BESSEL MODELS FOR GSp(4): BACKGROUND AND MOTIVATION

DIPENDRA PRASAD AND RAMIN TAKLOO-BIGHASH

To Steve Gelbart

Abstract. This is an informal companion to our paper *Bessel models for* GSp(4) written for people new to the subject.

Contents

1. Introduction

1.1. Saito-Tunnell. A well-known theorem of Tunnell and Saito [T, S] asserts that if π is an irreducible non-one-dimensional admissible representation of GL_2 over a local field k, and χ a quasi-character of a two-dimensional torus T/k with $\omega_\pi = \chi|_{k^\times}$, then

$$\dim_{\mathbb{C}} \mathrm{Hom}_{T(k)}(\pi, \chi) + \dim_{\mathbb{C}} \mathrm{Hom}_{T(k)}(\pi^{JL}, \chi) = 1.$$

Key words and phrases. Bessel models, Weil representation, Theta correspondence, local epsilon factors, central critical L-value.

The first author thanks the Institute for Advanced Study, as well as the University of California at San Diego, where this work was done, and gratefully acknowledges receiving support through grants to the Institute by the Friends of the Institute, and the von Neumann Fund. The second author is partially supported by a Young Investigator's Award from the NSA and a grant from the NSF. We are grateful to W.T. Gan for many suggestions; in fact, his help has been invaluable. We also thank N. Wallach for his help with the Archimedean theory, and Masaaki Furusawa for various useful discussions. Both of the authors have received warm encouragement from Steve Gelbart at various stages and would like to dedicate this work to him.

In this formula, π^{JL} is the Jacquet-Langlands transfer of π to $D^\times$, the unique quaternion algebra over k, and it is defined to be zero if the π is not discrete series. Also we define the second term above to be zero if T does not embed in $D^\times$, e.g. when T is a split torus. The Saito-Tunnell theorem also give necessary and sufficient conditions involving root numbers to determine which of the above terms contribute a non-zero functional. An interesting feature of this formula is that it relates the branching laws of representations of $\mathrm{GL}_2(k)$ to similar branching laws for an inner form $D^\times(k)$. The conjectures of Gross and Prasad [G-P1, G-P2] give conjectural generalizations of the Saito-Tunnell theorem to the context of arbitrary orthogonal groups and their orthogonal subgroups. In the Gross-Prasad conjecture it is assumed that the orthogonal group $\mathrm{SO}(V)$ and the orthogonal subgroup $\mathrm{SO}(W)$ form a relevant pair, i.e. $W^\perp$, the orthogonal complement of W in V, is a split orthogonal space. Also the two term identity above is replaced by a sum over L-packets of *pure inner forms* of the pair (V, W). This is a generalization of the Saito-Tunnell theorem as $\mathrm{PGL}_2(k)$ is a $\mathrm{SO}(2,1)(k)$ and $D^\times(k)/k^\times$ is $\mathrm{SO}(3)(k)$. In the Gross-Prasad conjecture there is a contrast between the situations where $\dim W = \dim V - 1$ and $\dim W < \dim V - 1$. In the situation where $\dim W = \dim V - 1$, the subgroup $\mathrm{SO}(W)$ will be a spherical subgroup of $\mathrm{SO}(V)$, and the general belief is that in this situation the branching laws should be more manageable. In contrast, when $\dim W < \dim V - 1$, the subgroup $\mathrm{SO}(W)$ is no longer a spherical subgroup of $\mathrm{SO}(V)$, and it will be too small to afford a multiplicity one statement of the type considered in the Saito-Tunnell theorem. Gross and Prasad's idea in this case is to "fatten up" the subgroup $\mathrm{SO}(W)$ by adding a unipotent subgroup U given by the splitting of $W^\perp$ to make the resulting subgroup $\mathrm{SO}(W) \ltimes U$ a spherical subgroup of $\mathrm{SO}(V)$. The smallest orthogonal group $\mathrm{SO}(V)$ with a non-trivial subgroup $\mathrm{SO}(W)$ with $\dim W < \dim V - 1$ will be a four dimensional space V. In this case, however, $\mathrm{SO}(V)$ can be easily described in terms of groups related to GL_2, and for that reason establishing branching laws will be trivial. The first non-trivial situation happens when $\dim V = 5$ and $\dim W = 2$. This is the situation which we consider in this paper.

1.2. **Orthogonal groups.** Let V be a finite dimensional vector space over k, and let $\mathrm{O}(V)$ be the orthogonal group of V, and $\mathrm{SO}(V)$ those orthogonal transformations T for which $\det T = 1$. The group $\mathrm{SO}(V)$ is split if and only if V is split. Similarly, if $\dim V$ is even, then $\mathrm{SO}(V)$ is quasi-split if and only if V is quasi-split.

Let $\overline{k}$ be a separable closure of k, and put $\Gamma = \mathrm{Gal}(\overline{k}/k)$. If $\underline{G}$ is an algebraic group, we denote the set $H^1(\Gamma, \underline{G}(\overline{k})$ in non-abelian Galois cohomology by $H^1(k, \underline{G})$. Then the classes in $H^1(k, \mathrm{O}(V))$ (resp. $H^1(k, \mathrm{SO}(V)))$ correspond bijectively to the isomorphism classes of quadratic spaces V' over k with $\dim V' = \dim V$ (resp. $\dim V' = \dim V$ and $d(V') = d(V)$)(cf. [G-P2], proposition 2.1).

Let V be a quadratic space over k and let W be a quadratic space which embeds as a subspace of V. Fix an embedding and define

$$W^{\perp} = \{v \in V \mid <v, w>= 0 \text{ for all } w \in W\}.$$

This is a quadratic space, whose isomorphism class depends only on the pair (V, W) and not on the chosen embedding. We have

$$V \simeq W \oplus W^{\perp}.$$

We say that the pair (V, W) is *relevant* if W embeds in V and $W^{\perp}$ is a split quadratic space of odd dimension over k. Let $\dim V = 2n+1$ and $\dim W = 2m$, and let $d_1, d_2 \in k^*/k^{*2}$. Then there is exactly one relevant pair of quasi-split spaces (V, W) with $d(V) \equiv d_1$ and $d(W) \equiv d_2$. In fact, a construction is this:

$$\begin{cases} V = (X_1 + X_1') \oplus <d_1> \\ W = (X_2 + X_2') \oplus d_1 E, \end{cases}$$

where X_1 and X_1' are dual isotropic subspaces of dimension n, X_2 and X_2' are dual isotropic subspaces of dimension $m-1$, and $E = k[x]/(x^2 - d_2)$. Also it is not hard to see that in the same situation the pairs $(U_1, 0)$ and $(U_2, <d_1>)$ are also relevant.

Let (V, W) be a relevant pair of quadratic spaces over k, and let $\mathbb{G}$ be the algebraic group $\mathrm{SO}(V) \times \mathrm{SO}(W)$. We start by defining a connected algebraic subgroup $\mathbb{H}$ of $\mathbb{G}$, and a homomorphism of algebraic groups $\ell : \mathbb{H} \to \mathbb{G}_a$. Write $V = W \oplus W^{\perp}$, $W^{\perp}$ split of dimension $2r+1$. We may write $W^{\perp} = (X + X') \oplus <a>$ with X and X' dual isotropic spaces of dimension r. Let P be the parabolic subgroup of $\mathrm{SO}(V)$ which fixes the isotropic subspace X, and let M be the Levi subgroup of P which fixes both X and X'. Then M acts on the quadratic space $Y = X + X'$. We have $P = M \ltimes N_P$, where N_P is the unipotent radical of P. The group M is isomorphic to $\mathrm{GL}(X) \times \mathrm{SO}(Y^{\perp})$, and N_P sits in an exact sequence of M-modules

$$0 \longrightarrow \wedge^2 X \longrightarrow N_P \longrightarrow X \otimes Y^{\perp} \longrightarrow 0.$$

The subspace W has codimension one in $Y^{\perp}$.

If $r = 0$, we put $\mathbb{H} = \mathrm{SO}(W)$ embedded diagonally in $\mathbb{G}$ and $\ell = 0$. If $r \geq 1$, let $X_1 \subset X$ be a hyperplane, and $\ell_1 : X \to \mathbb{G}_a$ a non-zero homomorphism which vanishes on X_1. Let $\ell_W : Y^\perp \to \mathbb{G}_a$ be a non-zero homomorphism which is zero on the hyperplane W. Consider the composite map

$$m : N_P \longrightarrow N_P^{ab} = X \otimes Y^\perp \longrightarrow_{\ell_1 \otimes \ell_W} \mathbb{G}_a.$$

The subgroup of M which fixes the map m is $\mathrm{GL}(X)_{\ell_1} \times \mathrm{SO}(W)$, where $\mathrm{GL}(X)_{\ell_1}$ is subgroup of $\mathrm{GL}(X) \simeq \mathrm{GL}_r$ contains a maximal unipotent subgroup N_r. We define the subgroup $\mathbb{H}$ of P by

$$\mathbb{H} = (N_r \times \mathrm{SO}(W)) \ltimes N_P.$$

Then $\mathbb{H}$ embeds in $\mathbb{G} = \mathrm{SO}(V) \times \mathrm{SO}(W)$, using the obvious projection from $\mathbb{H}$ onto the second factor $\mathrm{SO}(W)$. Let $\ell_r : N_r \to \mathbb{G}_a$ be a homomorphism which is non-trivial when restricted to each simple root space for GL_r in N_r. There is then a unique homomorphism

$$\ell : \mathbb{H} \longrightarrow \mathbb{G}_a$$

which is equal to $\ell_r$on the subgroup N_r, equal to zero on the subgroup $\mathrm{SO}(W)$, and equal to m on the subgroup N_P. The pair $(\mathbb{H}, \ell)$ is uniquely determined up to conjugation in the group $\mathbb{G}$ by the pair (V, W). Also if $\mathbb{G}$ is quasi-split, the group $\mathbb{H}$ is a spherical subgroup in the sense of Brion, i.e. it has an open orbit on the flag variety $F = \mathbb{G}/B$ of Borel subgroups of $\mathbb{G}$, with trivial stability subgroup.

1.3. **Gross-Prasad Conjecture.** Let G be a general reductive group. Let $\bar{k}$ be a separable algebraic closure of k, and $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Let ${}^L G$ be the Langlands dual of G. Let $W(k)'$ be the Weil-Deligne group of k. Vogan has refined the Langlands parametrization as follows. Assume G is quasi-split over k, and fix a generic character Θ_0 of the unipotent radical of a Borel subgroup. We say a group G' over k is a *pure inner form* of G if it is an inner form and the associated cohomology class in $H^1(k, G/Z)$ lifts to $H^1(k, G)$. To specify G' one must specify the lifted class in $H^1(k, G)$. The Langlands parameter $\varphi : W(k)' \to^G$ may also be a parameter for G'. If so, we let $\Pi_\varphi(G')$ be the Langlands L-packet; if not, we let $\Pi_\varphi(G')$ be the empty set. The Vogan L-packet Π_φ of φ is the disjoint union of representations of distinct groups:

$$\Pi_\varphi = \bigcup_{H^1(k,G)} \Pi_\varphi(G').$$

For the remainder of the this paragraph we assume that k is a local field with $\mathrm{char}\, k \neq 2$. Let (V, W) be a relevant pair of orthogonal spaces over k. There is no loss of generality in assuming that the spaces V and W are both quasi-split, as we will consider all pure inner forms of the group $\mathrm{SO}(V) \times \mathrm{SO}(W)$. We denote by G and H the group of k-rational points of the groups $\mathbb{G}$ and $\mathbb{H}$. Also we think of the homomorphism ℓ as a homomorphism $H \to k$ on k-rational points. Let $\psi : k \to S^1$ be a non-trivial additive character, and let

$$\Theta = \psi \circ \ell : H \to S^1$$

the corresponding homomorphism of H. The pair (H, Θ) is well-defined up to conjugacy. We will think of Θ as a one-dimensional representation of the locally compact group H. Let $\pi : G \to \mathrm{GL}(E)$ be an admissible representation of G. We define

$$\mathrm{Hom}_H(\pi, \Theta)$$

to be the vector space of all H-invariant continuous linear maps from E to Θ. Equivalently, $\mathrm{Hom}_H(\pi, \Theta)$ is the subspace of all continuous linear forms on E on which H acts by Θ^{-1}. The integer $\dim \mathrm{Hom}_H(\pi, \Theta)$ is an invariant of the isomorphism class of π. It does, however, depend on the particular quasi-split relevant pair (V, W) used to define H. Indeed the isomorphism class of $G = \mathrm{SO}(V) \times \mathrm{SO}(W)$ determines the invariants $2n + 1$, $2m$, and d_2, but the choice of d_1 is arbitrary. If (V', W') is a quasi-split relevant pair with $G' \simeq G$ but $d_1' \neq d_1$, the subgroup H' in G' will usually not be conjugate to H.

If $G_\alpha = \mathrm{SO}(V_\alpha) \times \mathrm{SO}(W_\alpha)$ is a pure inner form of G, then have

$$\begin{cases} \dim V_\alpha = \dim V & d(V_\alpha) \equiv d(V) \\ \dim W_\alpha = \dim W & d(W_\alpha) \equiv d(W). \end{cases}$$

If the pair (V_α, W_α) is relevant, we may define the spherical subgroup H_α of G_α, and a character Θ_α of H_α as above. Hence we have a well-defined vector space $\mathrm{Hom}_{H_\alpha}(\pi_\alpha, \Theta_\alpha)$ for any irreducible representation π_α of G_α which occurs in Π_φ. If G_α does not come from a relevant pair, we adopt the convention that $\mathrm{Hom}_{H_\alpha}(\pi_\alpha, \Theta_\alpha) = 0$ for all π_α, as H_α and Θ_α are not defined.

Conjecture 1 (Conjecture 6.9 of [G-P2])**.** *Let φ be a Langlands parameter for $G = \mathrm{SO}(V) \times \mathrm{SO}(W)$ and let Π_φ be its Vogan L-packet.*

(1) The complex vector space $L_\varphi = \oplus_{\pi_\alpha \in \Pi_\varphi} \mathrm{Hom}_{H_\alpha}(\pi_\alpha, \Theta_\alpha)$ has dimension ≤ 1.
(2) If φ is generic, then $\dim L_\varphi = 1$.

In fact, the second part of Conjecture 6.9 of [G-P2] states exactly which representation contributes to the one-dimensional vector space, but the statement requires more notation.

1.4. **Symplectic groups and orthogonal groups.** Let k be a field of characteristic not equal to two. The space

$$V = \{T \in \mathrm{M}_4(k) \,|\, TJ_2 \text{ is skew-symmetric and } \mathrm{tr}\, T = 0\}$$

is a five dimensional vector space over k. Here $J_n = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}$. The group $\mathrm{GSp}(4,k)$ acts on V by

$$g : T \mapsto g^{-1}Tg.$$

A symmetric non-degenerate form on V is given by

$$(T_1, T_2) = \frac{1}{4}\mathrm{tr}\, T_1T_2.$$

If we set $Q(T) = (T,T)$, then we have $Q(g.T) = Q(T)$ for all $g \in \mathrm{GSp}(4)$ and $T \in V$. In fact, we have an isomorphism $\mathrm{PGSp}(4) \simeq \mathrm{SO}(5)$.

More explicitly, the vector space V can be given in the following way:

$$V = \left\{ \begin{pmatrix} M & xJ_1 \\ yJ_1 & {}^TM \end{pmatrix} \;|\; M \in \mathrm{M}_2(k), \mathrm{tr}\, M = 0, x, y \in k \right\}.$$

Also, the quadratic form Q is given by

$$Q(T) = -\det M - xy,$$

for $T = \begin{pmatrix} M & xJ_1 \\ yJ_1 & {}^TM \end{pmatrix}$. We will denote the element T by $[M, x, y]$. The action of the group $\mathrm{GSp}(4)$ on the space V is explicitly given by the following relations:

$$\begin{pmatrix} A & \\ & \lambda\, {}^TA^{-1} \end{pmatrix}.T = [A^{-1}MA, x\lambda(\det A)^{-1}, y\lambda^{-1}\det A],$$

$$\begin{pmatrix} I_2 & S \\ & I_2 \end{pmatrix}.T = [M - ySJ_1, (x - y\det S)J_1 + MS - {}^T(MS), y]$$

where $S = {}^TS$, and

$$\begin{pmatrix} & -I_2 \\ I_2 & \end{pmatrix}.T = [{}^TM, -y, -x].$$

We will also need the following even more explicit realization. The space is given by

$$V = \left\{ T = \begin{pmatrix} t & v+w & 0 & x \\ v-w & -t & -x & 0 \\ 0 & y & t & v-w \\ -y & 0 & v+w & -t \end{pmatrix} \right\}$$

equipped with the quadratic form

$$Q(T) = t^2 + v^2 - w^2 - xy.$$

Then V has a two dimensional quadratic subspace

$$W = \left\{ T' = \begin{pmatrix} t & v & & \\ v & -t & & \\ & & t & v \\ & & v & -t \end{pmatrix} \right\}$$

equipped with

$$Q'(T') = t^2 + v^2.$$

The pair (V, W) is relevant, as $\dim V - \dim W = 3$ is odd, and

$$W^{\perp} = \left\{ T'' = \begin{pmatrix} 0 & w & 0 & x \\ -w & 0 & -x & 0 \\ 0 & y & 0 & -w \\ -y & 0 & w & 0 \end{pmatrix} \right\}$$

equipped with $Q''(T'') = -w^2 - xy$ is split. We set

$$X = \{[0_2, x, 0], x \in k\},$$
$$X' = \{[0_2, 0, y], y \in k\}.$$

Then X and X' are isotropic dual spaces. The stabilizer of X is the Siegel parabolic subgroup given by

$$P = \left\{ \begin{pmatrix} A & \\ & \lambda\, {}^T A^{-1} \end{pmatrix} \begin{pmatrix} I_2 & S \\ & I_2 \end{pmatrix} \right\}$$

The group $\mathrm{SO}(W)$ has the following realization

$$\left\{ \begin{pmatrix} A & \\ & {}^T A^{-1} \end{pmatrix}, A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \text{ with } a^2 + b^2 = 1 \right\}.$$

We note that $\mathrm{SO}(W) \subset P$. Then we consider a character of the unipotent radical N of the parabolic subgroup P. The subgroup N is abelian. The stabilizer of this character will just be a subgroup isomorphic to $\mathrm{SO}(W)$ in M. So in this case, the spherical subgroup is simply $\mathrm{SO}(W) \rtimes N_P$. We know from the discussion following corollary 8.10 of [G-P2] that the only relevant inner form of the above $G = \mathrm{SO}(V) \times \mathrm{SO}(W)$, which is $\mathrm{SO}(3,2) \times \mathrm{SO}(2,0)$, is $G' = \mathrm{SO}(1,4) \times \mathrm{SO}(0,2)$. We will later give a realization of this as a unitary group over a division algebra.

References

[B] S. Böcherer, *Bemerkungen über due Dirichletreihen von Koecher und Maaβ*, Preprint Math. Gottingensis Heft **68**(1986).

[B-S] S. Böcherer and R. Schulze-Pillot, *The Dirichlet series of Koecher and Maaßand modular forms of weight* 3/2, Math. Z. **209** (1992), 273-287.

[F] M. Furusawa, *On L-functions for* $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ *and their special values.* J. Reine Angew. Math. 438 (1993), 187–218.

[F-S] M. Furusawa, and J. A. Shalika, *On central critical values of the degree four L-function for* $\mathrm{GSp}(4)$*: The fundamental lemma*, Mem. of the AMS, vol 782, 2003.

[G-T1] W.T. Gan and S. Takeda, *On Shalika Periods and a theorem of Jacquet-Martin*, preprint.

[G-T2] W.T. Gan and S. Takeda, *The local Langlands conjecture for* $\mathrm{GSp}(4)$, preprint.

[G-J-R] D. Ginzburg, D. Jiang and S. Rallis, *On the nonvanishing of the central value of the Rankin-Selberg L-functions, II*, Automorphic representations, L-functions and applications: progress and prospects, 157-191, Ohio State Univ. Math. Research Inst. publication 11, de Guyter, Berlin (2005).

[G-P1] B. Gross and D. Prasad, *On the decomposition of a representation of* SO_n *when restricted to* SO_{n-1}. Canad. J. Math. 44 (1992), no. 5, 974–1002.

[G-P2] B. Gross and D. Prasad,*On irreducible representations of* $\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2m}$. Canad. J. Math. 46 (1994), no. 5, 930–950.

[G] J. Guo, *Uniqueness of Generalized Waldspurger model for* $\mathrm{GL}(2n)$. Pacific J. of Math, vol 180, No. 2, 1997, 273-289.

[H-K] M. Harris and S.S. Kudla, *Arithmetic automorphic forms for the nonholomorphic discrete series of* $\mathrm{GSp}(2)$. Duke Math. J. 66 (1992), no. 1, 59–121.

[H-S-T] M.Harris, D. Soudry, R. Taylor, *ℓ-adic representations associated to modular forms over quadratic imaginary fields I, lifting to* $\mathrm{GSp}_4(\mathbb{Q})$, Inventiones Math. 112 (1993), No. 2, 377-411.

[I-I] A. Ichino and T. Ikeda, *On periods of Automorphic forms on Special Orthogonal groups and the Gross-Prasad conjecture*, preprint (2007).

[J-M] H. Jacquet and K. Martin, *Shalika periods on GL(2,D) and GL(4)*, Pacific J. Math., Vol. 233, No. 2 (2007), pp. 341-370.

[L-P-T-Z] J.-S. Li, A. Paul, E.-C. Tan, and C.-B. Zhu, *The explicit duality correspondence of* $(\mathrm{Sp}(p,q), \mathrm{O}^*(2n))$, J. Funct. Anal. 200 (2003), no. 1, 71–100.

[M] G. Muic, *Intertwining operators and Jordan-Holder Series for* $Sp_4(\mathbb{R})$., preprint.

[N-PS] M. Novodvorsky and I. Piatetski-Shapiro, *Generalized Bessel models for the symplectic group of rank* 2. (Russian) Mat. Sb. (N.S.) 90(132) (1973), 246–256, 326.

[Pi-S] A. Pitale and R. Schmidt, *Integral representation for L-functions for* $\mathrm{GSp}(4) \times \mathrm{GL}(2)$, preprint.

[P] D. Prasad, *Trilinear forms for reprepresentations of* $GL(2)$*, and local epsilon factors*, Compositio Math. 75 (1990) 1-46.

[P1] D. Prasad, *Some applications of seesaw duality to branching laws*, Math. Annalen. 304 (1996), 1-20.

[P2] D. Prasad, *Some remarks on representations of a division algebra and of the Galois group of a local field*, J. Number Theory 74 (1999), no. 1, 73–97.

[P3] D. Prasad, *Relating invariant linear form and local epsilon factors via global methods*, with an appendix by H. Saito; Duke J. of Math. vol. 138, No.2, 233-261 (2007).

[P-S] D. Prasad and Rainer Schulze-Pillot, *Generalized form of a conjecture of Jacquet, and a local consequence*, to appear in the Crelle Journal (2008).

[Pr] T. Przebinda, *The duality correspondence of infinitesimal characters*, Colloq. Math. 70 (1996), no. 1, 93–102.

[Ra] S. Rallis, *Langlands' Functoriality and the Weil Representation*, American J. Math, 104 (1982), no. 3, pp. 469-515

[Ro1] B. Roberts, *The nonarchimdean theta correspondence for* GSp(2) *and* $GO(4)$, Transactions of the AMS, 351 (1999),781-811.

[Ro2] B. Roberts, *Global L-packets for* GSp(2) *and theta lifts*, Doc. Math. 6 (2001), 247–314.

[Sc] W. Scharlau, Quadratic and Hermitian Forms, Grundlehren der Mathematischen Wissenschaften, 270. Springer-Verlag, Berlin, 1985. x+421 pp.

[S] H. Saito, *On Tunnell's formula for characters of* GL(2), Compositio Math 85 (1993) 99-108.

[Sh] J. A. Shalika, *Multiplicity One for* GL(n), Ann. of Math. (2) 100 (1974), 171–193.

[TB] R. Takloo-Bighash, *Theta functions, Bessel coefficients, and Sprinor L-functions*, with an appendix by Philippe Michel, Forum Math. 19 (2007), no. 3, 487-554.

[T] J. Tunnell, *Local ϵ-factors, and characters of* GL(2), American J. Math 105 (1983)1277-1307.

[W] J.-L. Waldspurger,*Sur les valeurs de certaines fonctions L automorphes en leur centre de symtrie*. Compositio Math. 54 (1985), no. 2, 173–242.

[Wa1] N. Wallach, *Generalized Whittaker vectors for holomorphic and quatertionic representations*, Comment. Math. Helv. **78** (2003), 266-307.

[Wa2] N. Wallach, *Holomorphic continuation of generalized Jacquet integrals for degenerate principal series*, Representation Theory 10 (2006), 380–398.

[War] G. Warner *Harmonic Analysis on Semi-Simple Lie groups I*, Die Grundlehren der mathemtischen Wissenschaften in Einzeldarstellungen, Band 188, Springer-Verlag (1972).

D. P.: School of Mathematics, Tata Institute of Fundamental Research, Colaba, Mumbai-400005, INDIA
E-mail address: dprasad@math.tifr.res.in

R. T.-B.: Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, 851 S. Morgan St, Chicago, IL 60607
E-mail address: rtakloo@math.uic.edu