On Rankin-Cohen Brackets of Eigenforms

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1 Introduction

Let \( f \) and \( g \) be two modular forms of weights \( k \) and \( l \) on a congruence subgroup \( \Gamma \). The \( n \)th Rankin-Cohen bracket of \( f \) and \( g \) is defined by the formula

\[
[f, g]_n(z) = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+l-1}{r} f^{(r)}(z) g^{(s)}(z),
\]

where as in [8] we denote

\[
f^{(r)}(z) = \left( \frac{1}{2\pi i} \frac{d}{dz} \right)^r f(z) = \left( \frac{d}{dq} \right)^r f(z).
\]

for \( q = e^{2\pi i z} \).

For example, we have

\[
[f, g]_0 = fg,
\]
\[
[f, g]_1 = kf'g - lf'g,
\]
\[ \vdots \]

Differential operators on modular forms were studied in [5] and the Rankin-Cohen brackets were introduced by H. Cohen [1] and further studied by D. Zagier [7, 8]. Here we use the normalization used in [8] to guarantee that for all \( n \) we have \( [f, g]_n \in \mathbb{Z}[[q]] \) when \( f, g \in \mathbb{Z}[[q]] \).

The purpose of this note is to prove the following theorem:
Theorem 1.1 There are only a finite number of triples \((F, G, n)\) with the property that \(F\) and \(G\) are normalized eigenforms and \([F, G]_n\) is again an eigenform. The following describes all the possibilities:

1. We have \([E_4, E_6]_0 = E_{10}\) and \([E_4, E_{10}]_0 = [E_6, E_8]_0 = E_{14}\).

2. If \(k, l \in \{4, 6, 8, 10, 14\}\) and \(n \geq 1\) with \(k+l+2n \in \{12, 16, 18, 20, 22, 26\}\), then
   \[
   [E_k, E_l]_n = c_n(k, l) \Delta_{k+l+2n}
   \]
   where
   \[
   c_n(k, l) = -\frac{2l}{B_l} \binom{n + l - 1}{n} + (-1)^{n+1} \frac{2k}{B_k} \binom{n + k - 1}{n}.
   \]

3. If \(k \in \{4, 6, 8, 10, 14\}\) and \(n \geq 0\) with \(l, k+l+2n \in \{12, 16, 18, 20, 22, 26\}\), then
   \[
   [E_k, \Delta_l]_n = c_n(l) \Delta_{k+l+2n}
   \]
   where
   \[
   c_n(l) = \binom{n + l - 1}{n}.
   \]

This theorem generalizes the results of Duke [2] and Ghate [3]. Their result is included in the \(n = 0\) case of our theorem. The results of Zagier [7] allow the argument of Ghate to go through with slight modifications. We expect that a similar argument, along the lines of [4], would work for non-trivial level.

This paper is organized as follows. In the second section we recall two theorems from [8] that will be used in the proof of the main theorem. The proof of the main theorem and also the list of all possible cases for the level 1 case is included in the third section.

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2 Connection to L-functions

Let $\Gamma$ be a congruence subgroup, $k$ an integer, and $\mathcal{H}$ the upper-half-plane. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \mathcal{H}$ set

$$J_k(\gamma, z) = (cz + d)^{-k}.$$ 

If $k > 2$ we have the Eisenstein series

$$E^\Gamma_k(z) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} J_k(\gamma, z),$$

where $\Gamma_\infty = \{ \gamma \in \Gamma \mid \gamma \cdot i\infty = i\infty \}$. When $\Gamma = \text{SL}_2(\mathbb{Z})$ we will drop the superscript $\Gamma$. In this case, $E_k$ has an explicit Fourier expansion given by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)e^{2\pi inz}.$$ 

With appropriate normalization, Proposition 6 of [7] now reads as follows:

**Theorem 2.1** Let $k_1, k_2, k$ and $n$ be integers satisfying $k_2 \geq k_1 + 2 > 2$ and $k = k_1 + k_2 + 2n$. Let $f(z) = \sum_{j=1}^{\infty} a_j e^{2\pi i jz/w}$ and $g(z) = \sum_{j=0}^{\infty} b_j e^{2\pi i jz/w}$ be two modular forms for $\Gamma$ of respective weights $k$ and $k_1$. We have

$$\langle f, [E^\Gamma_{k_2}]_n \rangle = \frac{\Gamma(k - 1)\Gamma(k_2 + n)w^{k-n}}{(4\pi)^{k_{1/2} - 1}n!\Gamma(k_2)} \sum_{j=1}^{\infty} \frac{a_j b_j}{j^{k_1 + k_2 + n - 1}}.$$ 

Where $\langle \cdot, \cdot \rangle$ is the usual Petersson inner product

$$\langle f, g \rangle = \int_{\text{SL}_2(\mathbb{Z}) \setminus \mathcal{H}} f(z)\overline{g(z)} \frac{dx \, dy}{y^2}$$

and $w = [\Gamma'_\infty : \Gamma_\infty]$ for $\Gamma' = \text{SL}_2(\mathbb{Z})$. We also have the following theorem for the case where $\Gamma = \text{SL}_2(\mathbb{Z})$ and $g = E_{k_1}$:

**Theorem 2.2** Let $k_1, k_2 \geq 4$ be even integers with $k_1 \neq k_2$. Let $n$ and $f$ be as above. Then

$$\langle f, [E_{k_1}, E_{k_2}]_n \rangle = (-1)^{k_2/2} \frac{2k_1 2k_2}{B_{k_1} B_{k_2}} \frac{\Gamma(k - 1)}{n!^{2^k - 1} \Gamma(k - n - 1)} \times L^*(f, k - n - 1) L^*(f, k_2 + n).$$

Where $L^*(f, s) = (2\pi)^{-s}\Gamma(s)L(f, s)$ and $L(f, s)$ is the standard $L$-function of $f$. This theorem is the corollary to Proposition 6 of [7]. We note that when $n$ is even the identity is valid for $k_1 = k_2$. See also [6].
3 Proof of the theorem

We now prove the theorem. Suppose the triple \((F, G, n)\) satisfies the conditions of the theorem. Let the weights of \(F\) and \(G\) be \(u, v\) respectively. Set \(k = u + v + 2n\). Let \(H = [F,G]_n\) and

\[
F(z) = \sum_{j=0}^{\infty} A_j e^{2\pi i j z}
\]

\[
G(z) = \sum_{j=0}^{\infty} B_j e^{2\pi i j z}
\]

\[
H(z) = \sum_{j=0}^{\infty} C_j e^{2\pi i j z}.
\]

It is then clear from the definition that

\[
C_j = \sum_{m+l=j} A_m B_l \sum_{r+s=n} m^r t^s (-1)^r \binom{n+u-1}{r} \binom{n+v-1}{s}.
\]

In particular, if \(A_0 = B_0 = 0\) then \(C_0 = C_1 = 0\). The assumption that \(H\) is an eigenform then implies that \(H \equiv 0\). Hence at least one of \(F\) and \(G\) must not be a cusp form. There are two cases to be considered:

First Case. In this case \(F\) is a cusp form and \(G = E_v\). Then \(H\) will be an eigenform in \(S_k(SL_2(\mathbb{Z}))\). Suppose \(\dim S_k(SL_2(\mathbb{Z})) \neq 1\). Then there will be a cusp form \(U \in S_k(SL_2(\mathbb{Z}))\) with the property that \(\langle U, H \rangle = 0\). Let \(U(z) = \sum_{j=1}^{\infty} D_j e^{2\pi i j z}\). Then Theorem 2.1 implies

\[
\sum_{j=1}^{\infty} \frac{D_j A_j}{j^{u+v+n-1}} = 0.
\]

However, since \(U\) and \(F\) are cuspidal, we know the series \(\sum_{j=1}^{\infty} \frac{D_j A_j}{j^s}\) has an Euler product which is absolutely convergent for \(\Re(s) > u + \frac{v}{2} + n\). Since \(v > 2\) we have \(u + v + n - 1 > u + \frac{v}{2} + n\). Consequently, there is not a cusp form \(U\) with the above property, and so \(H\) can be an eigenform only when \(\dim S_k(SL_2(\mathbb{Z})) = 1\).
Second Case. Here $F = E_u$ and $G = E_v$. If $n = 0$, the result is already contained in $[3]$. So we assume $n > 0$. First we consider $u \neq v$. In this case it is easily seen that the function $H$ must be a cusp form of weight $u + v + 2n$. If $\dim S_k(\text{SL}_2(\mathbb{Z})) \neq 1$ then we can choose an eigenform $U \in S_k(\text{SL}_2(\mathbb{Z}))$ with the property that $\langle U, [E_u, E_v]_n \rangle = 0$. This combined with Theorem 2.2 implies

$$L^*(U, u + v + n - 1)L^*(U, v + n) = 0.$$ 

Now, it is well-known that $L^*(U, s)$ does not vanish for $\Re(s) > \frac{u + v + 2n + 1}{2}$, this region being the domain of absolute convergence of the Euler product. Since $s = u + v + n - 1$ is in the domain of the absolute convergence, the first term does not vanish. For the second term, if $v + n$ is not in the domain of absolute convergence then we use the functional equation to obtain $L^*(U, k - v - n) = (-1)^{k/2}L^*(U, u + n)$. Since $u \neq v$ are even numbers then $u + n$ must belong to the domain of absolute convergence.

Now we consider the case where $u = v$. Since $\langle f, g \rangle_n = (-1)^n[g, f]_n$ then for $n$ odd we must have $[E_u, E_u]_n = 0$. Hence we assume that $n$ is even. We separate the $n = 2$ case as a lemma:

**Lemma 3.1** The cusp form $[E_u, E_u]_2$ is not an eigenform unless $u \in \{4, 6, 8\}$.

**Proof.** For this we proceed with an explicit calculation of the Fourier expansion of $[E_u, E_u]_2$. We have

$$E_u(z) = 1 - \frac{2u}{B_u} \sum_{n=1}^{\infty} \sigma_{u-1}(n)q^n,$$

$$E'_u(z) = -\frac{2u}{B_u} \sum_{n=1}^{\infty} n \sigma_{u-1}(n)q^n,$$

$$E''_u(z) = -\frac{2u}{B_u} \sum_{n=1}^{\infty} n^2 \sigma_{u-1}(n)q^n.$$
Then a straightforward calculation shows that the function given by

\[ f(z) = \frac{-B_u}{2u^2(u+1)}[E_u, E_u]_2(z) \]

\[ = q + \sum_{N=2}^{\infty} \left\{ \frac{N^2 \sigma_{u-1}(N)}{\frac{2}{B_u}} \sum_{m+n=N, m,n \geq 1} n\sigma_{u-1}(n)\sigma_{u-1}(m) \left[ u(m-n)+m \right] \right\} q^N, \]

is the normalized form associated with \([E_u, E_u]_2\). Denote the \(n\)th Fourier coefficient of \(f\) by \(E(n)\). If \(f\) is a normalized eigenform then

\[ E(4) = E(2)^2 - 2^{2u+3}. \quad (3.2) \]

From equation 3.1 we have

\[ E(2) = 4 + 2^{u+1} + \frac{2}{B_u} \]

and

\[ E(4) = 16\sigma_{u-1}(4) + \frac{2}{B_u} \left( \sigma_{u-1}(3)(-4u+6) + 4\sigma_{u-1}(2)^2 \right). \]

Then 3.2 translates to an equation for \(X = 2/B_u\):

\[ X^2 + \left( 8\sigma_{u-1}(2) - \sigma_{u-1}(3)(-4u+6) - 4\sigma_{u-1}(2)^2 \right) X \]
\[ + \left( 16\sigma_{u-1}(2)^2 - 2^{2u+3} - 16\sigma_{u-1}(4) \right) = 0. \quad (3.3) \]

In order for this equation to have a rational solution, the discriminant \(D\) must be a perfect square of an integer. The discriminant of 3.3 is

\[ D = \left( 4\sigma_{u-1}(2)^2 + \sigma_{u-1}(3)(-4u+6) - 8\sigma_{u-1}(2)^2 \right)^2 \]
\[ - 4 \left( 16\sigma_{u-1}(2)^2 - 2^{2u+3} - 16\sigma_{u-1}(4) \right) \]
\[ = \left( 4^u + 3^{u-1}(-4u+6) + (2-4u) \right)^2 + 2^{2u+5} - 2^{u+5}. \]

Let \(Y = 4^u + 3^{u-1}(-4u+6) + (2-4u)\), and then \(D = Y^2 + 2^{2u+5} - 2^{u+5}\). As \(2^7u > 2^8\) for \(u \geq 4\) then it easily follows that \(2^{2u+5} - 2^{u+5} > 2^5Y + 2^8\). Thus \(D > (Y+16)^2\) for \(u \geq 4\).

As \((4/3)^u > 4 \cdot 17u/3c\) for some \(c < 1\) and \(u \geq 22\) then it follows that

\[ 17 \cdot 2^{2u+1} > 2^{2u+5} + 2^{2u+1}c > 2^{2u+5} + 2^3 \cdot 17u3^{u-1}. \]
This gives $2 \cdot 17Y + 17^2 > 2^{u+5} - 2^{u+5}$ and so $(Y + 17)^2 > D$. So for $u \geq 22$ we have

$$(Y + 17)^2 > D > (Y + 16)^2$$

and so $D$ cannot be a square of an integer. For $u \in \{10, 12, 14, 16, 18, 20\}$ direct calculation shows that the discriminant is not a square. It follows that equation 3.3 has no rational solution and so $f$ cannot be an eigenform, for $u \geq 10$.

If $u \in \{4, 6, 8\}$ then the respective $f$ is in fact an eigenform as for such $u$ the space $S_{2u+4}(SL_2(\mathbb{Z}))$ is one-dimensional. □

We have the following interesting non-vanishing result:

**Corollary 3.2** Suppose $k > 20$ and $k \equiv 0 \pmod{4}$. Then there are two eigenforms $f, g \in S_k(SL_2(\mathbb{Z}))$ with $L^*(f, \frac{k}{2}) \neq 0$ and $L^*(g, \frac{k}{2}) \neq 0$.

**Proof.** Let $u = \frac{1}{2}k - 2$. By Lemma 3.1 we know that $[E_u, E_u]_2$ is not an eigenform. This implies that there must exist at least two eigenforms $f$ and $g$ in $S_k$ such that $\langle f, [E_u, E_u]_2 \rangle \neq 0$ and $\langle g, [E_u, E_u]_2 \rangle \neq 0$. An application of Theorem 2.2 finishes the proof. □

We can now treat $[E_u, E_u]_n$ for general even $n$. By Theorem 2.2, $[E_u, E_u]_n$ will have non-zero projection of an eigenform $f \in S_{2u+2n}(SL_2(\mathbb{Z}))$ if and only if $L^*(f, u+n) \neq 0$. By the corollary, if $2u+2n > 20$ then there are at least two eigenforms $f$ and $g$ with this property, implying that $[E_u, E_u]_n$ cannot be an eigenform. The numbers $c_n(k, l)$ and $c_n(l)$ are easily calculated from 3.1.

**References**


