. For each $j \in I$, define a divisor D_j by $\{(U_i, f_i f_j^{-1})\}_{i\in I}$. It is clear that $D = D - \mathrm{div} f_j$, and consequently $D_j \sim D$. Furthermore, $U_j \cap \mathrm{Supp}D_j = \varnothing$. □

The lemma implies that given $f : X \to Y$ as above, there is an associated map $f^* : \mathrm{Pic}Y \to \mathrm{Pic}X$.

Example 2.4. Consider the Segre embedding $\phi_{n,m} : \mathsf{P}^n \times \mathsf{P}^m \to \mathsf{P}^{mn+m+n}$. We have seen that $\mathrm{Pic}(\mathsf{P}^{mn+m+n}) = \mathbb{Z}$ and $\mathrm{Pic}(\mathsf{P}^n + \mathsf{P}^m) = \mathbb{Z}^2$. The induced map $\phi^*_{m,n} : \mathbb{Z} \to \mathbb{Z}^2$ is simply $z \mapsto (z, z)$. Next, consider the d-uple embedding $\phi_d : \mathsf{P}^n \to \mathsf{P}^N$. Then the resulting map $\phi_d^* : \mathbb{Z} \to \mathbb{Z}$ is multiplication by d.

2.1.3. *Linear systems.* For $D \in \mathrm{Div}X$, we set

$$L(D) := \{f \in k(X)^\times ; D + \mathrm{div} f \geq 0\} \cup \{0\}.$$

It is seen that $L(D)$ is a vector space and we define $\ell(D) := \dim L(D)$. It is not hard to see that $k \subset L(D)$ if and only if D is effective. Also if $D \leq D'$ then $L(D) \subset L(D')$. Also if $D' = D + \mathrm{div} g$, then there is an isomorphism $L(D') \to L(D)$ given by $f \mapsto fg$. Note that this implies that $\ell(D)$ depends only on the class of D in $\mathrm{Pic}X$.

Given a divisor D, the collection of all effective divisors linearly equivalent to D, denoted by $|D|$, is parametrized by $\mathsf{P}(L(D))$. In fact, $\mathrm{div} f = \mathrm{div}(rf)$ for each $r \in k^\times$. The parametrization is given explicitly by

$$\kappa : f \mod k^\times \mapsto D + \mathrm{div} f.$$

We call $|D|$ the *complete linear system associated to* D. An arbitrary *linear system* is obtained by considering $\kappa(V)$ for a linear subspace V of $L(D)$.

Example 2.5. Let $X = \mathbb{A}^n$, and $D = 0$. Then $L(D) = k[X_1, \dots, X_n]$, and $|D|$ is the collection of all hypersurfaces.

Example 2.6. Let d be a positive, and let H be the hyperplane $\{x_0 = 0\}$ in $\mathrm{Div}\mathsf{P}^n$. Then $L(dH) = \{\frac{F}{x_0^d}; F \text{ homogeneous of degree } d\}$. $L(dH)$ has dimension $\binom{n+d}{n}$. The linear system $|dH|$ is the linear system of all hypersurfaces of degree d in P^n.

Example 2.7. Let $X \hookrightarrow \mathsf{P}^n$ be a projective variety, and I_X the homogeneous ideal of X. Fix a positive integer d and suppose F is a homogeneous polynomial of degree d not in I_X. Cover X by affine open sets $U_i := X \backslash \{X_i = 0\}$. Now define a divisor $(F)_X = \{(U_i, \frac{F}{X_i^d})\}$. Note that if G is any other homogeneous polynomial of degree d on X, then F/G is a rational function and $(F)_X - (G)_X = \mathrm{div}(F/G)$, i.e. $(F)_X \sim (G)_X$. Note that the resulting map $\mathbb{Z} \to \mathrm{Pic}X$ is the map associated to the embedding $X \hookrightarrow \mathsf{P}^n$. Also note that each of the divisors $(F)_X$ is effective. We can now define a linear system $L_X(d)$ as the collection of all $(F)_X$ such that $F \in (k[X_0, \dots, X_n]/I_X)_d \backslash \{0\}$.

We define the set of *base-points* of a linear system L to be the intersection of the supports of all divisors in L. We say L is *base-point free* if this intersection is empty. We call a divisor D *base-point free* if $|D|$ is base-point free. Let us also define the notion of the *fixed component*. The fixed component of a linear system L is the largest divisor D_0 such that $D \geq D_0$ for all $D \in L$. If $D_0 = 0$, we say the linear system has no fixed components.

2.1.4. *Rational maps associated to a linear system.* Let L be a linear system of dimension n parametrized by a projective space $\mathsf{P}(V) \subset \mathsf{P}(L(D))$. Select a basis $f_0, \dots, f_n$ for $V \subset L(D)$. The rational map associated to L and the chosen basis, denoted by ϕ_L, is the map $X \to \mathsf{P}^n$ given by

$$x \mapsto (f_0(x) : \cdots : f_n(x)).$$

ϕ_L is defined outside of the poles of individual f_i's and the set of common zeroes. There is a bijection between the following two collections:

- Linear systems L of dimension n without fixed components; and
- Morphisms $\phi : X \to \mathsf{P}^n$ with image not contained in a hyperplane, up to the action of $\mathrm{PGL}(n+1)$.

Example 2.8. Let $i : X \to \mathsf{P}^n$ be a projective variety not contained in any hyperplane. Then the morphism associated to $L_X(1)$ is i.

A linear system L on a projective variety X is *very ample* if the associated rational map $\phi_L : X \to \mathsf{P}^n$ is an embedding. A divisor D is said to be *very ample* if $|D|$ is very ample; it is called *ample* if some positive integral multiple of it is very ample. It is then not very hard to see that very ample divisors are hyper-plane sections for some embedding. The linear systems $L_X(d)$ are very ample. The following theorem will be fundamental in our applications to the height function:

Theorem 2.9. *Every divisor can be written as the difference of two (very) ample divisors. In fact, if D is an arbitrary divisor and H is very ample, then there exists $m \geq 0$ such that $D + mH$ is base-point free, and if D is base-point free, $D + H$ is very ample.*

Proof. See [15], Theorem A.3.2.3. □

Let us also recall the celebrated

Theorem 2.10 (Enrique-Severi-Zariski)**.** *Let $X \hookrightarrow \mathsf{P}^n$ be normal projective variety. There exists an integer $d_0 = d_0(X)$ such that for all integers $d \geq d_0$ the linear system is a complete linear system. In other words, if D is an effective divisor on X such that is linear equivalent to d times a hyperplane section, then there is a homogeneous polynomial F of degree d such that $D = (F)_X$.*

Proof. [15], Theorem A.3.2.5. □

This theorem implies that for all divisors D on a projective variety, $\ell(D)$ is finite.

2.2. **Heights.**

2.2.1. *The pull-back of the height function.* Let V be a projective variety defined over $\overline{\mathbb{Q}}$, and suppose $\phi : V \to \mathsf{P}^n$ is a morphism. We define the *absolute logarithmic height with respect to ϕ* to be the function $h_\phi : V(\overline{\mathbb{Q}}) \to [0, \infty)$ given by $h_\phi(P) = h(\phi(P))$. We also have an analogous *multiplicative height* defined using H and denoted by H_ϕ. Given a variety V, we define an equivalence relation $\sim_V$ on the collection of functions on $V(\overline{\mathbb{Q}})$: We say $f \sim_V g$ if $f(P) = g(P) + O(1)$ for all $P \in V(\overline{\mathbb{Q}})$.

Theorem 2.11. *Let V be a projective variety defined over $\overline{\mathbb{Q}}$, and let $\phi : V \to \mathsf{P}^n$ and $\psi : V \to \mathsf{P}^m$ be morphisms again defined over $\overline{\mathbb{Q}}$. Suppose $\phi^* H$ and $\psi^* H'$, H and H' hyperplane sections in P^n and P^m respectively, are linearly equivalent. Then $h_\phi \sim_V h_\psi$.*

Proof. Let $D \in \mathrm{Div}V$ be any effective divisor linearly equivalent to ϕ^*H. By a theorem above, ϕ and ψ are determined by two subspaces V and V' in $L(D)$ and choices of bases for them. If we choose a basis $\{h_0, \dots, h_N\}$ for $L(D)$, there are certain linear combinations with constant coefficients

$$f_i = \sum_{j=0}^{N} a_{ij} h_j \quad (0 \leq i \leq n)$$

and

$$f_i = \sum_{j=0}^{N} b_{ij} h_j \quad (0 \leq i \leq m)$$

such that $\phi = (f_0, \dots, f_n)$ and $\psi = (g_0, \dots, g_m)$. We also have a morphism $\lambda : V \to \mathsf{P}^N$ corresponding to the complete linear system $|D|$ and the chosen basis. Let $A : \mathsf{P}^N \to \mathsf{P}^n$, respectively $B : \mathsf{P}^N \to \mathsf{P}^m$, be the linear map defined by the matrix (a_{ij}), respectively (b_{ij}). Then $\phi = A \circ \lambda$ and $\psi = B \circ \lambda$. A, B are defined on $\lambda(V(\overline{\mathbb{Q}}))$. By our results on the projective space, we know $h(A(Q)) = h(Q) + O(1)$ and $h(B(Q)) = h(Q) + O(1)$ for all $Q \in \lambda(V(\overline{\mathbb{Q}}))$. Set $Q = \lambda(P)$ for $P \in V(\overline{\mathbb{Q}})$ to get $h(\phi(P)) = h(\lambda(P)) + O(1) = h(B(\lambda(P))) + O(1) = h(\psi(P)) + O(1)$. $\square$

2.2.2. *Weil's height machine.* We have the following theorem:

Theorem 2.12. *Let k be a number field. For every smooth projective variety V/k there exists a map*

$$h_V : \mathrm{Div}V \to \{ \textit{ real valued functions on } V(\overline{k})\}$$

with the following properties:

1. *Let $H \subset \mathsf{P}^n$ be a hyperplane section. Then $h_{\mathsf{P}^n,H} \sim_{\mathsf{P}^n} h$.*
2. *Let $\phi : V \to W$ be a morphism, and $D \in \mathrm{Div}W$. Then $h_{V,\phi^*D} \sim_V h_{W,D} \circ \phi$.*
3. *Let $D, W \in \mathrm{Div}V$. Then $h_{V,D+E} \sim_V h_{V,D} + h_{V,E}$.*
4. *If $D \in \mathrm{Div}V$ is principal, then $h_{V,D} \sim_V 0$.*
5. *Let $D \in \mathrm{Div}V$ be an effective divisor, and let B be locus of linear system $|D|$. Then $h_{V,D}(P) \geq O(1)$ for all $P \in (V \backslash B)(\overline{\mathbb{Q}})$.*
6. *Let $D, E \in \mathrm{Div}V$. Suppose D is ample and E is algebraically equivalent to 0. Then*

$$\lim_{h_{V,D}(P) \to \infty} \frac{h_{V,E}(P)}{h_{V,D}(P)} = 0.$$

(7) *Let* $D \in \mathrm{Div}V$ *be ample. then for every finite extension* k'/k *and every constant* B*, the set*

$$\{P \in V(k'); h_{V,D}(P) \leq B\}$$

is finite.

(8) *The function* h_V *is unique up to* $\sim_V$*.*

Note that by (Linear equivalent), the function h_V *descends to a function* $\mathrm{Pic}(V)$ *to* $\{$*real functions on* $V(\overline{\mathbb{Q}})\}/\sim_V$*.*

For the notion of *algebraic equivalence* mentioned above see [25], Book 1, Chapter III, 4.4. Briefly, a family of divisors on a variety X with base T, is any map $f : T \to \mathrm{Div}X$. We say the family f is *algebraic* if there exists a divisor $C \in \mathrm{Div}(X \times T)$, T an algebraic variety, such that if for each $t \in T$, $(X \times \{t\}) \cap C = f(t) \times \{t\}$. Divisors D_1, D_2 are said to be *algebraically equivalence* if there is an algebraic family $f : T \to \mathrm{Div}X$ and $t_1, t_2 \in T$ such that $D_1 = f(t_1)$ and $D_2 = f(t_2)$.

The construction of the function h_V goes like the following. If D is very ample, we choose a morphism $\phi_D : V \to \mathsf{P}^n$ associated to D and we define

$$h_{V,D}(P) = h(\phi_D(P))$$

for all $P \in V(\overline{\mathbb{Q}})$. Next if D is an arbitrary divisor, by Theorem 2.9 we write D as $D_1 - D_2$, with D_1, D_2 very ample. We then define

$$h_{V,D}(P) = h_{V,D_1}(P) - h_{V,D_2}(P).$$

This theorem is stated for logarithmic heights; as easily we can define a multiplicative height by setting $H_V = \exp(h_V)$. Sometimes it is easier to work with the multiplicative height and sometimes with the logarithmic height.

2.2.3. *Counting points.* The finiteness statement of the last theorem lead us to the following question. Let k be a number field and let $[D] \in \mathrm{Pic}(V)$ be such that for all B

$$N(B) = \#\{P \in V(k); H_{V,[D]}(P) \leq B\}$$

is finite. We know that every ample class satisfies this requirement. Note that here we have made a choice for $H_{V,[D]}$. The question that we are going to consider in these lectures is how the number $N(B)$ changes as $B \to \infty$. The general feeling is that the geometry of V should determine the growth of $N(B)$ if one desensitize the arithmetic by passing to a sufficiently large ground field (via finite extensions of k), and perhaps deleting certain exceptional sub-varieties.

Example 2.13. Let the ground field be $\mathbb{Q}$. Let F be a homogeneous form of degree d on P^n which we are going to think of as the collection of primitive $(n+1)$-tuples of integers. We will see later that there are about B^{n+1} rational points of heights less than B in $\mathsf{P}^n(\mathbb{Q})$. Since F is of degree d, it takes about B^d values on the set of primitive integral points of height less than B. Let us assume that the values taken by F are equally likely so that the probability of getting 0 is about B^{-d}. This means that $F(X_0, \ldots, X_n)$ should be zero about B^{n+1-d} among the rational points of height less than B. Recall that if $X = (F = 0)$, then $-K_X = (n+1-d)H$. The guiding principle is that the occurrence of $n+1-d$ is more than a mere accident!

In this example, when $n + 1 > d$ one expects abundance of rational points, and otherwise scarcity. In geometric terms, when $n + 1 > 0$ the resulting hypersurface is *Fano*; a non-singular projective variety the negative of whose canonical class is ample is called *Fano*. Yuri Manin has formulated certain conjectures describing the behavior of $N(B)$ for Fano varieties; for non-Fano varieties one expects that rational points should be rare. For curves this is completely worked out, and the results can be described in terms of the genus of the curve. So let C be a non-singular projective curve of genus g defined over a number field k. Then

- When $g = 0$, the curve C is Fano. Such a curve will have a Zariski dense set of rational points if we go over a finite extension of k; over such extension, the number of rational points of height less than B grows at least like a positive power of B.
- When $g = 1$, then neither the canonical class, nor its negative is ample; this is the so-called *intermediate type*. In this case, if the curve has a rational point, it will be an elliptic curve. For any number field k, by Mordell-Weil's theorem, the set $C(k)$ is a finitely-generated abelian group. Any logarithmic height function is equivalent, in the sense of $\sim_C$, to a quadratic form on $C(k)$. It is then easily seen that $N(B)$ grows at most like a power of $\log B$. This is the scarcity alluded to above.
- When $g > 1$, then the curve is of *general type*. In this case, by Mordell's conjecture, proved by Faltings and Vojta, for any number field k, the set $C(k)$ is finite. These are clearly quite rare!

So clearly even for curves the problem of understanding the distribution of rational points is not trivial! Next, we make some comments on surfaces: Let V be a non-singular projective surface defined over a number field k. Then

- $-K_V$ ample. Such a surface is called a *del Pezo* surface. The study of such surfaces poses some serious arithmetical problems. There is a classification of del Pezo surfaces over $\mathbb{C}$; there is a P^2 and $\mathsf{P}^1 \times \mathsf{P}^1$, and the remaining classes all can be obtained as blow-ups of P^2 at at most eight points in general position. This classification is not valid over a non-algebraically closed field as $\mathrm{Gal}(\overline{k}/k)$ may act non-trivially on the set of exceptional curves. At any rate, such surfaces can have subvarieties isomorphic to P^1 embedded in such a way that $(-K_V)|_{\mathsf{P}^1} = \frac{1}{2}(-K_{\mathsf{P}^1})$, so that if we do the counting with respect to the height function given by the anti-canonical class, then there will be B^2 points of height less than B on the P^1 subvariety alone; whereas after throwing out such an embedded subvariety one would only have $B^{1+\epsilon}$ rational points of bounded height for any $\epsilon > 0$.
- Intermediate cases: There are various interesting class of such surfaces, e.g. abelian surfaces, $K3$ surfaces, etc. Some of which are easy to study and some others are hard.
- General type: There are the conjectures of Bombieri and Lang which are higher dimensional analogues of the Mordell's conjecture, i.e. one expects that over any number field k, the collection of rational points $V(k)$ should be contained in a proper Zariski closed set. In general, there is very little known about this. See [15], F.5.2 for more details.

2.2.4. *Metrizations.* It was mentioned in the definition of $N(B)$ in the last paragraph that one had to make a choice in the definition of $H_{V,[D]}$, because after all the height functions are well-defined only up to equivalence. In this paragraph, we try to make the choices made geometric. Let V be an algebraic variety, and L an invertible sheaf on V. We proceed to define the notion of an *L-height on* $V(k)$. First let L be a very ample sheaf, and let $(s_0, \dots, s_n)$ be a basis for $\Gamma(L)$. For every $x \in V$, there is j such that $s_j(x) \neq 0$. Set

$$H_{L,s}(x) = \prod_v \max_i \left\{ \left| \frac{s_i(x)}{s_j(x)} \right| \right\}.$$

This is clearly independent of the chosen j. If L is not very ample, we fix an isomorphism $\sigma : L \to L_1.L_2^{-1}$ and bases s_i of $\Gamma(L_i)$, and put

$$H_{L,\sigma,s}(x) = H_{L_1,s_1}(x) H_{L_2,s_2}(x)^{-1}.$$

Example 2.14. For $k = \mathbb{Q}$, $V = \mathsf{P}^n$, $L = O(1)$, and s a homogeneous coordinate system on V, then

$$H_{O(1),s}(x) = \max_i(|x_i|)$$

where $(x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$ is a primitive representative for x.

In certain situations, one can also define a *local height*. Note that in the product formula for the height function the local factors $\max_i \left\{ \left| \frac{s_i(x)}{s_j(x)} \right| \right\}$ are not well-defined; only the product is independent of j for example. Let L be very ample, and $t \in \Gamma(L)$. Let D be the divisor $(t = 0)$. Then for all $x \in V \backslash D$, we can define

$$H_{D,V}(x) = \max_i \left\{ \left| \frac{s_i(x)}{t(x)} \right| \right\}.$$

There is also a local version of this defined by

$$H_{\mathcal{O}(D),s,t}(t) = \prod_v h_{D,v}(x)$$

for $x \in V(k) \backslash D(k)$. For more [15] B.8 for more details.

For a moment, let L be a one-dimensional space over k. An *Adelic metrization of* L is a choice at each place v of k of a norm $\|.\|_v : L \to \mathbb{R}$ with the following properties

(1) $\|a\lambda\|_v = |a|_v.\|\lambda\|_v$ for all $a \in k$ and $\lambda \in L$; and
(2) For each $\lambda \in L \backslash \{0\}$, $\|\lambda\|_v = 1$ for all except finitely many v.

It is clear that $(k, \{|.|_v\}_v)$ is an adelic metrization.

Given a metrization on L, then the set Λ defined by

$$\Lambda = \{\lambda \in L; \|\lambda\|_v \leq 1 \text{ for all non-arch. } v\}$$

is lattice in L, i.e. a projective $\mathcal{O}_k$-module.

Going back to geometry, given an invertible sheaf L on V, we call a family $\{\|.\|_{x,v}\}_{x \in V(k_v)}$ of v-adic norms on L_x a *v-adic metric on* L if for every Zariski open $U \subset V$ and every section $s \in \Gamma(U, L)$ the map $x \mapsto \|s(x)\|_{x,v}$ is a continuous function $U(k_v) \to \mathbb{R}$. What we did earlier is in fact a v-adic metric in disguise: If L is very ample, it is generated by global sections, and we can choose a basis s_i of $\Gamma(L)$ defined over k. If s is a section such that $g(x) \neq 0$, we set

$$\|s(x)\|_{x,v} : \max_i \left| \frac{s_i(x)}{s(x)} \right|_v^{-1},$$

otherwise $\|s(x)\|_{x,v} = 0$. We can now define what we mean by an arbitrary adelic metric on an invertible sheaf. Again we assume that L is very ample. An adelic metric on L is a collection of v-adic metrics such that for all but finitely many v the v-adic metric on L is defined by means of some fixed basis s_i of $\Gamma(L)$. We call the data $\mathcal{L} = (L, \{\|.\|_v\}_v)$ an *adelically metrized linear bundle*. To extend the construction to arbitrary invertible sheaves, as before we represent an arbitrary sheaf

L as $L_1 \otimes L_2^{-1}$ with L_1, L_2 very ample. If L_1 and L_2 are adelically metrized, then their v-adic metrizations naturally extend to the tensor product. An adelic metrizations of L is any metrization which for all but finitely many v is induced from the metrizations of L_1, L_2.

Given an adelically metrized line bundle $\mathcal{L} = (L, \{\|.\|_v\}_v)$, we define local and global height functions as follows. Let s be a local section of L. Let $U \subset X$ be the maximal Zariski open subset of X where s is defined and is non-zero. For $x = (x_v)_v \in \mathsf{U}(\mathbb{A})$ we define the local height

$$H_{\mathcal{L},s,v}(x_v) := \|s(x_v)\|_{x_v,v}^{-1}$$

and the global height function

$$H_{\mathcal{L},s}(x) := \prod_v H_{\mathcal{L},s,v}(x_v).$$

It is important to note that by the *product formula* the restriction of global height to $U(k)$ does not depend on the choice of s.

3. Conjectures

In this section, we state Manin's conjectures on the distribution of rational points on Fano varieties. First we need some preparation on zeta functions.

3.1. **Zeta functions and counting.** Let S be any countable, possibly finite, set, and let $h : S \to \mathbb{R}_+$ be a function such that

$$N_S(h, B) = \#\{x \in S; h(x) \leq B\}$$

is finite for all B. Put

$$\mathcal{Z}_S(h; s) = \sum_{x \in S} h(x)^{-s},$$

and set $\beta_h = \inf\{\sigma \in \mathbb{R}; \mathcal{Z}_S(h; s) \text{ converges for } \Re s > \sigma\}$. It is important to note that if h, h' are two functions such that $\log h \sim_S \log h'$, then $\beta_h = \beta_{h'}$. Also, β_h is non-negative unless when S is finite, in which case it is $-\infty$. When $\beta_h \geq 0$, we have

$$\beta_h = \limsup_{B \to \infty} \frac{\log N_S(h, B)}{\log B}.$$

This in particular implies that for $\beta_h \geq 0$ we have $N_S(h, B) = O(B^{\beta_h + \epsilon})$, and $N_S(h, B) = \Omega(B^{\beta_h - \epsilon})$ for all $\epsilon > 0$. Finer growth properties of $N_S(h, B)$ can be obtained from the analytic properties of $\mathcal{Z}$ as the following theorem shows:

Theorem 3.1 (Ikehara's tauberian theorem)**.** *Suppose for some $t > 0$ we have $\mathcal{Z}_S(h, s) = (s - \beta_h)^{-t} G(s)$ where $G(\beta) \neq 0$ and $G(s)$ is holomorphic for $\Re s \geq \beta_h$. Then*

$$N_S(h, B) = \frac{G(\beta_h)}{\beta_h \Gamma(t)} B^{\beta_h} (\log B)^{t-1} (1 + o(1))$$

as $B \to \infty$.

Better understanding of the analytic properties of G, e.g. holomorphic continuation to a larger domain with growth conditions, usually leads to getting error terms in the asymptotic formula for N_U.

Remark 3.2. It was pointed out that if h, h' satisfy $\log h \sim_S \log h'$ then $\beta_h = \beta_{h'}$. The same is not true of the analytic properties of $\mathcal{Z}_S$. Here is an example. Choose a sequence of positive integers d_i such that $d_i \to \infty$ and $\frac{d_{i+1}}{d_i} \to \infty$. Set $S = \mathbb{N}$ and let h be any function satisfying the finiteness condition for $N_S(B, h)$ for all B. Set

$$h'(x) = \begin{cases} 2h(x) & d_{2i} \leq h(x) < d_{2i+1}; \\ \frac{h(x)}{2} & d_{2i+1} \leq h(x) < d_{2i+2}. \end{cases}$$

Then one can show that $N_S(h'; B)/B^{\beta_{h'}} (\log B)^{t-1}$ does not tend to a limit as $B \to \infty$, so that $\mathcal{Z}(h', s)(s - \beta_{h'})^t$ cannot be holomorphic for $\Re s \geq \beta_{h'}$.

3.2. **Height zeta function.**

3.2.1. *Néron-Severi group.* Let V be an algebraic variety defined over a number field k. We define the Néron-Severi group $NS(V)$ to be the quotient of Pic$V$ by algebraic equivalence. For any field F we set $NS_F = NS(V) \otimes F$. Given a divisor L in Div$V$, we denote by $[L]$ its class in any of the Néron-Severi groups. Ample classes form a semi-group in Pic$V$ (Think about the Veronese map!); this semi-group will generate a convex open cone N_+° in $NS_\mathbb{R}$. The closure of N_+°, denoted by N_+, is the collection of *nef* (*numerically effective*) classes. Now L an ample sheaf on V. Let h_L be a height function coming from a metrization of L. Let U be a subset in $V(k)$, and let $\beta_U(L)$ be the abscissa of convergence of $\mathcal{Z}_U(h_L, s)$. One can show that $\beta_U(L)$ depends only on $[L]$ in $NS_\mathbb{Q}$. Also if $\beta_U(L) \geq 0$ for one ample class L, then the same holds for all ample classes. In general, β_U extends uniquely to a continuous function on N_+° which is inverse linear on each half-line.

3.2.2. *Accumulating subvarieties.* A subset $X \subset U$ ($U \subset V(k)$) is called *accumulating* if

$$\beta_U(L) = \beta_X(L) > \beta_{U\setminus X}(L),$$

and *weakly accumulating* if

$$\beta_U(L) = \beta_X(L) = \beta_{U\setminus X}(L).$$

Example 3.3. Let V be P^2 blown up at a point. For any height function associated to the anti-canonical class, the exceptional divisor is accumulating.

3.2.3. *Conjectures.* In the following conjectures V is a Fano variety defined over a number field k.

Conjecture 3.4. *There is a finite extension k' of k such that $V(k')$ is Zariski dense in V.*

Conjecture 3.5. *Let k' be as in the previous conjecture. Then for sufficiently small Zariski open $U \subset V$ we have $\beta_{U(k')}(-K_V) = 1$.*

Conjecture 3.6. *Let U, k' be as above. Then*

$$N_{U(k')}(B) = CB(\log B)^{t-1}(1 + o(1))$$

as $B \to \infty$, where $t = \operatorname{rk} NS(V)$.

There is a conjectural description of the constant C in the last conjecture due to Peyre [22].

Remark 3.7. If a variety V has a rational curve on it, then clearly it will have an infinite number of rational points over some extension. The *Rational Curve Conjecture* says that if for a quasi-projective variety V and an open subset $U \subset V$ we have $\beta_{U(k)}(L) > 0$, then U contains a rational curve.

Remark 3.8. Baryrev and Tschinkel [4] have introduced new conjectures for arbitrary metrized line bundles. These conjectures aim to describe the behavior of $N(U)$ for any metrized line bundle $\mathcal{L}$ who underlying invertible sheaf is the interior of the cone of effective divisors. In such situations, the expectation is that

$$N_U(B, \mathcal{L}) \sim CB^a(\log B)^{b-1}$$

with $C \geq 0$, $a \in \mathbb{N}$, $b \in \frac{1}{2}\mathbb{N}$. Indeed one expects the following to hold. First off, $N_+(X)$ is believed to be a polyhedral cone, so for a point in the boundary of $N_+(X)$ it would make sense to talk about the *codimension of the face containing* it. So Given $\mathcal{L}$ as above, we set

$$a(L) = \inf\{a \in \mathbb{R}; a[L] + [K_X] \in N_+(X)\}$$

and $b(L)$ will be the codimension of the face containing $a(L)[L]+K_X$. We note that $a(L)$ and $b(L)$ are independent of the metrization and depend only on the underlying geometric data. The constant C however depends on the metrization.

3.3. **Results and methods.**

3.3.1. *Some bad news.* Before we get our hopes too high let us point out that of these conjectures at least the Conjecture 3.6 is wrong! Let us explain a family of counter-examples due to Batyrev and Tschinkel [3]. Let X_{n+2} be a hypersurface in $\mathsf{P}^n \times \mathsf{P}^3$ ($n \geq 1$) defined by the equation

$$\sum_{i=0}^{3} l_i(\underline{x}) y_i^3 = 0$$

where $\underline{x} = (x_0, \dots, x_n)$, and $l_0(\underline{x}, \dots, l_3(\underline{x})$ are homogeneous linear forms in $x_0, \dots, x_n$. We will assume that any $k = \min(n+1, 4)$ forms among the l_i's are linearly independent. Then one can check that X_{n+2} is a smooth Fano variety containing a Zariski open subset U_{n+2} which is isomorphic to $\mathbb{A}^{n+2}$, and that the Picard group of X_{n+2} over an arbitrary field containing $\mathbb{Q}$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Then the blow comes from the fact that for any open subset $U \subset X_{n+2}$ there exists a number field k_0 containing $\mathbb{Q}(\sqrt{-3})$ which may depend on U such that for any field k containing k_0 one has

$$N(U, -K_{X_{n+2}}, B) \geq cB(\log B)^3$$

for all $B > 0$ and some positive constant c. This counterexamples has led some mathematicians [21] to ask a variation of the conjecture: Does there exist a dense open subset U and a constant C such that the asymptotic formula predicted in the conjecture holds?

3.3.2. *But it's not all bad news...* There are many cases where the conjectures of Manin and their refinements and generalizations are proved. Here is an incomplete list:

- smooth complete intersections of small degree in P^n (circle method, e.g. [7]);
- generalized flag varieties [11];
- toric varieties [5], [6];
- horospherical varieties [30];
- equivariant compactifications of $\mathbb{G}_a^n$ [9];
- bi-equivariant compactifications of unipotent groups [29];
- wonderful compactifications of semi-simple groups of adjoint type [12, 28].

We expect that Manin's conjecture (and its refinements) should hold for equivariant compactifications of *all* linear algebraic groups G and their homogeneous spaces G/H.

3.3.3. *Methods.* There are a number of methods that have been employed to treat special cases of the conjectures 3.4, 3.5, and 3.6. This is a very casual review of the basic ideas:

Elementary methods: Sometimes elementary tricks can be used to treat some special examples. We will explain one of these in the next section.

Circle method: This method, invented by Hardy and Ramanujan, first showed up in mathematics in connection to classical problems in number theory [32, 18]: Waring's problems, Vinogradov's work on writing numbers as sums of three primes, partitions, etc. The idea is indeed quite simple: Suppose we have a set $\mathcal{A} = \{a_m\}$ of positive integers and we want to know whether or in how many ways a given integer n can be written as a sum of s integers from the set $\mathcal{A}$. In order to do this one forms the generating function $f(x) = \sum_m e^{2\pi i a_m}$, and considers

$$f(x)^s = \sum_n R_s(n) e^{2\pi i n x}$$

where $R_s(n)$ is the number of ways of writing n as a sum of s elements of $\mathcal{A}$. It is then clear that

$$R_s(n) = \int_0^1 f(x)^s e^{-2\pi i n x}\, dx.$$

This is called the circle method as $e^{2\pi i x}$ traces a circle in $\mathbb{C}$ when x is in $[0,1]$. One then writes $[0,1]$ as a union $\mathcal{M} \cup \mathfrak{m}$. $\mathcal{M}$ is where the function f is supposed to be large, i.e. small neighborhoods of rational numbers, and $\mathfrak{m}$ is the left-over. In a successful application of the circle method one expects the main term of $R_s(n)$ to come from $\int_{\mathcal{M}}$ and the rest $\int_{\mathfrak{m}}$ be error term. More on this in the next section.

Harmonic analysis: This method is most successful when the variety in question has a large group of automorphisms, and in this situation the harmonic analysis of the group of automorphisms enters the picture. This has been the case in applications to toric varieties, additive group compactifications, wonderful compactifications, and flag varieties stated above. For example in [28] the idea is to consider the height zeta function $\mathcal{Z}_{\mathcal{L}}(s)$ for a given metrized line bundle $\mathcal{L}$ and interpret it as a special value of an appropriately defined automorphic form. Then one uses the spectral theory of automorphic forms to derive the desired results.

Dynamics and ergodic theory: This is one of the emerging ideologies of the subject and number theory in general. The method has already yielded a major theorem [12] and is expected to produce more results. The basic idea in [12] is to first prove the equidistribution of rational points of bounded height using mixing techniques of ergodic theory. Then the asymptotics of the number of rational points follow from volume computations. See [31] for a motivated description of the method.

Universal torsors: The idea is to relate the rational points on a projective variety to the integral points of the *universal torsor* of the variety. For example, the points of $\mathsf{P}^n(\mathbb{Q})$ are in one to one correspondence with primitive $(n+1)$-tuples of integers. This is clearly an oversimplified example! See [14] for some computations and [8, 21] for a list of applications to various counting problems.

3.4. **Examples.** In this paragraph we collect several examples of various degrees of difficulty.

3.4.1. *Products of varieties.* If $V = V_1 \times V_2$ and $H_i : V_i(k) \to \mathbb{R}_+$, $U_i \subset V_i$ open subsets, then we have a function $H : V \to \mathbb{R}_+$ defined by $H(x_1, x_2) = H_1(x_1)H_2(x_2)$ corresponding to Segre embedding. Assume that

$$N_{U_i}(H_i, B) = C_i B(\log B)^{t_i-1} + O(B(\log B)^{t_i-2}).$$

Then by a proposition in [11]

$$N_{U_1\times U_2}(H, B) = \frac{(t_1-1)!(t_2-1)!}{(t_1+t_2-1)!} C_1C_2B(\log B)^{t_1+t_2-1}(1+o(1)).$$

3.4.2. *The Pythagorean curve.* This is a particularly elementary example. Take the curve C in P^2 defined by $x^2+y^2=z^2$, and we would like to understand the growth of

$$N(B) = \{P \in C(\mathbb{Q}); H(P) \le Q\}.$$

Here H is the P^2 height. In this case, $N(B) \sim CB$. Calculating C is an amusing exercise.

3.4.3. *Projective spaces.* Let us treat $\mathsf{P}^n(\mathbb{Q})$. Let us start by setting

$$\begin{aligned}\mathcal{Z}(s) &= \sum_{\gamma\in\mathsf{P}^n(\mathbb{Q})} \frac{1}{H(\gamma)^s} \\ &= \sum_{(x_0,\ldots,x_n)\in\mathbb{Z}^{n+1}_{\mathrm{prim}}\setminus\{(0,\ldots,0)\}} \frac{1}{\max(|x_0|,\ldots,|x_n|)^s}.\end{aligned}$$

Now

$$\begin{aligned}\zeta(s)\mathcal{Z}(s) &= \sum_{(x_0,\ldots,x_n)\in\mathbb{Z}^{n+1}\setminus\{(0,\ldots,0)\}} \frac{1}{\max(|x_0|,\ldots,|x_n|)^s} \\ &= \sum_{k=1}^{\infty} \frac{(2k+1)^{n+1}-(2k-1)^{n+1}}{k^s}.\end{aligned}$$

Consequently

$$\begin{aligned}\mathcal{Z}(s) &= \left(\sum_{r=1}^{\infty}\frac{\mu(r)}{r^s}\right)\cdot\left(\sum_{k=1}^{\infty}\frac{(2k+1)^{n+1}-(2k-1)^{n+1}}{k^s}\right) \\ &= \sum_{m=1}^{\infty}\frac{1}{m^s}\sum_{rk=m}\mu(r)\left\{(2k+1)^{n+1}-(2k-1)^{n+1}\right\}.\end{aligned}$$

So this means, and this we could also gotten out of the Mobius inversion formula as well, that we need to find the asymptotic growth of

$$\begin{aligned}&\sum_{m\le B}\sum_{rk=m}\mu(r)\left\{(2k+1)^{n+1}-(2k-1)^{n+1}\right\} \\ =&\sum_{j=0}^{n+1}\binom{n+1}{j}2^j\left[1-(-1)^{n+1-j}\right]\sum_{m\le B}\sum_{rk=m}\mu(r)k^j \\ =&\frac{2^n}{\zeta(n+1)}B^{n+1}+O(B^n).\end{aligned}$$

There is a famous theorem of Schanuel [23] that generalizes this to $\mathsf{P}^n(k)$ for any number field k.

3.4.4. *Blowup of* P^2 *at one point.* This example is [15], F.5.4.3 (originally [24] 2.12; also [17] for generalizations) Let $X \to \mathsf{P}^2$ be the blowup at one point. Let L be the pullback to X of a generic line on P^2, and E the exceptional divisor. Then one shows [24] that $\mathrm{Pic}X$ is freely generated by L, E. The canonical divisor is $K_X = -3L + E$. It is seen using the Nakai-Moishezon Criterion ([13], Chapter V, Theorem 1.10) that a divisor $D = aL - bE$ is ample if $a > b > 0$. For an ample divisor $D = aL - bE$ set

$$\alpha(D) = \max\left\{\frac{3}{a}, \frac{2}{a-b}\right\}.$$

Let $U = X \backslash E$. Then

$$N_{U(k)}(D, B) = \begin{cases} cB^{\alpha(D)} & \text{if } D \text{ is not proportional to } K_X \\ cB^{\alpha(D)} \log B & \text{if } D \text{ is proportional to} K_X. \end{cases}$$

3.4.5. *Blow-up of* P^2 *at three points.* This is example 2.4 of [21]. Let $V \to \mathsf{P}^2(\mathbb{Q})$ be the blowup of P^2 at $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$, and $P_3 = (0 : 0 : 1)$. Then V is a hypersurface in $\mathsf{P}^1 \times \mathsf{P}^1 \times \mathsf{P}^1$ given by the equation $x_1 x_2 x_3 = y_1 y_2 y_3$. Let's set

$$H(P_1, P_2, P_3) = H_{\mathsf{P}^1}(P_1) H_{\mathsf{P}^1}(P_2) H_{\mathsf{P}^1}(P_3)$$

where H_{P^1} is the standard height on P^1. This will define a height function on $V(\mathbb{Q})$. There are six exceptional lines on V defined by $E_{ij} = (x_i = 0, y_j = 0)$ for $i \neq j$. Let $U = V \backslash \bigcup_{i \neq j} E_{ij}$. We have

$$N_{E_{ij}}(B, H) \sim CB^2$$

and

$$N_U(B, H) \sim \frac{1}{6}\left(\prod_p \left(1 - \frac{1}{p}\right)^4 \left(1 + \frac{4}{p} + \frac{1}{p^2}\right)\right) B(\log B)^3.$$

4. Compactifications of Semi-Simple Groups

In this section, I will explain the results and methods of [28]. The reader might consider reading this section in conjunction with [27]. In [27] we considered the problem treated in this section for the group PGL(2) where the automorphic forms material needed is basically the theory of the upper half space; also as the group PGL(2) is split we do not have to deal with problems coming from the arithmetic of inner forms, e.g. Galois cohomology. The reader should be warned though that the treatment of Eisenstein series and integrals as presented in [27] uses the deep results of Werner Müller in his proof of the *trace class conjecture*; fortunately, these results were deemed unnecessary in the final analysis of [28]. [31] too provides another sketch of the proof of the main theorem of this section.

4.1. **A Concrete Counting Problem.** Let G be an algebraic group defined over a number field F. We will assume that G is semi-simple of adjoint type. Let $\varrho : \mathsf{G} \to \mathrm{PGL}_N$ be an F-representation. Think of PGL_N as an open in $\mathbb{P}^{N^2-1}$, and get a height function H on $\mathrm{PGL}_N(F)$. Pull the height function back to $\mathsf{G}(F)$ and call it H_ϱ. We want to find an asymptotic formula for the number of elements of the following set

$$N_\varrho(B) = \{\gamma \in \mathsf{G}(F) \text{ such that } H_\varrho(\gamma) \le B\}$$

as $B \to \infty$.

We will show that if ϱ is absolutely irreducible and faithful, then there are $\mathsf{a} \in \mathbb{Q}$ and $\mathsf{b} \in \mathbb{Z}$ such that

$$N_\varrho(B) = CB^{\mathsf{a}_\varrho}(\log B)^{\mathsf{b}_\varrho - 1}(1 + O(B^{-\delta})).$$

The constants $\mathsf{a}_\varrho, \mathsf{b}_\varrho$ are geometric, but C depends on some arithmetic data. As usual the idea of the proof is to use a Tauberian theorem. Inspired by the Tauberian theorem, and as indicated earlier, we form the following Height Zeta Function:

$$\mathcal{Z}_\varrho(s) = \sum_{\gamma \in \mathsf{G}(F)} \frac{1}{H_\varrho(\gamma)^s}.$$

Our goal is to determine the analytic properties of this zeta function. In order to do this, we start by defining a function on $\mathsf{G}(F)\backslash\mathsf{G}(\mathbb{A})$. For $\Re s >> 0$, set

$$\mathcal{Z}_\varrho(s, g) = \sum_{\gamma \in \mathsf{G}(F)} \frac{1}{H_\varrho(\gamma g)^s},$$

so that $\mathcal{Z}_\varrho(s) = \mathcal{Z}_\varrho(s, e)$. Then one shows that for $\Re s$ large, $\mathcal{Z}_\varrho(s, .)$ is convergent for all g. The convergence is uniform on compact sets, and consequently the resulting function is smooth, and one shows that it is indeed bounded. In particular it is in $\mathsf{L}^2(\mathsf{G}(F)\backslash\mathsf{G}(\mathbb{A}))$. So the idea is to bring in the spectral theory of automorphic forms.

4.1.1. *Spectral theory: Anisotropic case.* Let us first consider the case where G is *anisotropic*. In this case

$$\mathsf{L}^2(\mathsf{G}(F)\backslash\mathsf{G}(\mathbb{A})) = (\widehat{\bigoplus_\pi \mathcal{H}_\pi}) \oplus (\bigoplus_\chi \mathbb{C}_\chi),$$

as a Hilbert direct sum of irreducible subspaces.

We have next the following *Automorphic Fourier expansion*

$$\mathcal{Z}_\varrho(s, g) = \sum_\pi \sum_{\phi \in \mathcal{B}_\pi} \langle \mathcal{Z}_\varrho(s, .), \phi \rangle \phi(g) + \sum_\chi \langle \mathcal{Z}_\varrho(s, .), \chi \rangle \chi(g).$$

as an identity of L^2-functions.

First step: Show for fixed s with $\Re s \gg 0$ that the right hand side converges normally so that this is an identity of continuous functions. Set $g = e$. Get

$$\mathcal{Z}_\varrho(s) = \sum_\pi \sum_{\phi \in \mathcal{B}_\pi} \langle \mathcal{Z}_\varrho(s,.), \phi \rangle \phi(e) + \sum_\chi \langle \mathcal{Z}_\varrho(s,.), \chi \rangle.$$

Second step: Analytically continue $\langle \mathcal{Z}_\varrho(s,.), \phi \rangle$.

$$\begin{aligned}
\langle \mathcal{Z}_\varrho, \phi \rangle &= \int_{\mathsf{G}(F)\backslash \mathsf{G}(\mathbb{A})} \mathcal{Z}_\varrho(s,g) \overline{\phi(g)}\, dg \\
&= \int_{\mathsf{G}(\mathbb{A})} H_\varrho(g)^{-s} \overline{\phi(g)}\, dg \\
&= \int_{\mathsf{G}(\mathbb{A})} H_\varrho(g)^{-s} \int_{\mathsf{K}_0} \overline{\phi(kg)}\, dk\, dg.
\end{aligned}$$

Without loss of generality we can assume that

$$\mathsf{K}_0 = \prod_{v \notin S} \mathsf{K}_v \times \mathsf{K}_0^S,$$

for a finite set of places S. Here for $v \notin S$, K_v is a maximal special open compact subgroup in $\mathsf{G}(F_v)$. After enlarging S to contain all the places where G is not split, we can assume that $\mathsf{K}_v = \mathsf{G}(\mathcal{O}_v)$. In particular, for $v \notin S$ the local representations π_v are spherical. Thus we have a normalized local spherical function φ_v associated to π_v. We have assumed that each ϕ is right K_0-invariant. In conclusion,

$$\begin{aligned}
\langle \mathcal{Z}_\varrho, \phi \rangle = &\prod_{v \notin S} \int_{\mathsf{G}(F_v)} \varphi_v(g_v) H_\varrho(g_v)^{-s}\, dg_v \\
&\times \int_{\mathsf{G}(\mathbb{A}_S)} H_\varrho(\eta(g_S))^{-s} \int_{\mathsf{K}_0^S} \phi(k\eta(g_S))\, dk\, dg_S.
\end{aligned}$$

(Here $\eta : \mathsf{G}(\mathbb{A}_S) \to \mathsf{G}(\mathbb{A})$ is the natural inclusion map.) The integral over $\mathsf{G}(\mathbb{A}_S)$ is easy to handle. Our main concern here is the first factor

$$I_\varrho^S(s) = \prod_{v \notin S} \int_{\mathsf{G}(F_v)} \varphi_v(g_v) H_\varrho(g_v)^{-s}\, dg_v.$$

For this we will need bounds on matrix coefficients. More on this later.

Third step: Analytically continue and identify the poles of the finite sum $\sum_\chi \langle \mathcal{Z}_\varrho(s,.), \chi \rangle$. More on this later as well. Let us just point out that this step involves volume calculations and transfer of characters to quasi-split forms of the group.

Fourth step: Analytically continue $\sum_\pi \sum_{\phi\in\mathcal{B}_\pi} \langle \mathcal{Z}_\varrho(s,.),\phi\rangle\phi(e)$. This follows from the analytic behavior of the spectral zeta function of the Laplace operator.

4.1.2. *Spectral theory: General.* Let us explain the situation when the group in question is not anisotropic. Fix a Borel subgroup B. We will work with standard parabolic subgroups up to association. Let a typical parabolic subgroup be denoted by P, its Levi factor M_P, the unipotent radical by N_P, and connected component of the center A_P Let $X(\mathsf{M}_\mathsf{P})_\mathbb{Q}$ the group of characters of M_P defined over $\mathbb{Q}$, and $\mathfrak{a} = \mathrm{Hom}(X(\mathsf{M}_\mathsf{P})_\mathbb{Q},\mathbb{R})$, and $\mathfrak{a}^* = X(\mathsf{M}_\mathsf{P})_\mathbb{Q}\otimes\mathbb{R}$. Denote the set of simple roots of $(\mathsf{P},\mathsf{A}_\mathsf{P})$ by Δ_P.

If $m=(m_v)_v\in\mathsf{M}_\mathsf{P}(\mathbb{A})$, we define a vector $H_\mathsf{M}(m)\in\mathfrak{a}_\mathsf{P}$ by

$$e^{<H_\mathsf{M}(m),\chi>} = |\chi(m)| = \prod_v |\chi(m_v)|_v$$

for all $\chi\in X(\mathsf{M}_\mathsf{P})_\mathbb{Q}$. This is a homomorphism $\mathsf{M}(\mathbb{A})\to\mathfrak{a}_\mathsf{P}$. Let $\mathsf{M}(\mathbb{A})^1$ be the kernel of this homomorphism. Then $\mathsf{M}(\mathbb{A}) = \mathsf{M}(\mathbb{A})^1\times\mathsf{A}_\mathsf{P}(\mathbb{R})^0$. By Iwasawa decomposition, any $x\in\mathsf{G}(\mathbb{A})$ can be written as

$$nmak$$

with $n\in\mathbb{N}(\mathbb{A}), m\in\mathsf{M}(\mathbb{A})^1, a\in\mathsf{A}_\mathsf{P}(\mathbb{R})^0, k\in\mathsf{K}$. Define $H_\mathsf{P}(x) = H_\mathsf{M}(a)\in\mathfrak{a}_\mathsf{P}$. There is a vector $\rho_\mathsf{P}\in\mathfrak{a}_P^*$ such that

$$\delta_\mathsf{P}(p) = e^{2\rho_P(H_\mathsf{P}(p))}.$$

Let π be an automorphic representation of $\mathsf{M}(\mathbb{A})$ (trivial on A_M). Define $\mathcal{A}_\mathsf{P}$ (resp. $\mathcal{A}_\mathsf{P}^2$, $\mathcal{A}_\mathsf{P}^c$, $\mathcal{A}_\mathsf{P}^\pi$, $\mathcal{A}_\mathsf{P}^r$) to be the space of automorphic forms on $\mathsf{N}(\mathbb{A})\mathsf{M}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})$ such that

$$\varphi(ag) = e^{<\rho_\mathsf{P},H_\mathsf{M}(a)>}\varphi(g)$$

for every $a\in\mathsf{A}_\mathsf{M}$ and $g\in\mathsf{G}(\mathbb{A})$ (resp. for any $k\in\mathsf{K}$, the function $m\mapsto\varphi(mk)$ on $\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})^1$ is square-integrable, or is cuspidal, or belongs to the space of π, or is in the residual spectrum). For any $\zeta\in\mathfrak{a}_\mathsf{P}^*\otimes\mathbb{C}$ define

$$I(g,\zeta)\varphi(x) = e^{<-\zeta,H_\mathsf{M}(x)>}e^{<\zeta,H_\mathsf{M}(xg)>}\varphi(xg).$$

If $\varphi\in\mathcal{A}_\mathsf{P}$, we define an *Eisenstein Series* by

$$E(g,\varphi,\zeta) = \sum_{\gamma\in\mathsf{P}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{Q})} e^{<\zeta,H_\mathsf{M}(\gamma g)>}\varphi(\gamma g).$$

Let f be a function on $\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})$. Casually, by the *spectral expansion* of f we mean [1]

$$S(f,x) = \sum_{X\in\mathfrak{X}}\sum_{\mathsf{P}} n(A)^{-1} \int_{\Pi(M)} \sum_{\phi\in\mathcal{B}_{\mathsf{P}}(\pi)_X} E(x,\phi)\left(\int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} \overline{E(y,\phi)} f(y)\,dy\right) d\pi.$$

Note that in the particular case where $\mathsf{M}=\mathsf{G}$, the integrals have to be interpreted appropriately to give a discrete sum. Now the question is whether $\mathcal{Z}_\varrho(s,.)$ has a spectral decomposition. Here one needs to make sure that this spectral decomposition is an identity of continuous functions, because after all we need the value of $\mathcal{Z}(s,.)$ at the identity and a mere identity of L^2 functions is not sufficient! Note that our function is not rapidly decaying, so Arthur's results as stated in [1] would not be sufficient. We will show that our function will in fact have a spectral decomposition, but one has to be careful about the order of summation.

4.1.3. *A spectral decomposition theorem.* We will use the following theorem:

Theorem 4.1. *Let f be a smooth function on $\mathsf{G}(\mathbb{A})$ and suppose f is right invariant under a compact-open subgroup of $\mathsf{G}(\mathbb{A}_f)$. Define a function F on $\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})$ by $F(g)=\sum_{\gamma\in\mathsf{G}(\mathbb{Q})} f(\gamma g)$. Suppose F is absolutely convergent for all g, and it is smooth and bounded as a function of g. Then there is n,N such that if the integrals*

$$\int_{\mathsf{G}(\mathbb{A})} |f(y)|.\|y\|^N\,dy$$

and

$$\int_{\mathsf{G}(\mathbb{A})} |\Delta^n f(y)|.\|y\|^N\,dy$$

converge, then F has a spectral decomposition (equality of continuous functions).

In order to prove the theorem one needs to show that the sum $S(f,x)$ is uniformly convergent on compact sets. This is sufficient as one can show that for all pseudo-Eisenstein series θ_ϕ [19], we have

$$(S(f),\theta_\phi) = (f, S(\theta_\phi)).$$

On the other hand as θ_ϕ is rapidly decreasing, we get the identity $S(\theta_\phi)=\theta_\phi$. Consequently $(f-S(f),\theta_\phi)=0$, and the result now follows from the density of θ_ϕ in the space of all automorphic forms.

The result now follows from various results of Arthur and the following lemma

Lemma 4.2. *Given m there is n and functions $f_1 \in C_c^m(\mathsf{G}(\mathbb{R}))^{K_{\mathbb{R}}}$ and $f_2 \in C_c^\infty(\mathsf{G}(\mathbb{R}))^{K_{\mathbb{R}}}$ such that $\Delta^n * f_1 + f_2$ is the delta distribution at the identity.*

4.1.4. *Analytic continuation I.* Now we can set $g = e$. For $\Re s$ large we have (essentially)

$$\mathcal{Z}(s) = \sum_{X \in \mathfrak{X}} \sum_{\mathsf{P}} n(A)^{-1}$$
$$\int_{\Pi(M)} \sum_{\phi \in \mathcal{B}_{\mathsf{P}}(\pi)_X} E(e,\phi) \left(\int_{\mathsf{G}(\mathbb{A})} \overline{E(y,\phi)} H(y)^{-s} \, dy \right) d\pi.$$

The first step in the analytic continuation is to analytically continue

$$\int_{\mathsf{G}(\mathbb{A})} \overline{E(y,\phi)} H(y)^{-s} \, dy$$

Here the idea is to integrate against the stabilizer of H to get matrix coefficients (similar to Godement and Jacquet) and use bounds on matrix coefficients. If group is rank one, we use bounds towards Ramanujan for GL_2 and Rogawski's results for forms of $\mathsf{U}(3)$. If the rank is larger than one, use local bounds as in the next paragraph.

4.1.5. *Bounds for matrix coefficients.* The results of this paragraph are due to H. Oh [20]. Let k be a non-archimedean local field of $\mathrm{char}(k) \neq 2$, and residual degree q. Let H be the group of k-rational points of a connected reductive split or quasi-split group with $\mathsf{H}/\mathsf{Z}(\mathsf{H})$ almost k-simple. Let S be a maximal k-split torus, B a minimal parabolic subgroup of H containing S and K a good maximal compact subgroup of H with Cartan decomposition $\mathsf{G} = \mathsf{K}\mathsf{S}(k)^+\mathsf{K}$. Let Φ be the set of non-multipliable roots of the relative root system $\Phi(\mathsf{H},\mathsf{S})$, and Φ^+ the set of positive roots in Φ. A subset $\mathcal{S}$ of Φ^+ is called a strongly orthogonal system of Φ if any two distinct elements α and α' of $\mathcal{S}$ are strongly orthogonal, that is, neither of $\alpha \pm \alpha'$ belongs to Φ. Define a bi-K-invariant function $\xi_{\mathcal{S}}$ on H as follows: first set

$$n_{\mathcal{S}}(g) = \frac{1}{2} \sum_{\alpha \in \mathcal{S}} \log_q |\alpha(g)|,$$

then

$$\xi_{\mathcal{S}}(g) = q^{-n_{\mathcal{S}}(g)} \prod_{\alpha \in \mathcal{S}} \left(\frac{(\log_q |\alpha(g)|)(q-1) + (q+1)}{q+1}\right).$$

Theorem 4.3 (H. Oh)**.** *Assume that the semi-simple k-rank of H is at least 2. Let $\mathcal{S}$ be any strongly orthogonal system of Φ. Then for any unitary representation ϱ of H without an invariant vector and with K-finite unit vectors ν and ν',*

$$|(\varrho(g)\nu, \nu')| \leq (\dim(\mathsf{K}\nu)\dim(\mathsf{K}\nu'))^{\frac{1}{2}} \cdot \xi_{\mathcal{S}}(g),$$

for any $g \in \mathsf{H}$.

4.1.6. *Analytic continuation II.* The main ingredient of this step is the following theorem:

Theorem 4.4. *Let $\Lambda(\phi)$ for each $\phi \in \mathcal{B}_{\mathsf{P}}(\pi)_X$ be defined by $\Delta.\phi = \Lambda(\phi).\phi$. Then there is $m > 0$ such that*

$$\sum_{X \in \mathfrak{X}} \sum_{\mathsf{P}} n(A)^{-1} \int_{\Pi(M)} \left(\sum_{\phi \in \mathcal{B}_{\mathsf{P}}(\pi)_X} \Lambda(\phi)^{-m} |E(e,\phi)|^2 \right) d\pi$$

is convergent. The outermost summation is only over those classes for which $\Lambda(\phi) \neq 0$ and fixed by K.

This follows from a theorem of Arthur [2] combined with an observation of Lapid [16].

4.1.7. *Analytic continuation III.* The first and the second steps imply that the first pole of the height zeta function must come from one dimensional contributions to the spectral expansion. So let χ be an automorphic character of $\mathsf{G}(\mathbb{A})$, and consider the integral

$$\int_{\mathsf{G}(\mathbb{A})} \chi(g) H(g)^{-s} \, dg.$$

One then proves that this is regularized in a certain domain by a product of Hecke L-functions, and that's how one finishes the proof. The regularization process is indeed pretty substantial. What goes into the proof is Cartan Decomposition plus two important ingredients:

- Volume estimates
- Transfer of characters

4.1.8. *Volume estimate.*

Proposition 4.5. *There exist c, independent of v, such that for all t in the positive Weyl chamber in $T(F_v)$ we have*

$$\mathrm{vol}(\mathsf{K}_v t \mathsf{K}_v) \leq \delta_{\mathsf{B}}(t)(1 + \frac{c}{q_v}).$$

The proof of this fact is geometric and uses p-adic integration techniques of Batyrev, Denef, Igusa, and Loeser. More on this in 4.2.3. Benedict Gross has an argument based on Bruhat-Tits theory.

4.1.9. *Transfer of characters between inner forms.*

Proposition 4.6. *Let G and G' be two groups defined over a global field which are inner forms of each other, and let χ be an automorphic character of $\mathsf{G}(\mathbb{A})$. Then there is an automorphic character χ' of $\mathsf{G}'(\mathbb{A})$ such that for almost all places v, we have $\chi_v = \chi'_v$. Here we are identifying G_v and G'_v.*

The proof of this uses Galois cohomology. Let us first describe the local transfer. Let G and G' as above. Let

$$j_v \,:\, \mathsf{G}^{sc}(F_v) \to \mathsf{G}(F_v)$$

be the canonical homomorphism from its simply connected covering. Let v be a finite place of F and $\hat{\mathsf{G}}_v$ the group of characters χ_v of $\mathsf{G}(F_v)$ which are trivial on $j_v(\mathsf{G}^{sc}(F_v))$. By Kneser's theorem, $H^1(F_v, \mathsf{G}^{sc}(\bar{F}_v)) = 1$. We have an exact sequence

$$1 \to \mathsf{Z}(F_v) \to \mathsf{G}^{sc}(F_v) \xrightarrow{j_v} \mathsf{G}(F_v) \xrightarrow{\delta_v} H^1(F_v, \mathsf{Z}(\bar{F}_v)) \to 1$$

and a similar sequence for G'. Here Z is the center of G^{sc}. We may identify

$$\hat{\mathsf{G}}_v = H^1(F_v, \mathsf{Z}(\bar{F}_v))^\vee,$$

(the character group of $H^1(F_v, \mathsf{Z}(\bar{F}_v))$. Recall that G is obtained from G' by replacing the standard Galois action by a twisted action, via an element of $H^1(\mathrm{Gal}(\bar{F}/F), \mathrm{Inn}(\mathsf{G}'(\bar{F})))$. Since inner automorphisms of $\mathsf{G}'(\bar{F})$ fix $\mathsf{Z}(\bar{F})$, these two actions coincide on $\mathsf{Z}(F)$, and similarly for $\mathsf{Z}(\bar{F}_v)$. We get then, for non-archimedian v, a natural isomorphism

$$\mathrm{tr}_v \,:\, \hat{\mathsf{G}}_v \to \hat{\mathsf{G}}'_v.$$

Now let us describe the global transfer. Let $\chi = \prod_v \chi_v$ be a one-dimensional automorphic representation of $\mathsf{G}(\mathbb{A})$ such that χ_v is trivial on $j_v(\mathsf{G}^{sc}(F_v))$, for all $v \in \mathrm{Val}(F)$. In particular, $\chi_\infty := \prod_{v|\infty} \chi_v$ is trivial on G^*_∞. Here we have written $\mathsf{G}^*_\infty \subset \mathsf{G}_\infty$ for the image of $\prod_{v|\infty} \mathsf{G}^{sc}(F_v)$ under $\prod_{v|\infty} j_v$. We define

$$\chi'_0 = \prod_{v\nmid\infty} \mathrm{tr}_v(\chi_v).$$

We also have

$$\mathsf{G}'(\mathbb{A}) = \mathsf{G}'(F) \cdot \mathsf{G}'_0 \cdot (\mathsf{G}'_\infty)^*.$$

We extend χ_0' to a character χ' of $\mathsf{G}'(\mathbb{A})$ by setting $\chi' = 1$ on $\mathsf{G}'(F) \cdot (\mathsf{G}'_\infty)^*$. Then one shows that χ' thus defined is an automorphic character.

Remark 4.7. One can use the results of the one-dimensional regularization to get an asymptotic formula for the adelic volume of the compact set

$$\{g \in \mathsf{G}(\mathbb{A}) \mid H(g) \leq B\}.$$

This is fundamental in the work of Gorodnik, Macourant, and Oh [12].

4.2. **Connection to Manin's conjecture.** Our purpose here is to connect our concrete counting problem to Manin's conjecture for a certain class of varieties.

4.2.1. *Wonderful compactifications.* Let us first work over an algebraically closed field of characteristic zero.

Proposition 4.8 (De Concini-Procesi [10])**.** *There exists a canonical compactification of a connected adjoint group* G*: a smooth projective variety* X *over* F *such that*

- $\mathsf{G} \subset X$ *is a Zariski open subvariety and the action of* $\mathsf{G} \times \mathsf{G}$ *on* G *(by* $(g_1, g_2)(g) = g_1 g g_2^{-1}$*) extends to an action of* $\mathsf{G} \times \mathsf{G}$ *on* X*;*
- *The boundary* $X \backslash \mathsf{G}$ *is a union of strict normal crossings divisors* D_i *(for* $i = 1, ..., \mathsf{r}$*). For every* $I \subset [1, ..., \mathsf{r}]$ *the subvariety* $D_I = \cap_{i \in I} D_i$ *is a* $\mathsf{G} \times \mathsf{G}$*-orbit closure. All* $\mathsf{G} \times \mathsf{G}$*-orbit closures are obtained this way;*
- X *contains a unique closed* $\mathsf{G} \times \mathsf{G}$*-orbit* $Y = \mathsf{G}/\mathsf{B} \times \mathsf{G}/\mathsf{B}$*;*
- *The components* D_I *are isomorphic to fibrations over* $\mathsf{G}/\mathsf{P}_I \times \mathsf{G}/\mathsf{P}_I$ *with fibers canonical compactifications of the adjoint form of the associated Levi groups.*

There are various way to construct the *wonderful compactification.* This is my favorite construction: Let $\mathfrak{g} = \mathrm{Lie}(\mathsf{G})$ (a vector group defined over F) and $\mathsf{n} = \dim(\mathsf{G}) = \dim(\mathfrak{g})$. The variety $\mathbb{L}$ of Lie subalgebras of the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ can be regarded as a subvariety of the Grassmannian $\mathrm{Gr}(n, 2n)$. It contains $\mathfrak{g}$, embedded diagonally. Moreover, $\mathbb{L}$ is a projective $\mathsf{G} \times \mathsf{G}$-variety. Taking the closure of the $\mathsf{G} \times \mathsf{G}$-orbit X° through $\mathfrak{g}$ we obtain $X \subset \mathbb{L}$. Since G is an adjoint group, the adjoint representation of G on $\mathsf{G} \times \mathsf{G}$ is faithful and we may identify X° with the variety G, or more precisely $\mathsf{G} \times \mathsf{G}/\mathrm{Diag}(\mathsf{G})$.

4.2.2. *Picard group of* X*.* Again F is an algebraically closed field of characteristic zero. For $\mathsf{w} \in \mathsf{W}$ we denote by $X(\mathsf{w})$ the closure of $\mathsf{BwB} \subset \mathsf{G}$ in X. The $\mathsf{B} \times \mathsf{B}$ - stable boundary components D_i correspond to $X(s\mathsf{w}_0 s_i)$. Every line bundle L on X restricts to the unique closed $\mathsf{G}\times\mathsf{G}$-orbit $Y = \mathsf{G}/\mathsf{B}\times\mathsf{G}/\mathsf{B}$; we get a restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$. Recall that one can identify the Picard group of G/B with $\mathfrak{X}^*(\mathsf{T}^{sc})$.

Proposition 4.9. *Let X be the canonical compactification of G as above.*

- *The Picard group $\mathrm{Pic}(X)$ is freely generated by the classes $[D_i]$. In particular,* $\mathrm{rk}\ \mathrm{Pic}(X) = \mathrm{rk}_F\, \mathsf{G}$.*;*
- *Let*
$$Y := \cap_i D_i \ .$$
Then Y is the unique closed $\mathsf{G}\times\mathsf{G}$-orbit in X and $Y \simeq \mathsf{G}/\mathsf{B}\times \mathsf{G}/\mathsf{B}^-$, with B^- the opposite Borel. The restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is injective. The image of $\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}(Y)$ consists of classes
$$L(\lambda) = (\lambda, -\mathsf{w}_0\lambda) \subset \mathrm{Pic}(Y) = \mathrm{Pic}(\mathsf{G}/\mathsf{B}) \times \mathrm{Pic}(\mathsf{G}/\mathsf{B});$$
- *The (closed) cone of effective divisors is given by*
$$\Lambda_{\mathrm{eff}}(X) := \oplus_{i=1}^{\mathsf{r}} \mathbb{R}_{\geq 0}[D_i].$$
More precisely, if $\lambda \in \mathfrak{X}(\mathsf{T}^{sc})$ is a dominant weight then the line bundle $L(\lambda)$ on X has a unique (up to scalars) global section f_λ with divisor
$$\mathrm{div}(f_\lambda) = \sum_{i=1}^{\mathsf{r}} \langle \lambda, \alpha_i^\vee \rangle D_i.$$
Moreover, f_λ is an eigenvector of $\mathsf{B}^{sc}\times\mathsf{B}^{sc}$ with weight $(-\mathsf{w}_0\lambda, \lambda)$.
- *The anticanonical class is given by*
$$-K_X = L(2\rho + \sum_{i=1}^{r} \alpha_i).$$

Remark 4.10. If the group G is defined over a non-algebraically closed field F, then the wonderful compactification is defined over F. There is a Galois action on the boundary components which is compatible with the Galois action on the root system.

4.2.3. *An integral computation.* Here we use the structure of the wonderful compactification to calculate a certain integral which is used in the proof of the volume estimates mentioned earlier. This computation may be of independent interest in automorphic forms. Let A be a subset of $\Delta(\mathsf{G}, \mathsf{T})$. Set
$$D_A = \bigcap_{\alpha\in A} D_\alpha,$$

and

$$D_A^0 = D_A \backslash \left(\bigcup_{A \subsetneq A'} D_{A'} \right),$$

for D_α as above. Given $g \in \mathsf{G}(F_v)$, write $g = k_1 t k_2$ with t in the positive Weyl chamber (everything split and unramified). For $\mathsf{s} = (s_\alpha)_\alpha$, set $H(\mathsf{s}, g) = \prod_\alpha |\alpha(t)|^{s_\alpha}$. Then

Theorem 4.11. *We have*

$$\int_{\mathsf{G}(F_v)} H(\mathsf{s}, g)^{-1}\, dg = \frac{1}{\#\mathsf{G}(k_v)} \sum_A \# D_A^0(k_v) \prod_{\alpha \in A} \frac{q_v - 1}{q_v^{s_\alpha - \kappa_\alpha} - 1}.$$

This is the main ingredient in the proof of the volume estimates mentioned earlier. In fact, one writes the left hand side as a sum over the elements t of the positive Weyl chamber of expressions of the form $\mathsf{H}(\mathsf{s}, g)^{-1} \mathrm{vol}(\mathsf{K}_v t \mathsf{K}_v)$. Comparing the two expressions combined with Bruhat decomposition over a finite field gives the result.

4.2.4. *Final theorem.* This is our main result:

Theorem 4.12. *Let X be the wonderful compactification of a semi-simple group G over F of adjoint type and $\mathcal{L} = (L, \|\cdot\|_v)$ an adèlically metrized line bundle such that its class $[L] \in \mathrm{Pic}(X)$ is contained in the interior of the cone of effective divisors $\Lambda_{\mathrm{eff}}(X)$. Then*

$$\mathcal{Z}(\mathcal{L}, s) := \sum_{x \in \mathsf{G}(F)} H_{\mathcal{L}}(x)^{-s} = \frac{c(\mathcal{L})}{(s - a(L))^{b(L)}} + \frac{h(s)}{(s - a(L))^{b(L)-1}}$$

and, consequently,

$$N(\mathcal{L}, B) := \#\{x \in \mathsf{G}(F) \mid H_{\mathcal{L}}(x) \le B\} \sim c(\mathcal{L}) B^{a(L)} \log(B)^{b(L)-1}$$

as $B \to \infty$. Here

- $a(L) = \inf\{a \mid a[L] + [K_X] \in \Lambda_{\mathrm{eff}}(X)\}$ *(where K_X is the canonical line bundle of X);*
- $b(L)$ *is the (maximal) codimension of the face of $\Lambda_{\mathrm{eff}}(X)$ containing $a(L)[L] + [K_X]$;*
- $c(\mathcal{L}) \in \mathbb{R}_{>0}$ *and*
- $h(s)$ *is a holomorphic function (for $\Re(s) > a(L) - \epsilon$, some $\epsilon > 0$).*

Moreover, $c(-\mathcal{K}_X)$ is the constant defined in Peyre.

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