On Rankin-Cohen Brackets of Eigenforms

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1 Introduction

Let f and g be two modular forms of weights k and l on a congruence subgroup Γ. The n^{th} Rankin-Cohen bracket of f and g is defined by the formula

$$[f,g]_n(z) = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+l-1}{r} f^{(r)}(z) g^{(s)}(z),$$

where as in [8] we denote

$$\begin{aligned} f^{(r)}(z) &= \left(\frac{1}{2\pi i} \frac{d}{dz} \right)^r f(z) \\ &= \left(q \frac{d}{dq} \right)^r f(z). \end{aligned}$$

for $q = e^{2\pi i z}$.

For example, we have

$$\begin{aligned} [f,g]_0 &= fg, \\ [f,g]_1 &= kfg' - lf'g, \\ &\vdots \end{aligned}$$

Differential operators on modular forms were studied in [5] and the Rankin-Cohen brackets were introduced by H. Cohen [1] and further studied by D. Zagier [7, 8]. Here we use the normalization used in [8] to guarantee that for all n we have $[f,g]_n \in \mathbb{Z}[[q]]$ when $f, g \in \mathbb{Z}[[q]]$.

The purpose of this note is to prove the following theorem:

Theorem 1.1 *There are only a finite number of triples* (F, G, n) *with the property that* F *and* G *are normalized eigenforms and* $[F, G]_n$ *is again an eigenform. The following describes all the possibilities:*

1. *We have* $[E_4, E_6]_0 = E_{10}$ *and* $[E_4, E_{10}]_0 = [E_6, E_8]_0 = E_{14}$.

2. *If* $k, l \in \{4, 6, 8, 10, 14\}$ *and* $n \geq 1$ *with* $k+l+2n \in \{12, 16, 18, 20, 22, 26\}$, *then*
$$[E_k, E_l]_n = c_n(k, l)\Delta_{k+l+2n}$$
where
$$c_n(k, l) = -\frac{2l}{B_l}\binom{n+l-1}{n} + (-1)^{n+1}\frac{2k}{B_k}\binom{n+k-1}{n}.$$

3. *If* $k \in \{4, 6, 8, 10, 14\}$ *and* $n \geq 0$ *with* $l, k+l+2n \in \{12, 16, 18, 20, 22, 26\}$, *then*
$$[E_k, \Delta_l]_n = c_n(l)\Delta_{k+l+2n}$$
where
$$c_n(l) = \binom{n+l-1}{n}.$$

This theorem generalizes the results of Duke [2] and Ghate [3]. Their result is included in the $n = 0$ case of our theorem. The results of Zagier [7] allow the argument of Ghate to go through with slight modifications. We expect that a similar argument, along the lines of [4], would work for non-trivial level.

This paper is organized as follows. In the second section we recall two theorems from [8] that will be used in the proof of the main theorem. The proof of the main theorem and also the list of all possible cases for the level 1 case is included in the third section.

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2 Connection to L-functions

Let Γ be a congruence subgroup, k an integer, and $\mathfrak{H}$ the upper-half-plane. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \mathfrak{H}$ set

$$J_k(\gamma, z) = (cz + d)^{-k}.$$

If $k > 2$ we have the Eisenstein series

$$E_k^{\Gamma}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} J_k(\gamma, z),$$

where $\Gamma_\infty = \{\gamma \in \Gamma \,|\, \gamma.i\infty = i\infty\}$. When $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ we will drop the superscript Γ. In this case, E_k has an explicit Fourier expansion given by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z}.$$

With appropriate normalization, Proposition 6 of [7] now reads as follows:

Theorem 2.1 *Let k_1, k_2, k and n be integers satisfying $k_2 \geq k_1 + 2 > 2$ and $k = k_1 + k_2 + 2n$. Let $f(z) = \sum_{j=1}^{\infty} a_j e^{2\pi i j z / w}$ and $g(z) = \sum_{j=0}^{\infty} b_j e^{2\pi i j z / w}$ be two modular forms for Γ of respective weights k and k_1. We have*

$$\langle f, [g, E_{k_2}^{\Gamma}]_n \rangle = \frac{\Gamma(k-1)\Gamma(k_2+n) w^{k-n}}{(4\pi)^{k-1} n! \Gamma(k_2)} \sum_{j=1}^{\infty} \frac{a_j \overline{b_j}}{j^{k_1+k_2+n-1}}.$$

Where $\langle \cdot, \cdot \rangle$ is the usual Petersson inner product

$$\langle f, g \rangle = \int_{SL_2(\mathbb{Z}) \backslash \mathfrak{H}} f(z) \overline{g(z)} \, \frac{dx\, dy}{y^2}$$

and $w = [\Gamma'_\infty : \Gamma_\infty]$ for $\Gamma' = SL_2(\mathbb{Z})$. We also have the following theorem for the case where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $g = E_{k_1}$:

Theorem 2.2 *Let $k_1, k_2 \geq 4$ be even integers with $k_1 \neq k_2$. Let n and f be as above. Then*

$$\langle f, [E_{k_1}, E_{k_2}]_n \rangle = (-1)^{k_2/2} \frac{2k_1}{B_{k_1}} \frac{2k_2}{B_{k_2}} \frac{\Gamma(k-1)}{n! 2^{k-1} \Gamma(k-n-1)} \times L^*(f, k-n-1) L^*(f, k_2+n).$$

Where $L^*(f, s) = (2\pi)^{-s} \Gamma(s) L(f, s)$ and $L(f, s)$ is the standard L-function of f. This theorem is the corollary to Proposition 6 of [7]. We note that when n is even the identity is valid for $k_1 = k_2$. See also [6].

3 Proof of the theorem

We now prove the theorem. Suppose the triple (F, G, n) satisfies the conditions of the theorem. Let the weights of F and G be u, v respectively. Set $k = u + v + 2n$. Let $H = [F, G]_n$ and

$$F(z) = \sum_{j=0}^{\infty} A_j e^{2\pi i j z}$$
$$G(z) = \sum_{j=0}^{\infty} B_j e^{2\pi i j z}$$
$$H(z) = \sum_{j=0}^{\infty} C_j e^{2\pi i j z}.$$

It is then clear from the definition that

$$C_j = \sum_{m+t=j} A_m B_t \sum_{r+s=n} m^r t^s (-1)^r \binom{n+u-1}{r}\binom{n+v-1}{s}. \tag{3.1}$$

In particular, if $A_0 = B_0 = 0$ then $C_0 = C_1 = 0$. The assumption that H is an eigenform then implies that $H \equiv 0$. Hence at least one of F and G must not be a cusp form. There are two cases to be considered:

First Case. In this case F is a cusp form and $G = E_v$. Then H will be an eigenform in $S_k(\mathrm{SL}_2(\mathbb{Z}))$. Suppose $\dim S_k(\mathrm{SL}_2(\mathbb{Z})) \neq 1$. Then there will be a cusp form $U \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ with the property that $\langle U, H\rangle = 0$. Let $U(z) = \sum_{j=1}^{\infty} D_j e^{2\pi i j z}$. Then Theorem 2.1 implies

$$\sum_{j=1}^{\infty} \frac{D_j \overline{A_j}}{j^{u+v+n-1}} = 0.$$

However, since U and F are cuspidal, we know the series $\sum_{j=1}^{\infty} \frac{D_j \overline{A_j}}{j^s}$ has an Euler product which is absolutely convergent for $\Re(s) > u + \frac{v}{2} + n$. Since $v > 2$ we have $u + v + n - 1 > u + \frac{v}{2} + n$. Consequently, there is not a cusp form U with the above property, and so H can be an eigenform only when $\dim S_k(\mathrm{SL}_2(\mathbb{Z})) = 1$.

Second Case. Here $F = E_u$ and $G = E_v$. If $n = 0$, the result is already contained in [3]. So we assume $n > 0$. First we consider $u \neq v$. In this case it is easily seen that the function H must be a cusp form of weight $u+v+2n$. If $\dim S_k(\mathrm{SL}_2(\mathbb{Z})) \neq 1$ then we can choose an eigenform $U \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ with the property that $\langle U, [E_u, E_v]_n \rangle = 0$. This combined with Theorem 2.2 implies

$$L^*(U, u+v+n-1)L^*(U, v+n) = 0.$$

Now, it is well-known that $L^*(U, s)$ does not vanish for $\Re(s) > \frac{u+v+2n+1}{2}$, this region being the domain of absolute convergence of the Euler product. Since $s = u+v+n-1$ is in the domain of the absolute convergence, the first term does not vanish. For the second term, if $v+n$ is not in the domain of absolute convergence then we use the functional equation to obtain $L^*(U, k-v-n) = (-1)^{k/2} L^*(U, u+n)$. Since $u \neq v$ are even numbers then $u+n$ must belong to the domain of absolute convergence.

Now we consider the case where $u = v$. Since

$$[f, g]_n = (-1)^n [g, f]_n$$

then for n odd we must have $[E_u, E_u]_n = 0$. Hence we assume that n is even. We separate the $n = 2$ case as a lemma:

Lemma 3.1 *The cusp form $[E_u, E_u]_2$ is not an eigenform unless $u \in \{4, 6, 8\}$.*

Proof. For this we proceed with an explicit calculation of the Fourier expansion of $[E_u, E_u]_2$. We have

$$E_u(z) = 1 - \frac{2u}{B_u} \sum_{n=1}^{\infty} \sigma_{u-1}(n) q^n,$$

$$E'_u(z) = -\frac{2u}{B_u} \sum_{n=1}^{\infty} n\sigma_{u-1}(n) q^n,$$

$$E''_u(z) = -\frac{2u}{B_u} \sum_{n=1}^{\infty} n^2\sigma_{u-1}(n) q^n.$$

Then a straightforward calculation shows that the function given by

$$
\begin{aligned}
f(z) &= \frac{-B_u}{2u^2(u+1)}[E_u, E_u]_2(z) \\
&= q + \sum_{N=2}^{\infty} \left\{ N^2\sigma_{u-1}(N) + \frac{2}{B_u} \sum_{\substack{m+n=N \\ m,n\geq 1}} n\sigma_{u-1}(n)\sigma_{u-1}(m)\,[u(m-n)+m] \right\} q^N,
\end{aligned}
$$

is the normalized form associated with $[E_u, E_u]_2$. Denote the n^{th} Fourier coefficient of f by $E(n)$. If f is a normalized eigenform then

$$E(4) = E(2)^2 - 2^{2u+3}. \tag{3.2}$$

From equation 3.1 we have

$$E(2) = 4 + 2^{u+1} + \frac{2}{B_u}$$

and

$$E(4) = 16\sigma_{u-1}(4) + \frac{2}{B_u}\left(\sigma_{u-1}(3)(-4u+6) + 4\sigma_{u-1}(2)^2\right).$$

Then 3.2 translates to an equation for $X = 2/B_u$:

$$
\begin{aligned}
X^2 + \left(8\sigma_{u-1}(2) - \sigma_{u-1}(3)(-4u+6) - 4\sigma_{u-1}(2)^2\right) X& \\
+ \left(16\sigma_{u-1}(2)^2 - 2^{2u+3} - 16\sigma_{u-1}(4)\right) &= 0.
\end{aligned} \tag{3.3}
$$

In order for this equation to have a rational solution, the discriminant D must be a perfect square of an integer. The discriminant of 3.3 is

$$
\begin{aligned}
D &= \left(4\sigma_{u-1}(2)^2 + \sigma_{u-1}(3)(-4u+6) - 8\sigma_{u-1}(2)\right)^2 \\
&\qquad - 4\left(16\sigma_{u-1}(2)^2 - 2^{2u+3} - 16\sigma_{u-1}(4)\right) \\
&= \left(4^u + 3^{u-1}(-4u+6) + (2-4u)\right)^2 + 2^{2u+5} - 2^{u+5}.
\end{aligned}
$$

Let $Y = 4^u + 3^{u-1}(-4u+6) + (2-4u)$, and then $D = Y^2 + 2^{2u+5} - 2^{u+5}$. As $2^7u > 2^8$ for $u \geq 4$ then it easily follows that $2^{2u+5} - 2^{u+5} > 2^5Y + 2^8$. Thus $D > (Y+16)^2$ for $u \geq 4$.

As $(4/3)^u > 4 \cdot 17u/3c$ for some $c < 1$ and $u \geq 22$ then it follows that

$$17 \cdot 2^{2u+1} > 2^{2u+5} + 2^{2u+1}c > 2^{2u+5} + 2^3 \cdot 17u3^{u-1}.$$

This gives $2 \cdot 17Y + 17^2 > 2^{2u+5} - 2^{u+5}$ and so $(Y+17)^2 > D$. So for $u \geq 22$ we have

$$(Y+17)^2 > D > (Y+16)^2$$

and so D cannot be a square of an integer. For $u \in \{10, 12, 14, 16, 18, 20\}$ direct calculation shows that the discriminant is not a square. It follows that equation 3.3 has no rational solution and so f cannot be an eigenform, for $u \geq 10$.

If $u \in \{4, 6, 8\}$ then the respective f is in fact an eigenform as for such u the space $S_{2u+4}(\mathrm{SL}_2(\mathbb{Z}))$ is one-dimensional. □

We have the following interesting non-vanishing result:

Corollary 3.2 *Suppose $k > 20$ and $k \equiv 0 \pmod 4$. Then there are two eigenforms $f, g \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ with $L^*(f, \frac{k}{2}) \neq 0$ and $L^*(g, \frac{k}{2}) \neq 0$.*

Proof. Let $u = \frac{1}{2}k - 2$. By Lemma 3.1 we know that $[E_u, E_u]_2$ is not an eigenform. This implies that there must exist at least two eigenforms f and g in S_k such that $\langle f, [E_u, E_u]_2 \rangle \neq 0$ and $\langle g, [E_u, E_u]_2 \rangle \neq 0$. An application of Theorem 2.2 finishes the proof. □

We can now treat $[E_u, E_u]_n$ for general even n. By Theorem 2.2, $[E_u, E_u]_n$ will have non-zero projection of an eigenform $f \in S_{2u+2n}(\mathrm{SL}_2(\mathbb{Z}))$ if and only if $L^*(f, u+n) \neq 0$. By the corollary, if $2u + 2n > 20$ then there are at least two eigenforms f and g with this property, implying that $[E_u, E_u]_n$ cannot be an eigenform. The numbers $c_n(k, l)$ and $c_n(l)$ are easily calculated from 3.1.

References

[1] H. Cohen, *Sums involving the values at negative integers of L functions of quadratic characters*, Math. Ann. **217** (1975), 271–285.

[2] W. Duke, *When is the product of two eigenforms an eigenform?* in Elementary and Analytic Number Theory, ed. by Kálmán Győry, Henryk Iwaniec, and Jerzy Urbanowicz, Number theory in progress Vol. **2**, de Gruyter, Berlin, 1999, 737–741.

[3] E. Ghate, *On monomial relations between Eisenstein series*, J. Ramanujan Math. Soc. **15** (2000), 71–79.

[4] E. Ghate, *On products of eigenforms*, Acta Arith. **102** (2002), 27–44.

[5] R. A. Rankin, *The construction of automorphic forms from the derivatives of a given form*, J. Indian Math. Soc. **20** (1956), 103–116.

[6] G. Shimura, *The special values of the zeta functions associated with cusp forms*, Comm. Pure Appl. Math. **29** (1976), 783–804.

[7] D. Zagier, *Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields*, in Modular Functions of One Variable IV, Lect. Notes in Math. **627**, Springer, Berlin, 1977, 105–169.

[8] D. Zagier, *Modular forms and differential operators*, Proc. Indian Acad. Sci. Math. Sci. **104**, K.G. Ramanuthan memorial issue, (1994), 57–75.