L-FUNCTIONS FOR THE p-ADIC GROUP $GSp(4)$

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Abstract. In this paper we compute the local L-factors for Novodvorsky integrals for all generic representations of the group $GSp(4)$ over a nonarchemidean local field.

1. Introduction. In this paper we will study the p-adic theory of Novodvorsky integrals for the similitude symplectic group $GSp(4)$, and will present the computation of the nonarchemidean L-factors given by these integrals for all generic representations of the group. These integrals which were introduced by M. Novodvorsky in the Corvallis conference [17] serve as one of the few available integral representations for the Spin L-function of $GSp(4)$. Unfortunately, Novodvorsky's paper is somewhat sketchy, and skips the proofs. Some of the details missing in Novodvorsky's original paper have been reproduced in Daniel Bump's survey article [1]. The reader is advised to consult Bump's paper for local unramified computations, the proof of the Euler product decomposition of the global integral and other interesting results. David Soudry has generalized the integrals considered here to orthogonal groups of arbitrary odd degree. The local theory of Soudry's integrals appear in his *Memoire* paper [27]. Our motivation for this work comes from the work of Masaaki Furusawa and Joseph Shalika on the special values of L-functions of $GSp(4)$ using the relative trace formula [8], where they need precise information about the local behavior of Novodvorsky integrals.

We now describe our method. Following Godement [9], the main idea is to determine the germ expansions of Whittaker functions when restricted to the maximal torus in the Siegel parabolic subgroup. The functionals that appear as coefficients in these germ expansions are easily seen to belong to the dual of a certain twisted Jacquet module. We are most interested in the eigenspace decomposition of this dual module under the action of the Siegel torus. This decomposition essentially determines the germ expansion and naturally the local L-factor. To do explicit computations one uses the fact that germ expansions must be invariant under intertwining operators by the multiplicity one theorem of Shalika [25], up to certain constants, i.e. local coefficients. These local coefficients have been extensively studied by Shahidi in [22], [23], [15]. Away from the poles of intertwining operators this simple argument determines the germ expansion. To extend

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the results to singular cases, one needs a careful description of the eigenspaces mentioned above. Then one uses the analytic continuation of all the ingredients. The problem is slightly harder in these cases because the dual module is not semi-simple. For reducible representations one uses the classification theorems of Sally-Tadic [20] and Shahidi [23]. The final results appear as Theorems 4.1 and 5.1. The above method can be used for Whittaker functions of other p-adic groups, as well as other unique models such as Bessel models. For this though a study of local coefficients associated to these models is indispensable [6].

There are other problems that are closely related to the subject matter of this work. In this article we have computed the local gcd for each generic representation of the similitude symplectic group: one would naturally want to determine explicitly a vector in the space of the given generic representation that gives this gcd. This is particularly important in trace formula applications. Another problem that is yet to be solved is the problem of performing the archemidean computations. It is also essential to compute the ϵ-factor defined by Novodvorsky's integrals. We hope to address these issues in a future work.

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Notation. Throughout this paper F will denote a nonarchemidean local field of residue characteristic p and residue degree q, and $\mathcal{O}$ will denote its ring of integers. Also ψ will be a fixed additive character of F, trivial on $\mathcal{O}$, and nontrivial on every larger ideal. We will use the notation of [3] when working with arbitrary reductive groups. The group $GSp(4)$ over an arbitrary field K is the group of all matrices $g \in GL_4(K)$ that satisfy the following equation for some scalar $\nu(g) \in K$:

$$ {}^t g J g = \nu(g) J, $$

where $J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$. It is standard that $G = GSp(4)$ is a reductive group. The map $(F^\times)^3 \longrightarrow G$, given by

$$ (a, b, \lambda) \mapsto \operatorname{diag}(a, b, \lambda b^{-1}, \lambda a^{-1}) $$

gives a parameterization of the maximal torus T in G. Let χ_1, χ_2 and χ_3 be quasi-characters of $F^\times$. We define the character $\chi_1 \otimes \chi_2 \otimes \chi_3$ of T by

$$(\chi_1 \otimes \chi_2 \otimes \chi_3)(\mathrm{diag}(a, b, \lambda b^{-1}, \lambda a^{-1})) = \chi_1(a)\chi_2(b)\chi_3(\lambda).$$

The Weyl group is a dihedral group of order eight. It has generators

$$w_1 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix}.$$

It will also be useful to know the realization of the Weyl group as a group of permutations on four letters. Explicitly, we have the following permutations:

$$\{\text{identity}, (1\,2)(3\,4), (2\,3), (1\,4), (1\,2\,4\,3), (1\,3\,4\,2), (1\,3)(2\,4), (1\,4)(2\,3)\}.$$

w_l will always denote the longest Weyl element. We have three standard parabolic subgroups: the Borel subgroup B, the Siegel subgroup P, and the Klingen subgroup Q with the following Levi decompositions:

$$B = \left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & b^{-1}\lambda & \\ & & & a^{-1}\lambda \end{pmatrix} \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & r & s \\ & 1 & t & r \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\},$$

$$P = \left\{ \begin{pmatrix} g & \\ & \alpha {}^\tau g^{-1} \end{pmatrix} \begin{pmatrix} 1 & & r & s \\ & 1 & t & r \\ & & 1 & \\ & & & 1 \end{pmatrix} \mid g \in GL(2) \right\}.$$

Here ${}^\tau g$ is the transposed matrix with respect to the second diagonal, and finally

$$Q = \left\{ \begin{pmatrix} \alpha & & \\ & g & \\ & & \alpha^{-1} \det g \end{pmatrix} \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & r & s \\ & 1 & & r \\ & & 1 & \\ & & & 1 \end{pmatrix} \mid g \in GSp(2) \right\}.$$

For these parabolic subgroups, the modular functions are explicitly given by the following:

$$\delta_B(p) = |a^4 b^2 \lambda^{-3}|,$$
$$\delta_P(p) = |(\det g)^3 \lambda^{-3}|,$$

and

$$\delta_Q(p) = |\alpha^4 (\det g)^{-2}|,$$

for typical elements as above. We will use the notation $\chi_1 \times \chi_2 \rtimes \chi_3$ for the parabolically induced representation from the minimal parabolic subgroup, by the character $\chi_1 \otimes \chi_2 \otimes \chi_3$. If π is a smooth representation of $GL(2)$, and χ a quasi-character of $F^\times$, then $\pi \rtimes \chi$ (resp. $\chi \rtimes \pi$) is the parabolically induced representation from the Levi subgroup of the Siegel (resp. Klingen) parabolic subgroup. We define a character of the unipotent radical $N(B)$ of the Borel subgroup by the following:

$$\theta\left(\begin{pmatrix}1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1\end{pmatrix}\begin{pmatrix}1 & & r & s \\ & 1 & t & r \\ & & 1 & \\ & & & 1\end{pmatrix}\right) = \psi(x+t).$$

We call an irreducible representation (Π, V_Π) generic, if there is a functional λ_Π on V_Π such that

$$\lambda_\Pi(\Pi(n)v) = \theta(n)v,$$

for all $v \in V_\Pi$ and $n \in N(B)$. If such a functional exists, it is unique up to a constant [25]. Shahidi has given canonical constructions of these functionals in [22] for representations induced from generic representations. We define Whittaker functions on $G \times V_\Pi$ by

$$W(\Pi, v, g) = \lambda_\Pi(\Pi(g)v).$$

When there is no danger of confusion, after fixing v and suppressing Π, we write $W(g)$ instead of $W(\Pi, v, g)$. For a character Ψ of the unipotent radical of the Borel subgroup, we denote by $\pi_{N,\Psi}$ the Jacquet module, twisted by Ψ, of the representation π. We will also use Shahidi's notations on intertwining operators and local coefficients from [22]. These objects have been explicitly written out whenever we have used them. We will also use Sally and Tadic's notations for Langlands parameters.

2. The integral of Novodvorsky.

2.1. The L-function. We recall that ${}^L GSp(4)^\circ = GSp(4, \mathbb{C})$, the connected component of the L-group of $GSp(4)$. Let ${}^L T$ be the maximal torus of $GSp(4, \mathbb{C})$ of elements of the form

$$t(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \begin{pmatrix}\alpha_1 & & & \\ & \alpha_2 & & \\ & & \alpha_3 & \\ & & & \alpha_4\end{pmatrix},$$

where $\alpha_1\alpha_4 = \alpha_2\alpha_3$. The fundamental dominant weights of the torus are λ_1 and λ_2 where

$$\begin{aligned}\lambda_1 t(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= \alpha_1,\\ \lambda_2 t(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= \alpha_1\alpha_3^{-1}.\end{aligned}$$

The dimensions of the representation spaces associated with these dominant weights are 4 and 5 respectively. If π is an automorphic cuspidal representation of $GSp(4)$, the Langlands L-function $L(s,\pi,V(\lambda_1))$ is usually referred to as th *Spin L-function*. A typical Euler factor of this L-function will have degree 4.

2.2. Global integral. Unlike the rest of the paper this subsection is concerned with global theory. In this subsection we will follow the exposition of Bump [1]. Let k be an arbitrary global field, and let (π, V_π) be an irreducible cuspidal automorphic representation of $GSp(4)$ over k. Suppose that π is *generic*, i.e., there exists $\phi \in V_\pi$ such that the Whittaker function of ϕ

$$\begin{aligned}W_\phi(g) = \int_{(\mathbb{A}/k)^4} \phi&\left(\begin{pmatrix}1 & x_2 & & \\ & 1 & & \\ & & 1 & -x_2\\ & & & 1\end{pmatrix}\begin{pmatrix}1 & & x_3 & x_4\\ & 1 & x_1 & x_3\\ & & 1 & \\ & & & 1\end{pmatrix} g\right)\\ &\times \psi^{-1}(x_1+x_2)\,dx_1\,dx_2\,dx_3\,dx_4\end{aligned}$$

is not identically zero, where ψ is a nontrivial character of $\mathbb{A}/k$.

Let μ be a character of $\mathbb{A}^\times/k^\times$. Then the Hecke type integral considered by Novodvorsky is

$$\begin{aligned}Z_N(s,\phi,\mu) = \int_{\mathbb{A}^\times/k^\times}\int_{(\mathbb{A}/k)^3} \phi&\left(\begin{pmatrix}1 & x_2 & & x_4\\ & 1 & & \\ & z & 1 & -x_2\\ & & & 1\end{pmatrix}\begin{pmatrix}y & & & \\ & y & & \\ & & 1 & \\ & & & 1\end{pmatrix}\right)\\ &\times \psi(-x_2)\mu(y)|y|^{s-\frac{1}{2}}\,dz\,dx_2\,dx_4\,d^\times y.\end{aligned}$$

Since ϕ is left invariant under the matrix

$$\begin{pmatrix} & & 1 & \\ & & & 1\\ -1 & & & \\ & -1 & & \end{pmatrix},$$

this integral has a functional equation $s \to 1-s$.

A usual unfolding process as sketched in [1] then shows that

$$Z_N(s,\phi,\mu) = \int_{\mathbb{A}^\times}\int_{\mathbb{A}} W_\phi\begin{pmatrix} y & & & \\ & y & & \\ & x & 1 & \\ & & & 1\end{pmatrix}\mu(y)|y|^{s-\frac{3}{2}}\,dx\,d^\times y.$$

If ϕ is chosen correctly, the Whittaker function may be assumed to decompose locally as $W_\phi(g) = \prod_v W_v(g_v)$, a product of local Whittaker functions. Then

$$\begin{aligned} Z_N(s,\phi,\mu) &= \prod_v Z_{v,N}(s, W_v, \mu_v) \\ &= \prod_v \int_{k_v^\times}\int_{k_v} W_v\begin{pmatrix} y & & & \\ & y & & \\ & x & 1 & \\ & & & 1\end{pmatrix}\mu_v(y)|y|_v^{s-\frac{3}{2}}\,dx_v\,d^\times y_v.\end{aligned}$$

As we will see later each of the local factors $Z_{v,N}$ for nonarchemidean v is a rational function in Nv^{-s}. Also if we fix v, for different choices of ϕ the rational functions Z_v form a principal fractional ideal in $\mathbb{C}(Nv^{-s})$. We are most interested in this local generator. The importance of these factors comes from the fact that if all the local data at v are unramified, the generator is exactly the local v-factor of the Spin L-function of ϕ.

2.3. Local integral. Suppose π is a generic representation of $GSp(4)$ over a local field F, and σ a quasi-character of $F^\times$. For $W \in \mathbb{W}(\pi,\psi)$ define the following function:

$$Z_N(s,W,\sigma) = \int_{F^\times}\int_F W\begin{pmatrix} y & & & \\ & y & & \\ & x & 1 & \\ & & & 1\end{pmatrix}\sigma(y)|y|^{s-\frac{3}{2}}\,dx\,d^\times y.$$

Then we have the following theorem:

THEOREM 2.1. *Z_N is a rational function of q^{-s}, and satisfies the following functional equation:*

$$Z_N(s,W,\sigma) = \gamma(s)Z_N(1-s,\pi(w)W,\sigma^{-1}\omega_\pi^{-1}),$$

for $w = \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}$, *and for a fixed function γ which depends only on the class of π and σ.*

Proof. Rationality follows from Theorem 3.6. For the functional equation we proceed as follows. For g of the following form

$$\begin{pmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & t & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & r & s \\ & 1 & & r \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

it is easy to check that

$$Z_N(s, \pi(g)W, \sigma) = \sigma^{-1}(a)|a|^{\frac{1}{2}-s}\psi(x)Z_N(s, W, \sigma).$$

The other side of the functional equation satisfies the same invariance equation. Let L be the set of all matrices of the above form. L is a subgroup. If we change coordinates by (2 3), this subgroup is sent to the Novodvorsky subgroup [8]. Thus by Furusawa's notation [7] both sides of the functional equation are (S, Λ, ψ)-Bessel functionals, where $S = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. But the space of these functionals is at most one dimensional ([18], also [7]). □

Remark 2.2. This is the "right" functional equation, as the following proposition shows.

PROPOSITION 2.3. *$\tilde{\pi}$ is equivalent to $\pi \otimes \omega_\pi^{-1}$.*

Proof. Let Θ_π be the locally integrable function which gives the character of π. It is easily seen that $\Theta_{\tilde{\pi}}(x) = \Theta_\pi(x^{-1})$, $\forall x \in G$. Also $\Theta_{\pi\otimes\omega^{-1}} = \Theta_{\pi'\otimes\omega^{-1}}$ where $\pi'(x) = \pi(JxJ^{-1})$, where J is the defining matrix of the group. Then one checks that $\Theta_{\pi'\otimes\omega^{-1}}$ is equal to $\Theta_\pi({}^t x^{-1})$. Thus we just need to show that $\Theta_\pi(x) = \Theta_\pi({}^t x)$ a.e. on G. We need the following lemma:

LEMMA 2.4. *Let X be the set of regular semi-simple elements in G. Then every invariant under conjugation distribution on X is invariant under transposition.*

Proof. One easily verifies that the action of G on X by conjugation and the homeomorphism of X given by transposition satisfy the conditions of the theorem of p. 91 of [16]. □

This lemma finishes the proof of the proposition. □

3. Asymptotic expansions and $L(s, \pi, \sigma)$.

3.1. We first prove that the integrals of Novodvorsky admit a "greatest common divisor."

LEMMA 3.1. *The support of the map* $x \mapsto W\begin{pmatrix} y &&& \\ & y && \\ & x & 1 & \\ &&& 1 \end{pmatrix}$ *lies in a compact set* Ω *that does not depend on* y.

Proof. The same as in [12]. □

Now we define a new zeta function by the following:

$$Z(s,W,\sigma)=\int_{F^\times} W\begin{pmatrix} y &&& \\ & y && \\ && 1 & \\ &&& 1 \end{pmatrix}\sigma(y)|y|^{s-\frac{3}{2}}\,d^\times y.$$

PROPOSITION 3.2. *The vector space generated by the* $Z_N(s,W,\sigma)$ *for various* W *is the same as the space generated by the* $Z(s,W,\sigma)$.

Proof. First we prove the equality of the spaces generated by the zeta integrals. Fix W, and let K be the compact open subset of G that fixes W. Let $U=\Omega\cap K$. Then for a finite number of elements $x_1,x_2,\dots,x_n$, $\Omega\subset\cup_i x_iU$. Then

$$\begin{aligned} Z_N(s,W,\sigma) &= \int_{F^\times}\sum_{i=1}^{n}\int_{x_iU} W\left[\begin{pmatrix} y &&& \\ & y && \\ && 1 & \\ &&& 1 \end{pmatrix} h\right]\sigma(y)|y|^{s-\frac{3}{2}}\,dh\,d^\times y \\ &= \sum_{i=1}^{n}\operatorname{vol}(x_iU)\int_{F^\times}\pi(x_i)W\begin{pmatrix} y &&& \\ & y && \\ && 1 & \\ &&& 1 \end{pmatrix}|y|^{s-\frac{3}{2}}\,d^\times y \\ &= \sum_{i=1}^{n}\operatorname{vol}(x_iU)Z(s,\pi(x_i)W,\sigma). \end{aligned}$$

This proves one of the inclusions. For the other side, suppose ϕ is a Schwartz function whose Fourier transform $\hat{\phi}$ is supported in a small neighborhood of 0, and $\phi(0)\neq 0$. Then

$$\begin{aligned} &\phi(0)W\left(\begin{pmatrix} y &&& \\ & y && \\ && 1 & \\ &&& 1 \end{pmatrix}\right) \\ &\qquad = \int_F \hat{\phi}(x)W\left(\begin{pmatrix} y &&& \\ & y && \\ && 1 & \\ &&& 1 \end{pmatrix}\begin{pmatrix} 1 &&& \\ & 1 && \\ & x & 1 & \\ &&& 1 \end{pmatrix}\right)dx \end{aligned}$$

$$= \int\int \phi(s)\psi(-xs)W\left(\begin{pmatrix} y &&& \\ & y && \\ && 1 & \\ &&& 1\end{pmatrix}\begin{pmatrix} 1 &&& \\ & 1 && \\ & x & 1 & \\ &&& 1\end{pmatrix}\right) ds\,dx$$

$$= \int\int \phi(s)W\left(\begin{pmatrix} y &&& \\ & y && \\ && 1 & \\ &&& 1\end{pmatrix}\begin{pmatrix} 1 &&& \\ & 1 && \\ & x & 1 & \\ &&& 1\end{pmatrix}\begin{pmatrix} 1 && s & \\ & 1 && s \\ && 1 & \\ &&& 1\end{pmatrix}\right) ds\,dx$$

$$= \int\int \phi(s)\left(\pi\begin{pmatrix} 1 && s & \\ & 1 && s \\ && 1 & \\ &&& 1\end{pmatrix}W\right)\begin{pmatrix} y &&& \\ & y && \\ & x & 1 & \\ &&& 1\end{pmatrix} dx\,ds$$

$$= \sum_i \int W_i\begin{pmatrix} y &&& \\ & y && \\ & x & 1 & \\ &&& 1\end{pmatrix} dx.$$

This finishes the proof. □

Now we will show that the functions $Z(s, W, \sigma)$ admit a gcd. The method is basically that of [11]. We first need a lemma.

LEMMA 3.3. *The function* f_W: $y \mapsto W\begin{pmatrix} y &&& \\ & y && \\ && 1 & \\ &&& 1\end{pmatrix}$ *has bounded support in* F.

Proof. Choose $t \in F$ with $|t|$ small enough, so that $W\left(g\begin{pmatrix} 1 &&& \\ & 1 & t & \\ && 1 & \\ &&& 1\end{pmatrix}\right) = W(g)$. Then

$$W\begin{pmatrix} y &&& \\ & y && \\ && 1 & \\ &&& 1\end{pmatrix} = W\left(\begin{pmatrix} y &&& \\ & y && \\ && 1 & \\ &&& 1\end{pmatrix}\begin{pmatrix} 1 &&& \\ & 1 & t & \\ && 1 & \\ &&& 1\end{pmatrix}\right)$$

$$= \psi(ty)W\begin{pmatrix} y &&& \\ & y && \\ && 1 & \\ &&& 1\end{pmatrix}$$

So in order for the function to be nonzero, we need $\psi(yt) \neq 0$. □

Let $\mathbb{K}(\pi, \psi)$ be the space of all functions f_W.

LEMMA 3.4. *$\mathcal{S}(F^\times) \subset \mathbb{K}(\pi, \psi)$. When π is super-cuspidal, $\mathbb{K}(\pi, \psi) = \mathcal{S}(F^\times)$*

Proof. See Theorem 1 of Chapter 1 of [9]. □

The following proposition is fundamental.

PROPOSITION 3.5. *There is a finite set of finite functions S with the following property: for any $W \in \mathbb{W}(\pi, \psi)$, and $\chi \in S$ there is a Schwartz function $\Phi_{\chi,W}$ on F such that the following equality holds for every $y \in F^\times$*

$$W\begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \sum_{\chi \in S} \Phi_{\chi,W}(y)\chi(y)|y|^{\frac{3}{2}}.$$

Proof. For super-cuspidal representations, this follows from Lemma 3.4. For the Whittaker functional of an induced representation of an arbitrary quasi-split group, this is equation (3.4.2) of [4]. □

THEOREM 3.6. *For every W in $\mathbb{W}(\pi, \psi)$, $\zeta(s, W, \sigma)$ is a rational function in $X = q^{-s}$. The ideal generated by these functions is principal.*

Proof. The rationality assertion is obvious. It is seen that each χ in the germ expansion is either a quasi-character itself, or the product of a quasi-character and some power of the function $\log_q |\ |$. We will denote the quasi-character associated with χ by $\tilde{\chi}$. Similar to the argument on pages 1.46–47 of [9] we see that if $\chi = \tilde{\chi}.(\log_q |\ |)^{n(\chi)}$ then

$$\frac{\int_F \Phi_{\chi,W}(y)\chi(y)\sigma(y)|y|^s \, d^\times y}{L(s, \tilde{\chi}\sigma)^{n(\chi)}}$$

is an entire function. Now Theorem 4.3 of [19] says that these are all the possible denominators. This finishes the proof. □

We will denote the generator of the above ideal by $L(s, \pi, \sigma)$.

PROPOSITION 3.7. *$L(s, \pi, \sigma)^{-1} \in \mathbb{C}[q^{-s}]$.*

Proof. Suppose

$$L(s, \pi, \sigma) = \frac{P(q^{-s})}{Q(q^{-s})} \text{ with } (P, Q) = 1.$$

If $P(q^{-s})$ is not a monomial in q^{-s} , for some values of s, say s_0, $P(q^{-s_0}) = 0$. This implies for all $W \in \mathbb{W}(\pi, \psi)$, $Z(s_0, W, \sigma) = 0$ which is impossible, because

$\mathcal{S}(F^\times) \subset \mathbb{K}(\pi, \psi)$. Now the proposition follows from this identity:

$$Z\left(s, \pi\left(\begin{pmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\right) W, \sigma\right) = |a|^{\frac{3}{2}-s}\sigma^{-1}(a)Z(s, W, \sigma). \qquad \square$$

COROLLARY 3.8. $L(s, \pi, \sigma) = \gcd_{\chi \in S} L(s, \sigma\tilde{\chi})^{n(\chi)}$. □

Note that this is basically the same as Theorem 4.1 of [19]. The L-functions appearing on the right-hand side are Tate's L-factors.

PROPOSITION 3.9. *$L(s, \pi, \sigma) = 1$ when π is super-cuspidal, or when π is a sub-quotient of the induction of a supercuspidal representation of the Levi factor of the Klingen parabolic subgroup. Also deg $L(s, \pi, \sigma)^{-1} \leq 4$ (≤ 2), when π is a sub-quotient of a representation coming from the minimal (resp. Siegel) parabolic subgroup.*

Proof. When π is supercuspidal, the assertion is obvious by Lemma 3.4. Now let π be a sub-quotient of a supercuspidal representation Π of a standard parabolic subgroup $P = P_\theta = MN$, $\theta \subset \Delta$. The idea is to bound the size of the set S in Proposition 3.5. Fix a $\chi \in S$, and define a functional Λ_χ on $\mathbb{W}(\pi, \psi)$ by

$$\Lambda_\chi(W) = \Phi_{\chi,W}(0).$$

This functional satisfies the following identity, for all $n \in N$

$$\Lambda_\chi(\pi(n)W) = \bar{\theta}(n)\Lambda_\chi(W),$$

where $\bar{\theta}$ is defined by the following:

$$\bar{\theta}\left(\begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & r & s \\ & 1 & t & r \\ & & 1 & \\ & & & 1 \end{pmatrix}\right) = \psi(x).$$

This in particular shows that $\chi \in \pi^*_{N,\bar{\theta}}$, the twisted Jacquet module [5]. It follows from the proof of Proposition 3.5 that the functionals Λ_χ are linearly independent. Thus, as the twisted Jacquet module is an exact functor, we just need to bound the dimension of $\mathrm{Ind}(\Pi|P, G)^*_{N,\bar{\theta}}$. Let P_Π be the usual map from $\mathcal{S}(G) \otimes V_\Pi$ to $\mathrm{Ind}(\Pi|P, G)$ defined by the following

$$P_\Pi(\phi)(g) = \int_P \delta_P^{\frac{1}{2}}(p)\Pi(p^{-1})\phi(pg)d_r p,$$

for $\phi \in \mathcal{S}(G)$. It's standard [3] that P_Π is surjective and commutes with right translations. Identify $\mathrm{Ind}(\Pi|P, G)$ with its Whittaker model, which we will denote by $\mathbb{W}(\Pi)$, and let T be the pullback of the functional Λ_χ to G via P_Π. Then T satisfies

$$T(L_p \circ R_u \phi) = \delta_P^{-1/2}(p)\bar{\theta}(u)T(\Pi^{-1}(p)\phi),$$

for $\phi \in \mathcal{S}(G) \otimes V_\Pi$, $p \in P$, and $u \in U$ the unipotent radical of the Borel subgroup. Similar to the method of [25] we will work with functions ϕ, whose support are subsets of PwU, for various w in $W_\theta \backslash W$. Fix one of these double cosets. The space of distributions on PwU satisfying the invariance properties of T is at most one dimensional [10]. Consider the natural map $P \times U \longrightarrow PwU$ defined by $(p, u) \mapsto pwu^{-1}$. Next, $P \times U$ has a natural action on PwU by $(p, u).x = pxu^{-1}$. The group $P \times U$ also acts on $\mathcal{S}(PwU) \otimes V_\Pi$ by duality:

$$(p, u).\phi(x) = \phi(pxu^{-1}).$$

By using this action

$$T((p, u)\phi) = \delta_P^{1/2}(p)\bar{\theta}(u)T(\Pi(p)\phi).$$

Now we have a map $\mathcal{S}(P \times U) \otimes V_\Pi \longrightarrow \mathcal{S}(PwU) \otimes V_\Pi$ defined by

$$\phi_f(x) = \int_{\Delta_w} f(x\delta)\, d\delta,$$

where $\Delta_w = \{(p, u)|pwu^{-1} = w\}$. Indeed Δ_w is the isotropy group of the action of $P \times U$ on PwU at w and we have $P \times U/\Delta_w = PwU$. Let T^* be the pullback of T to $P \times U$ via the above map. One can easily see that

$$T^*(L_{(p,u)}f) = T((p^{-1}, u^{-1}).\phi_f) = \delta_p^{-1/2}(p)\bar{\theta}(u^{-1})T^*(\Pi^{-1}(p)f).$$

Then there exists a functional $\lambda \in V_\Pi^*$ such that

$$T^*(f) = \lambda\left(\int_P\int_U \delta_p^{-1/2}(p)\bar{\theta}(u^{-1})\Pi^{-1}(p)f(p, u)\, dp\, du\right).$$

This is because of the following natural generalization of Lemma 17 of [10] on left-invariant distributions.

LEMMA 3.10. *Suppose G is a p-adic group, ρ: $G \longrightarrow GL(V)$ is a smooth representation of G on some complex vector space V. Let T be a functional on $\mathcal{S}(G; V)$ which satisfies*

$$T(L_g f) = T(\rho(g)f),$$

for all $f \in \mathcal{S}(G;V)$ *and* $g \in G$. *Then there exists a* $\lambda \in V^*$ *in such a way that*

$$T(f) = \lambda\left(\int \rho(g)f(g)\,d_lg\right).$$

Proof. Define a map $\gamma\colon\ \mathcal{S}(G;V) \longrightarrow \mathcal{S}(G;V)$ by

$$(\gamma f)(g) = \rho(g^{-1})f(g).$$

Note that this map is well defined because ρ is smooth. It can now be checked that the functional $\tilde{T} = \gamma^* T$ is left-invariant. Now if we use the indentification $\mathcal{S}(G;V) = \mathcal{S}(G) \otimes V$, the proof of Lemma 17 of [10] shows that there exists a functional λ such that

$$\tilde{T}(f) = \lambda\left(\int f(g)\,d_lg\right).$$

Writing this equation in terms of T proves the lemma. □

On the other hand for all $\delta \in \Delta_w$,

$$T^*(R_\delta f) = T(F_{R_\delta f}) = T(F_f) = T^*(f).$$

Using the integral representation for T^* we have:

$$\begin{aligned} T^*(R_\delta f) &= \lambda\left(\int_P\int_U \delta_p^{-1/2}(p)\bar{\theta}(u^{-1})\Pi^{-1}(p)R_\delta f(p,u)\,dp\,du\right)\\ &= \lambda\left(\int_P\int_U \delta_p^{-1/2}(p)\bar{\theta}(u^{-1})\Pi^{-1}(p)f(pp_1,uu_1)\,dp\,du\right)\\ &= \lambda\left(\int_P\int_U \delta_p^{-1/2}(pp_1^{-1})\bar{\theta}^{-1}(uu_1^{-1})\Pi^{-1}(pp_1^{-1})f(p,u)\,dp\,du\right)\\ &= \lambda\left(\delta_p^{1/2}(p_1)\bar{\theta}(u_1)\Pi(p_1)\int_P\int_U \delta_p^{-1/2}(p)\bar{\theta}^{-1}(u)\Pi^{-1}(p)f(p,u)\,dp\,du\right), \end{aligned}$$

for $\delta = (p_1, u_1)$. Finally, it is trivial that for every $v \in V_\sigma$ there exists $f \in \mathcal{S}(P \times U) \otimes V_\Pi$ such that

$$v = \int_P\int_U \delta_p^{-1/2}(p)\bar{\theta}^{-1}(u)\Pi^{-1}(p)f(p,u)\,dp\,du.$$

Thus for all $(p,u) \in \Delta_w$, λ satisfies the following equation:

$$\lambda(\bar{\theta}(u)\Pi(p)v - v) = 0.$$

The rest of the proof consists of a case-by-case analysis of all the possibilities. The Borel subgroup case is simple. For fixed w, if we can find p such that the corresponding u satisfies $\bar{\theta}(u) \neq 1$, then it follows that

$$(\bar{\theta}(u) - 1)\lambda(v) = 0.$$

It follows from this equation, then, that $\lambda(v) = 0$ for all v. Finding such p is possible when w sends the element

$$\begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \tag{1}$$

to an upper-triangular matrix. This happens for exactly four elements of the Weyl group, hence the first assertion of the proposition. For the other two parabolics, first we give representatives for the right cosets of W_θ in W. Note that by Proposition 1.3.1 of [3], $G = \cup PwU$, $w \in W_\theta \backslash W$. Next by Lemma 1.1.2 of [3], in any right cosets of W_θ in W there exists a unique element w characterized by $w^{-1}\theta > 0$. Hence for the two parabolics, we get the following sets of representatives:

(1) Siegel parabolic subgroup:

$$\{\text{identity}, (2\,3), (1\,2\,4\,3), (1\,3)(2\,4)\}.$$

(2) Klingen parabolic subgroup:

$$\{\text{identity}, (1\,4), (1\,2)(3\,4), (1\,3\,4\,2)\}.$$

For the Siegel parabolic case, for the two elements $(2\,3)$ and $(1\,2\,4\,3)$, there exist $(p, u) \in \Delta_w$ satisfying $\Pi(p) = 1$ and $\bar{\theta}(u) \neq 1$. Hence the same argument shows that their corresponding functional λ is identically zero. For the Klingen parabolic subgroup, though, we have to use the super-cuspidality of the inducing representation. Let N_M be the unipotent radical of the Levi factor of P. Then it can be checked that for the three Weyl elements identity, $(1\,4)$, and $(1\,2)(2\,3)$ we have

$$\bar{\theta}(w^{-1} N_M w) = \{1\}.$$

From this it follows that $\lambda(v) = 0$ for all $v \in V_\Pi(N_M)$. But since Π is supercuspidal $V_\Pi(N_M)$ generates V_Π as a vector space, hence the result. It remains to study the element $(1\,3\,4\,2)$. We have the following identity:

$$(1\,3\,4\,2)\left\{\begin{pmatrix} 1 & & & t \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\right\} = \left\{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & t & 1 & \\ & & & 1 \end{pmatrix}\right\}.$$

This identity implies that

$$\lambda\left(\Pi\begin{pmatrix}1 & \\ t & 1\end{pmatrix}v - v\right) = 0.$$

Here we have used the same notation for the restriction of Π to the $GL(2)$ part of the Levi factor of P. This now implies that $\lambda = 0$, again because Π is supercuspidal. We have completed the proof of the proposition. □

3.2. From this point on, we concentrate on representations induced from the minimal parabolic subgroup and the Siegel parabolic subgroup. We will use the same notations as in the proof of Proposition 3.9. We will also assume that $\pi = \mathrm{Ind}(\Pi|P, G)$ is irreducible. Proposition 3.5 says that for all $W \in \mathbb{W}(\Pi)$ there is a $\delta(W) > 0$ such that if $|a| < \delta(W)$, then

$$W\begin{pmatrix}a & & & \\ & a & & \\ & & 1 & \\ & & & 1\end{pmatrix} = \sum_{\chi\in S}\Lambda_\chi(W)\chi(a)|a|^{\frac{3}{2}}.$$

The reason for $\delta(W) > 0$ is that the functions $\Phi_{\chi,W}$ in Proposition 3.5 are all Schwartz functions. Now we would like to study the behavior of $\delta(W)$, when W holomorphically depends on a parameter, or a space of parameters, in the sense we will now explain. For $f \in \mathrm{Ind}(\Pi|P\cap K, K)$ and $s \in \mathbb{C}$, define a function f_s by the following:

$$f_s(pk) = \delta_P(p)^{s+\frac{1}{2}}\Pi(p)f(k),$$

for $p \in P$ and $k \in K$. Note that f_s is well defined, and belongs to the space of a certain induced representation π_s. Let λ_{π_s} be the Whittaker functional of π_s. We know from above that there exists a number $\delta = \delta(f, s)$ such that for $|a| < \delta$ we will have the required asymptotic expansion for $\lambda_{\pi_s}(\pi_s(\hat{a})f_s)$, where $\hat{a} = \mathrm{diag}(a, a, 1, 1)$. We will refer to the following proposition as the Uniformity Proposition [26].

PROPOSITION 3.11. $\delta(f, s) = \delta(f)$.

In [26], Shalika proves the corresponding proposition for $GL(2)$. Namely, suppose μ_1 and μ_2 are two quasi-characters of $F^\times$, in such a way that $\mathrm{Ind}(\mu_1 \otimes \mu_2|B, G)$ is irreducible. Then for complex parameters (s_1, s_2), one considers the representation $I(s_1, s_2) = \mathrm{Ind}(\mu_1\nu^{s_1} \otimes \mu_2\nu^{s_2}|B, GL(2))$. Now for $f \in \mathrm{Ind}(\mu_1 \otimes \mu_2|B \cap K, K)$, one can define the number $\delta(f, s_1, s_2)$ the way we defined our $\delta(f, s)$. Then Shalika's uniformity proposition asserts that $\delta(f, s_1, s_2) = \delta(f, 0, 0)$. Our method of proof of Proposition 3.11 which we are about to present can be used to give a proof for Shalika's theorem different from his original method.

Proof. We will use the following identity repeatedly [5]

$$\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} t^{-1} & -1 \\ & t \end{pmatrix} \begin{pmatrix} 1 & \\ t^{-1} & 1 \end{pmatrix}, \tag{2}$$

for t nonzero. We will prove the proposition for the two cases of interest separately.

Case 1. Siegel parabolic subgroup: Supercuspidal representations. Let $|a| < 1$. We know from [22] that in this case

$$W(\pi_s, f_s, \hat{a}) = |a|^{\frac{3}{2}-s}\chi(a) \int_{F^3} f_s \left[\begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix} \begin{pmatrix} 1 & & r & u \\ & 1 & t & r \\ & & 1 & \\ & & & 1 \end{pmatrix}; e \right] \psi(-at)\, dt\, dr\, du,$$

as a principal value integral. Let ϕ be the characteristic function of the ring of integers. We have two different cases:

(1) t integer. As $|a| < 1$, we have $\psi(at) = 1$. Thus the integral is independent of a.

(2) t noninteger. By using the fundamental identity above write the integral as the following:

$$I_f^s(a) = \int_{F^3} |t|^{-\frac{3}{2}-s}(1-\phi(t))\psi(-at) f_s \left[\begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix} \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & t^{-1} & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & r & u \\ & 1 & & r \\ & & 1 & \\ & & & 1 \end{pmatrix}; \begin{pmatrix} t^{-1} & \\ & 1 \end{pmatrix} \right] dt\, dr\, du.$$

Now we will use the following identity:

$$\begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix} \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & t^{-1} & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & r & u \\ & 1 & & r \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & -t^{-1} & 1 & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & r & & \\ & 1 & & \\ & & 1 & -r \\ & & & 1 \end{pmatrix} \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix} \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & -rt^{-1} & & u+r^2t^{-1} \\ & 1 & & \\ & & 1 & rt^{-1} \\ & & & 1 \end{pmatrix}.$$

This implies the following:

$$I_f^s(a) = \int_{F^3} |t|^{-\frac{1}{2}-s}(1-\phi(t))\psi(-at)\psi(-r)f_s$$
$$\left[\begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix} \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & r & & u \\ & 1 & & \\ & & 1 & -r \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & t^{-1} & 1 & \\ & & & 1 \end{pmatrix}; \begin{pmatrix} t^{-1} & \\ & 1 \end{pmatrix} \right] dt\,dr\,du.$$

Now we divide $I_f^s(a)$ to the following 4 integrals:

Integral 1. The integrand in $I_f^s(a)$ is multiplied by $\phi(u)\phi(r)$. This will then be a finite linear combination of integrals of the following form:

$$\int_F |t|^{-\frac{1}{2}-s}(1-\phi(t))\mathrm{ch}_{\mathcal{M}}(t^{-1})\psi(-at)v\begin{pmatrix} t^{-1} & \\ & 1 \end{pmatrix} dt,$$

for certain compact sets $\mathcal{M}$ in $\mathcal{O}$ independent of s. The function

$$t \mapsto v\begin{pmatrix} t^{-1} & \\ & 1 \end{pmatrix}$$

has compact support in $F^\times$. Thus it suffices to have $a(\mathrm{supp} v)^{-1} \subset \mathcal{O}$.

Integral 2. The integrand in $I_f^s(a)$ is multiplied by $\phi(u)(1-\phi(r))$. This time we obtain integrals of the following form:

$$\int_{F^2} |t|^{-\frac{1}{2}-s}|r|^{3-6s}\psi(-at-r)v\begin{pmatrix} t^{-1} & \\ & 1 \end{pmatrix} \mathrm{ch}_{\mathcal{M}_1}(t^{-1})\mathrm{ch}_{\mathcal{M}_2}(r^{-1})$$
$$(1-\phi(r))(1-\phi(t))\,dt\,dr.$$

This is taken care of in the same exact manner.

Integral 3. The integrand in $I_f^s(a)$ is multiplied by $\phi(r)(1-\phi(u))$. We have the following integral:

$$\int_{F^3} |t|^{-\frac{1}{2}-s}|u|^{-\frac{3}{2}-3s}(1-\phi(t))(1-\phi(u))\phi(r)\psi(-at-r) \tag{3}$$

$$f_s\left[\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}\begin{pmatrix} 1 & & & \\ & & 1 & \\ & & & 1 \\ u^{-1} & & & 1 \end{pmatrix}\begin{pmatrix} 1 & r & & \\ & 1 & & \\ & & 1 & -r \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & t^{-1} & 1 & \\ & & & 1 \end{pmatrix}\right.; \tag{4}$$

$$\left.\begin{pmatrix} t^{-1} & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & u^{-1} \end{pmatrix}\right] dt\, dr\, du. \tag{5}$$

This is now a finite linear combination of the following integrals:

$$\int_{F^2} |t|^{\frac{1}{2}-s}|u|^{-3s-\frac{1}{2}}\psi(-at)v\begin{pmatrix} t^{-1} & \\ & u^{-1} \end{pmatrix}\mathrm{ch}_{\mathcal{M}_1}(t^{-1})\mathrm{ch}_{\mathcal{M}_2}(u^{-1})$$
$$(1-\phi(u))(1-\phi(t))\,\frac{dt}{|t|}\,\frac{du}{|u|}.$$

Let $u = t\gamma$:

$$\int_{F^2} |t|^{-4s}|\gamma|^{-3s-\frac{1}{2}}\psi(-at)v\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix}\mathrm{ch}_{\mathcal{M}_1}(t^{-1})\mathrm{ch}_{\mathcal{M}_2}(t^{-1}\gamma^{-1})$$
$$(1-\phi(t\gamma))(1-\phi(t))\,\frac{dt}{|t|}\,\frac{d\gamma}{|\gamma|}$$
$$= \int_{\mathfrak{p}^2} |t|^{4s}|\gamma|^{3s+\frac{1}{2}}\psi(-at^{-1})v\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix}\mathrm{ch}_{\mathcal{M}_1}(t)\mathrm{ch}_{\mathfrak{p}\cap\mathcal{M}_2}(t\gamma)\,\frac{dt}{|t|}\,\frac{d\gamma}{|\gamma|}.$$

This too is a finite linear combination of the following integrals:

$$\int_K\int_{\mathfrak{p}} |t|^{4s}|\gamma|^{3s+\frac{1}{2}}\psi(-at^{-1})\mathrm{ch}_{\mathcal{M}_1}(t)\mathrm{ch}_{\mathfrak{p}\cap\mathcal{M}_2}(t\gamma)\,\frac{dt}{|t|}\,\frac{d\gamma}{|\gamma|},$$

for compact sets K in $F^\times$. Choose a large integer M in such a way that $\pi^M K \subset \mathfrak{p}\cap\mathcal{M}_2$, also $|a| < q^{-M}$. Then we get an integral which is independent of a and the following integral:

$$\int_{|t|<q^{-M}} |t|^{4s}\psi(-at^{-1})\,\frac{dt}{|t|},$$

which can be explicitly computed in terms of ord a.

Integral 4. The integrand in $I_f^s(a)$ is multiplied by $(1-\phi(r))(1-\phi(u))$. We start from equation (3) above. After simple manipulations, and a change of variable

$R = u^{-1}r$, we get the following:

$$\int_{F^3} |t|^{-\frac{1}{2}-s}|u|^{-\frac{1}{2}-3s}(1-\phi(t))(1-\phi(u))(1-\phi(uR))\psi(-at-Ru)$$
$$f_s\left[\begin{pmatrix}1&&&\\&1&&\\&&1&\\R^2u&&&1\end{pmatrix}\begin{pmatrix}&1&&\\1&&&\\&&&1\\&&1&\end{pmatrix}\begin{pmatrix}1&&&\\&1&&\\&&1&\\u^{-1}&&&1\end{pmatrix}\begin{pmatrix}1&&&\\&1&&\\&t^{-1}&1&\\&&&1\end{pmatrix};\right.$$
$$\left.\begin{pmatrix}t^{-1}&\\&u^{-1}\end{pmatrix}\begin{pmatrix}1&\\R(1+u)&1\end{pmatrix}\right]dt\,dR\,du.$$

We divide this integral again to integrals having $\phi(R^2u)$ and $1-\phi(R^2u)$ in them. The integral with $\phi(R^2u)$ is easy to deal with. We will now study the second integral. After using equation (2), the integral can be written as a finite linear combination of the following integrals:

$$\int_{F^3} |t|^{-\frac{1}{2}-s}|u|^{-\frac{1}{2}-3s}(1-\phi(R^2u))(1-\phi(t))(1-\phi(u))(1-\phi(uR))$$
$$\psi(-at-Ru)|R^2u|^{-3s-\frac{3}{2}}v\left[\begin{pmatrix}t^{-1}&\\&u^{-1}\end{pmatrix}\begin{pmatrix}1&\\R(1+u)&1\end{pmatrix}\begin{pmatrix}R^{-2}u^{-1}&\\&1\end{pmatrix}\right]$$
$$\mathrm{ch}_{\mathcal{M}_1}(t^{-1})\mathrm{ch}_{\mathcal{M}_2}(u^{-1})\,dt\,dR\,du.$$

This integral in turn is a finite linear combination of the following:

$$\int_{F^3} |t|^{-\frac{1}{2}-s}|u|^{-\frac{1}{2}-3s}(1-\phi(R^2u))(1-\phi(t))(1-\phi(u))(1-\phi(uR))$$
$$\psi(-at-Ru)|R^2u|^{-3s-\frac{3}{2}}v\begin{pmatrix}R^{-2}t^{-1}&\\&1\end{pmatrix}$$
$$\mathrm{ch}_{\mathcal{M}_1}(t^{-1})\mathrm{ch}_{\mathcal{M}_2}(u^{-1})\mathrm{ch}_{\mathcal{M}_3}(R^{-1}u^{-1}(1+u))\,dt\,dR\,du,$$

which can again be dealt with as before. This completes the proof of the uniformity proposition in this case.

Case 2. Siegel parabolic subgroup: nonsupercuspidal representations. Here we will show that the following integrals have asymptotics that are uniform in s:

$$\widetilde{W}(\pi_s, f_s, \hat{a}) = \int_{F^3} f_s\left[\begin{pmatrix}&&1&\\&&&1\\-1&&&\\&-1&&\end{pmatrix}\begin{pmatrix}1&&r&u\\&1&t&r\\&&1&\\&&&1\end{pmatrix};e\right]\psi(-at)\,dt\,dr\,du.$$

The same proof as above works in this case. One just needs to take into account the contribution of the germ expansion of Π.

Case 3. Borel parabolic subgroup. In this case we have the following:

$$\begin{aligned}
&W(\chi(s), f_s, \hat{a}) \\
&\quad = \chi_3(a)|a|^{\frac{3}{2}-3s}\int_{F^4}\psi(-x-at) \\
&\qquad f_s\left[\begin{pmatrix}1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1\end{pmatrix}\begin{pmatrix}1 & & r & u \\ & 1 & t & r \\ & & 1 & \\ & & & 1\end{pmatrix}\right]dx\,dr\,du\,dt \\
&\quad = \int_{\mathfrak{p}}\chi_1(x)\chi_2(x^{-1})\psi(-x^{-1})\lambda^a_{\text{Siegel}}\left(\chi\begin{pmatrix}1 & & & \\ x^{-1} & 1 & & \\ & & 1 & \\ & & -x^{-1} & 1\end{pmatrix}f_s\right)\frac{dx}{|x|} \\
&\qquad + \int_{\mathcal{O}}\psi(-x)\lambda^a_{\text{Siegel}}\left(\chi\begin{pmatrix}1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1\end{pmatrix}f_s\right)dx,
\end{aligned}$$

where

$$\lambda^a_{\text{Siegel}}(f) = \int_{F^3} f\left(w_2w_1w_2\begin{pmatrix}1 & & r & u \\ & 1 & t & r \\ & & 1 & \\ & & & 1\end{pmatrix}\right)\psi(-at)\,dr\,du\,dt.$$

This is the twisted-by-a Whittaker functional of $\text{Ind}(\text{Ind}(\chi_1 \otimes \chi_2) \otimes \chi_3 | P,\ G)$. Here χ_3 is the character of the similitude part of P. The result now follows from Case 2 and [26]. □

4. Local computations for the Borel parabolic subgroup.

4.1. Let χ_1, χ_2, χ_3 be three quasi-characters of $F^\times$. We call a character $\chi_1 \otimes \chi_2 \otimes \chi_3$ "regular" if it is not fixed by any Weyl element except for the trivial element. It can easily be checked that this is equivalent to the statement that no element of the set $\{\chi_1, \chi_2, \chi_1\chi_2^{\pm 1}\}$ is trivial. It is not hard to see when the representation $\chi_1 \times \chi_2 \rtimes \chi_3$ is reducible: It happens if and only if one of the quasi-characters in the set just mentioned is equal to $\nu^{\pm 1}$ [20], [23]. Sally and Tadic [20] have completed the classification for representations induced from the Borel subgroup. They have in particular determined the reducibilities of inductions. Here, for the convenience of the reader, we include a resume of their results. Following [20], L stands for the Langlands Quotient. Also $R(G)$ is the Grothendieck group of the category of all smooth representations of $GSp(4)$.

Let $\chi_1 \otimes \chi_2 \otimes \chi_3$ be regular, and suppose $\chi_1 \times \chi_2 \rtimes \chi_3$ is reducible. In the notation of [20] this means $s(\chi_1 \otimes \chi_2 \otimes \chi_3) > 0$. If $s(\chi_1 \otimes \chi_2 \otimes \chi_3) = 1$, then $\chi_1 \otimes \chi_2 \otimes \chi_3$ is associated either to a character of the form $\upsilon^{1/2}\chi \otimes \upsilon^{-1/2}\chi \otimes \sigma$ where $\chi \notin \{\xi, \upsilon^{\pm 1}\xi, \upsilon^{\pm 3/2}\}$ for any ξ unitary with $\xi^2 = 1$, or it is associated to a character of the form $\chi \otimes \upsilon \otimes \sigma$ where $\chi \notin \{1_{F^\times}, \upsilon^{\pm 1}, \upsilon^{\pm 2}\}$. Lemmas 3.3 and 3.4 of [20] combined with the exactness of Jacquet functor give the following:

(1) Let χ, ξ, and $\sigma \in (F^\times)\tilde{}$, where $\chi \notin \{\xi, \upsilon^{\pm 1}\xi, \upsilon^{\pm 3/2}\}$ for any ξ with $\xi^2 = 1$. Then $\chi \mathrm{St}_{GL(2)} \rtimes \sigma$ and $\chi 1_{GL(2)} \rtimes \sigma$ are irreducible representations. We have

$$\upsilon^{1/2}\chi \times \upsilon^{-1/2}\chi \rtimes \sigma = \chi \mathrm{St}_{GL(2)} \rtimes \sigma + \chi 1_{GL(2)} \rtimes \sigma$$

in $R(G)$. The representation $\chi \mathrm{St}_{GL(2)} \rtimes \sigma$ is generic.

(2) Let $\chi, \sigma \in (F^\times)\tilde{}$. Suppose that $\chi \notin \{1_{F^\times}, \upsilon^{\pm 1}, \upsilon^{\pm 2}\}$. Then $\chi \rtimes \sigma \mathrm{St}_{GSp(2)}$ and $\chi \rtimes \sigma 1_{GSp(2)}$ are irreducible representations. We have

$$\chi \times \upsilon \rtimes \upsilon^{-1/2}\sigma = \chi \rtimes \sigma \mathrm{St}_{GSp(2)} + \chi \rtimes 1_{GSp(2)}$$

in $R(G)$. The representation $\chi \rtimes \sigma \mathrm{St}_{GSp(2)}$ is generic.

Now we consider regular $\chi_1 \otimes \chi_2 \otimes \chi_3$ with $s(\chi_1 \otimes \chi_2 \otimes \chi_3) = 2$. Then $\chi_1 \otimes \chi_2 \otimes \chi_3$ is either associated with $\upsilon^2 \otimes \upsilon \otimes \sigma$, or $\upsilon\xi \otimes \xi \otimes \sigma$, with $\xi^2 = 1$. This situation is the subject of Lemmas 3.5 and 3.6 of [20].

(1) For $\sigma \in (F^\times)\tilde{}$ the following equalities holds in $R(G)$

$$\begin{aligned}\upsilon^2 \times \upsilon \rtimes \upsilon^{-1/2}\sigma &= \upsilon^{3/2}\mathrm{St}_{GL(2)} \rtimes \upsilon^{-1/2}\sigma + \upsilon^{3/2} 1_{GL(2)} \rtimes \upsilon^{-1/2}\sigma \\ &= \upsilon^2 \rtimes \sigma \mathrm{St}_{GSp(2)} + \upsilon^2 \rtimes \sigma 1_{GSp(2)}\end{aligned}$$

and

$$\begin{aligned}\upsilon^2 \rtimes \sigma \mathrm{St}_{GSp(2)} &= \upsilon\sigma \mathrm{St}_{GSp(4)} + L((\upsilon^2, \mathrm{St}_{GSp(2)})) \\ \upsilon^{3/2}\mathrm{St}_{GL(2)} \rtimes \upsilon^{-1/2}\sigma &= \upsilon\sigma \mathrm{St}_{GSp(4)} + L((\upsilon^{3/2}\mathrm{St}_{GL(2)}, \upsilon^{-1/2}\sigma)).\end{aligned}$$

In this case, the Steinberg representation $\mathrm{St}_{GSp(4)}$ is generic.

(2) Let ξ be of order two. Then the representation $\upsilon\xi \times \xi \rtimes \sigma$ contains a unique generic essentially square integrable sub-quotient. This sub-quotient will be denoted by $\delta([\xi, \upsilon\xi], \sigma)$. We have in $R(G)$

$$\begin{aligned}\upsilon\xi \times \xi \rtimes \sigma &= \upsilon^{1/2}\xi \mathrm{St}_{GL(2)} \rtimes \sigma + \upsilon^{1/2}\xi 1_{GL(2)} \rtimes \sigma \\ &= \upsilon^{1/2}\xi \mathrm{St}_{GL(2)} \rtimes \xi\sigma + \upsilon^{1/2}\xi 1_{GL(2)} \rtimes \xi\sigma\end{aligned}$$

and

$$\nu^{1/2}\xi\mathrm{St}_{GL(2)} \rtimes \sigma = \delta([\xi, \nu\xi], \sigma) + L((\nu^{1/2}\xi\mathrm{St}_{GL(2)}, \sigma)).$$

Now suppose $\chi_1 \otimes \chi_2 \otimes \chi_3$ is not regular and that $\chi_1 \times \chi_2 \rtimes \chi_3$ is not irreducible. Then $\chi_1 \otimes \chi_2 \otimes \chi_3$ is associated to a character of the form $\nu \otimes 1_{F^\times} \otimes \sigma$, or $\nu \otimes \nu \otimes \sigma$, or $\nu^{1/2}\xi \otimes \nu^{-1/2}\xi \otimes \sigma$, with ξ of order two. The following are the results of Lemmas 3.7, 3.8, and 3.9 of [20].

(1) Suppose that ξ is of order two. Then we have

$$\nu^{1/2}\xi \times \nu^{-1/2}\xi \rtimes \sigma = \xi\mathrm{St}_{GL(2)} \rtimes \sigma + \xi 1_{GL(2)} \rtimes \sigma$$

in $R(G)$. Both representations on the right-hand side are irreducible, and $\xi\mathrm{St}_{GL(2)} \rtimes \sigma$ is generic.

(2) We have in $R(G)$

$$\nu \times \nu \rtimes \nu^{-1/2}\sigma = \nu \rtimes \sigma\mathrm{St}_{GSp(2)} + \nu \rtimes \sigma 1_{GSp(2)}.$$

Both representations are irreducible, and $\nu \rtimes \sigma\mathrm{St}_{GSp(2)}$ is generic.

(3) We have in $R(G)$

$$\begin{aligned} \nu \times 1_{F^\times} \rtimes \nu^{-1/2}\sigma &= \nu^{1/2}\mathrm{St}_{GL(2)} \rtimes \nu^{-1/2}\sigma + \nu^{1/2} 1_{GL(2)} \rtimes \nu^{-1/2}\sigma \\ &= 1_{GL(2)} \times \nu \rtimes \nu^{-1/2}\sigma \\ &= 1_{F^\times} \rtimes \sigma\mathrm{St}_{GSp(2)} + 1_{F^\times} \rtimes \sigma 1_{GSp(2)}. \end{aligned}$$

The representations $1_{F^\times} \rtimes \sigma\mathrm{St}_{GSp(2)}$ and $\nu^{1/2}\mathrm{St}_{GL(2)} \rtimes \nu^{-1/2}\sigma$ (resp. $\nu^{1/2}1_{GL(2)} \rtimes \nu^{-1/2}\sigma$) have exactly one irreducible sub-quotient in common. That sub-quotient is essentially tempered and it will be denoted by $\tau(S, \nu^{-1/2}\sigma)$ (resp. $\tau(T, \nu^{-1/2}\sigma)$). These two essentially tempered representations are not equivalent. We have in $R(G)$

$$\nu^{1/2}\mathrm{St}_{GL(2)} \rtimes \nu^{-1/2}\sigma = \tau(S, \nu^{-1/2}\sigma) + L((\nu^{1/2}\mathrm{St}_{GL(2)}, \nu^{-1/2}\sigma)).$$

The representation $\tau(S, \nu^{-1/2}\sigma)$ is generic.

4.2. In this section we will prove the following theorem:

THEOREM 4.1.

(a) *If* $\chi_1 \times \chi_2 \rtimes \chi_3$ *is irreducible, then*

$$L(s, \chi_1 \times \chi_2 \rtimes \chi_3) = L(s, \chi_3)L(s, \chi_1\chi_3)L(s, \chi_2\chi_3)L(s, \chi_1\chi_2\chi_3).$$

(b) *Suppose* $\chi \notin \{\nu^{\pm 1/2}\xi, \nu^{\pm 3/2}\}$, *for any* ξ *of order two. Then* $\chi St_{GL(2)} \rtimes \sigma$ *is irreducible generic, and*

$$L(s, \chi St_{GL(2)} \rtimes \sigma) = L(s,\sigma)L(s,\sigma\chi^2)L\left(s+\frac{1}{2},\sigma\chi\right).$$

(c) *Suppose* $\chi \notin \{1_{F^\times}, \nu^{\pm 2}\}$. *Then* $\chi \rtimes \sigma St_{GSp(2)}$ *is irreducible generic, and*

$$L(s, \chi \rtimes \sigma St_{GSp(2)}) = L(s+\frac{1}{2},\sigma\chi)L\left(s+\frac{1}{2},\sigma\right).$$

(d) $L(s, St_{GSp(4)}) = L(s+\frac{3}{2}, 1_{F^\times})$.

(e) *For* ξ *of order two,*

$$L(s, \delta([\xi, \nu\xi], \sigma)) = L(s+1,\sigma)L(s+1,\sigma\xi).$$

(f) $L(s, \tau(S,\sigma)) = L(s+1,\sigma)^2$.

Proof. Let $\chi = \chi_1 \times \chi_2 \rtimes \chi_3$. Also let $\hat{a} = \begin{pmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{pmatrix}$.

Step 1. Suppose $\chi_1 \otimes \chi_2 \otimes \chi_3$ is regular. We have seen before that for $|a|$ small:

$$W(\chi, f, \hat{a}) = \chi_3(a)|a|^{3/2}(\lambda^1_\chi(f) + \lambda^2_\chi(f)\chi_1(a) + \lambda^3_\chi(f)\chi_2(a) + \lambda^4_\chi(f)\chi_1\chi_2(a)),$$

where λ^i_χ are functionals in $\chi^*_{N,\bar{\theta}}$. Most of this section is devoted to the careful analysis of the functionals λ^i_χ. By applying various intertwining operators, we get the following identities:

$$\begin{aligned} \lambda^2_\chi &= C(\chi, w_2w_1)\lambda^1_{w_2w_1\chi} \circ A(\chi, w_2w_1), \\ \lambda^3_\chi &= C(\chi, w_2)\lambda^1_{w_2\chi} \circ A(\chi, w_2), \end{aligned}$$

and

$$\lambda^4_\chi = C(\chi, w_2w_1w_2)\lambda^1_{w_2w_1w_2\chi} \circ A(\chi, w_2w_1w_2).$$

Also for f with support in the open cell we have:

$$\lambda^1_\chi(f) = \int_N f(w_l n)\bar{\theta}^{-1}(n)\, dn.$$

Step 2. We will now give alternative descriptions for the functionals λ^i_χ. We still assume that $\chi_1 \otimes \chi_2 \otimes \chi_3$ is regular.

For any Weyl element w, define the following functional:

$$\lambda^w_\chi(f) = \int_{N_w} f(w^{-1}n)\bar{\theta}(n^{-1})\, dn, \tag{6}$$

where N_w is the usual $N \cap w\bar{N}w^{-1}$. These functionals have the following invariance properties:

$$\begin{aligned}
\lambda^{w_1}_\chi(\chi(\hat{a})f) &= \chi_1\chi_2\chi_3(a)|a|^{3/2}\lambda^{w_1}_\chi(f),\\
\lambda^{w_1w_2}_\chi(\chi(\hat{a})f) &= \chi_1\chi_3(a)|a|^{3/2}\lambda^{w_1w_2}_\chi(f),\\
\lambda^{w_1w_2w_1}_\chi(\chi(\hat{a})f) &= \chi_2\chi_3(a)|a|^{3/2}\lambda^{w_1w_2w_1}_\chi(f),
\end{aligned}$$

and

$$\lambda^{w_1w_2w_1w_2}_\chi(\chi(\hat{a})f) = \chi_3(a)|a|^{3/2}\lambda^{w_1w_2w_1w_2}_\chi(f).$$

One notes that $\lambda^{w_1w_2w_1w_2}_\chi$ is the same as λ^1_χ. Furthermore

LEMMA 4.2. *One has the following identities:*

$$\lambda^{w_1w_2}_\chi = \lambda^{w_1}_{w_2\chi} \circ A(\chi, w_2), \tag{7}$$

$$\lambda^{w_1w_2w_1}_\chi = \lambda^{w_1w_2}_{w_1\chi} \circ A(\chi, w_1), \tag{8}$$

and

$$\lambda^{w_1w_2w_1w_2}_\chi = \lambda^{w_1w_2w_1}_{w_2\chi} \circ A(\chi, w_2). \tag{9}$$

Proof. Straightforward. □

As dim $\chi^*_{N,\bar{\theta}} \leq 4$, these identities combined with those of Step 1 imply that for w in $\{w_2, w_2w_1, w_2w_1w_2\}$ there exist scalars $A_{w_lw}(\chi)$ such that the following hold:

$$\lambda^{w_lw}_\chi = A_{w_lw}(\chi)\lambda^1_{w\chi} \circ A(\chi, w),$$

We have the following lemma:

LEMMA 4.3. *For w as above*

$$A_{w_lw}(\chi) = C(\chi, w)C(w\chi, w^{-1}).$$

Proof. The idea is to use the the following equation [22]:

$$A(w\pi, w^{-1}) \circ A(\pi, w) = C(\pi, w)^{-1}C(w\pi, w^{-1})^{-1}.$$

Let, for example, $w = w_2$. Then by Lemma 4.2

$$\begin{aligned}\lambda^1_{w_2\chi} \circ A(\chi, w_2) &= \lambda^{w_1w_2w_1w_2}_{w_2\chi} \circ A(\chi, w_2)\\ &= \lambda^{w_1w_2w_1}_{w_2w_2\chi} \circ A(w_2\chi, w_2) \circ A(\chi, w_2)\\ &= C(\chi, w_2)^{-1} C(w_2\chi, w_2)^{-1} \lambda^{w_1w_2w_1}_{\chi}.\end{aligned}$$

The same proof works for other w. □

We have then proven the following proposition:

PROPOSITION 4.4. *In the regular case, there exists $\delta = \delta(f)$ such that for all a with $|a| < \delta$ the following holds:*

$$\begin{aligned}W(\chi, f, \hat{a}) &= \frac{|a|^{3/2}\chi_1\chi_2\chi_3(a)}{C(w_2w_1w_2\chi, w_2w_1w_2)}\lambda^{w_1}_{\chi}(f) + \frac{|a|^{3/2}\chi_1\chi_3(a)}{C(w_2w_1\chi, w_1w_2)}\lambda^{w_1w_2}_{\chi}(f)\\ &\quad + \frac{|a|^{3/2}\chi_2\chi_3(a)}{C(w_2\chi, w_2)}\lambda^{w_1w_2w_1}_{\chi}(f) + |a|^{3/2}\chi_3(a)\lambda^{w_1w_2w_1w_2}_{\chi}(f).\end{aligned}$$ □

Step 3. Now we would like to extend our results form the previous sections to the nonregular case. Let us first fix some notations. Let $\chi(s)$ denote the representation $\chi_1 \upsilon^{4s} \times \chi_2 \upsilon^{2s} \rtimes \chi_3 \upsilon^{-3s}$. For $f \in Ind(\chi_1 \otimes \chi_2 \otimes \chi_3 | B \cap K, K)$ define the function f_s by the following:

$$f_s(bk) = \delta_B(b)^{s+\frac{1}{2}}(\chi_1 \otimes \chi_2 \otimes \chi_3)(b) f(k).$$

f_s is a well-defined function on G and it belongs to $\chi(s)$. The idea is the following. The representations $\chi(s)$ are regular. Our first task is to give a description of the meromorphy of the the complex functions $g_w(s) = \lambda^w_{\chi(s)}(f_s)$. We have the following proposition:

PROPOSITION 4.5. *$g_w(s)$ is a rational function of $X = q^{-s}$. Furthermore $g_{w_1}(s)$, $\frac{g_{w_1w_2}(s)}{L(2s,\chi_2)}$, $\frac{g_{w_1w_2w_1}(s)}{L(4s,\chi_1)L(2s,\chi_1\chi_2^{-1})}$, and $\frac{g_{w_1w_2w_1w_2}(s)}{L(2s,\chi_2)L(6s,\chi_1\chi_2)L(4s,\chi_1)}$ are polynomials of X and X^{-1}.*

We will denote these polynomials by P_1, P_2, P_3, and P_4 in the order they appear.

Proof. First the function g_{w_1}:

$$\lambda^{w_1}_{\chi(s)}(f_s) = \int_F \psi(-x) f_s \left[w_1 \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \right] dx$$

$$= \int_{\mathcal{O}} f\left[w_1\begin{pmatrix}1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1\end{pmatrix}\right] dx$$

$$+\chi_1(-1)\int_{\mathfrak{p}}\psi(-x^{-1})\chi_1(x)\chi_2(x^{-1})|x|^{2s}f\left[\begin{pmatrix}1 & & & \\ x & 1 & & \\ & & 1 & \\ & & -x & 1\end{pmatrix}\right]\frac{dx}{|x|}.$$

Choose $M > 1$ large enough so that $f\begin{pmatrix}1 & & & \\ x & 1 & & \\ & & 1 & \\ & & -x & 1\end{pmatrix} = f(e)$ for all $x \in \pi^k\mathcal{U}$ and $k > M$. Then we get the following as the final expression:

$$\lambda^{w_1}_{\chi(s)}(f_s) = \int_{\mathcal{O}} f\left[w_1\begin{pmatrix}1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1\end{pmatrix}\right] dx$$

$$+\chi_1(-1)\sum_{j=1}^{M}\chi_1(\pi^j)\chi_2(\pi^{-j})q^{-2ks}\int_{\pi^j\mathcal{U}}\psi(-x^{-1})f\begin{pmatrix}1 & & & \\ x & 1 & & \\ & & 1 & \\ & & -x & 1\end{pmatrix}\frac{dx}{|x|},$$

which is a polynomial. Terms corresponding to $j > M$ will not appear because

$$\int_{\pi^{-j}\mathcal{U}}\psi(x)\,dx = 0,$$

when $j > 1$. To prove the other assertions of the proposition, we use the fact that the rest of the functionals can be written as compositions of λ^{w_1} and $GL(2)$ intertwining operators as in Lemma 4.2:

$$\lambda^{w_1w_2}_{\chi(s)} = \lambda^{w_1}_{w_2\chi(s)} \circ A(\chi(s), w_2),$$
$$\lambda^{w_1w_2w_1}_{\chi(s)} = \lambda^{w_1w_2}_{w_1\chi(s)} \circ A(\chi(s), w_1),$$

and

$$\lambda^{w_1w_2w_1w_2}_{\chi(s)} = \lambda^{w_1w_2w_1}_{w_2\chi(s)} \circ A(\chi(s), w_2).$$

Now we proceed as follows. To prove the proposition, for $\lambda^{w_1w_2}$, we notice that, since λ^{w_1} is a polynomial, the poles of $\lambda^{w_1w_2}$ are among the poles of $A(\chi(s), w_2)$. The poles of this intertwining operator in turn are among those of $L(2s, \chi_2)$. To see this, we perform the following computation

$$
\begin{aligned}
A(\chi(s), w_2)f_s(e) &= \int_F f_s\left(\begin{pmatrix}1&&&\\&&1&\\&-1&&\\&&&1\end{pmatrix}\begin{pmatrix}1&&&\\&1&t&\\&&1&\\&&&1\end{pmatrix}\right)dt\\
&= \int_{\mathcal{O}} f\left(\begin{pmatrix}1&&&\\&&1&\\&-1&&\\&&&1\end{pmatrix}\begin{pmatrix}1&&&\\&1&t&\\&&1&\\&&&1\end{pmatrix}\right)dt\\
&\quad+\int_{F\setminus\mathcal{O}} f_s\left(\begin{pmatrix}1&&&\\&t^{-1}&-1&\\&&t&\\&&&1\end{pmatrix}\begin{pmatrix}1&&&\\&1&&\\&t^{-1}&1&\\&&&1\end{pmatrix}\right)dt\\
&= \int_{\mathcal{O}}+\int_{\mathfrak{p}}|t|^{2s}\chi_2(t)f_s\left(\begin{pmatrix}1&&&\\&1&&\\&t&1&\\&&&1\end{pmatrix}\right)\frac{dt}{|t|}\\
&= \text{polynomial}+\text{polynomial}.L(2s,\chi_2).
\end{aligned}
$$

A similar computation proves that the poles of $A(\chi(s), w_1)$ are among the poles of $L(s, \chi_1\chi_2^{-1})$. Repeating the same argument proves the rest of the assertions of the proposition. □

Combining everything that we have proven so far including those in the appendix and the uniformity proposition gives the following result:

PROPOSITION 4.6. *There exist $\delta = \delta(f)$ such that for all a with $|a| < \delta$ the following holds:*

$$
\begin{aligned}
&W(\chi(s), f_s, \hat{a})\\
&= |a|^{3/2}\chi_3(a)\Big[\chi_1\chi_2(a)\chi_2(-1)C(\chi_1^{-1}v^{-4s})C(\chi_2^{-1}v^{-2s})\\
&\qquad\qquad C(\chi_1^{-1}\chi_2^{-1}v^{-6s})P_1(f,s)|a|^{3s}\\
&\quad+\chi_1(a)\chi_2(-1)C(\chi_1^{-1}v^{-4s})C(\chi_1^{-1}\chi_2 v^{-2s})L(2s,\chi_2)P_2(f,s)|a|^{s}\\
&\quad+\chi_2(a)C(\chi_2^{-1}v^{-2s})L(4s,\chi_1)L(2s,\chi_1\chi_2^{-1})P_3(f,s)|a|^{-s}\\
&\quad+L(2s,\chi_2)L(6s,\chi_1\chi_2)L(4s,\chi_1)P_4(f,s)|a|^{-3s}\Big]
\end{aligned}
$$
□

Now we would like to use Proposition 4.6 to compute the gcd for irregular cases. For the time being we assume there are no reducibilities. There are three cases to deal with:

(1) $\chi_1 = 1$, and $\chi_2 \neq 1$.

(2) $\chi_1\chi_2 = 1$, and $\chi_1 \neq 1$.

(3) $\chi_1 = 1$ and $\chi_2 = 1$.

We first need a proposition. In what follows we will use q^{-s} and X interchangeably:

PROPOSITION 4.7. *Let*

$$P_i(X,f) = \sum_{j=0}^{\infty} a_j^i(f)(X-1)^j,$$

and also let η_i be the germ associated with P_i at $s = 0$. Then $\{a_1^i(f)\}_{i=1}^4$ generates a one dimensional space. Also we have the following relations:

$$a_j^i(\chi(\hat{a})f) = \eta_i(a) \sum_{l=0}^{j} \binom{(5-2i)\log_q |a|}{l} a_{j-l}^i(f).$$

Notice that we have not included the negative powers of $X-1$ for the simple reason that $\frac{1}{X}$ is holomorphic at $X = 1$!

Proof. Let Re $s > 0$.

$$\begin{aligned} P_i(X, \chi(\hat{a})f) &= \eta_i(a)|a|^{2i-5} P_i(X,f) \\ &= \eta_i(a) X^{(5-2i)\log_q |a|} P_i(X,f) \\ &= \eta_i(a)(1 + (X-1))^{(5-2i)\log_q |a|} P_i(X,f). \end{aligned}$$

Now a simple application of the binomial theorem proves the proposition. □

1. $\chi_1 = 1$, and $\chi_2 \neq 1$. In this case Proposition 4.6 implies that $W(\chi(s), f_s, \hat{a})$ is the sum of the following two expressions for $|a|$ small:

$$\frac{\chi_2(a)}{X^4-1}\Big[\chi_2(-1)(1-q^{-1}X^4)X^4 C(\chi_2^{-1}\upsilon^{-2s})C(\chi_2^{-1}\upsilon^{-6s})P_1(X)X^{3r} - C(\chi_2^{-1}\upsilon^{-2s})L(2s,\chi_2^{-1})P_3(X)X^{-r}\Big],$$

and

$$\frac{1}{X^4-1}\Big[\chi_2(-1)(1-q^{-1}X^4)X^4 C(\chi_2\upsilon^{-2s})L(2s,\chi_2)P_2(X)X^{r} - L(2s,\chi_2)L(6s,\chi_2)P_4(X)X^{-3r}\Big].$$

The expressions inside the brackets must vanish for $X = 1$. This is because according to [22] Whittaker functions are holomorphic with respect to the variable s:

$$P_3(1) = \frac{\chi_2(-1)(1-q^{-1})C(\chi_2^{-1})P_1(1)}{L(0,\chi_2^{-1})},$$

and

$$P_4(1) = \frac{\chi_2(-1)(1-q^{-1})C(\chi_2)P_2(1)}{L(0,\chi_2)}.$$

Now we compute the limit when X approaches 1 by using l'Hôpital's rule. It follows that to prove the appearance of the terms $\log_q|a|$ and $log_q|a|\chi_2(a)$ in the asymptotic expansion, we just need to find functions f in $1 \times \chi_2 \rtimes 1$ such that $P_1(f,s) \neq 0$ and $P_2(f,0) \neq 0$. These are both obvious. So in this case:

$$L(s, 1 \times \chi_2 \rtimes \chi_3) = L(s,\chi_3)^2 L(s,\chi_2\chi_3)^2.$$

2. $\chi_1\chi_2 = 1$, and $\chi_1 \neq 1$. In this case, to prove the existence of the term $\chi_1\chi_3$ and $\chi_2\chi_3$ we need to prove the existence of functions f such that $P_2(1) \neq 0$ and $P_3(1) \neq 0$. We also need to find f such that $P_1(f) \neq 0$. Both assertions are obvious. Thus we have proven in this case that we get the following identity:

$$L(s,\chi_1 \times \chi_1^{-1} \rtimes \chi_3) = L(s,\chi_3)^2 L(s,\chi_1\chi_3)L(s,\chi_2\chi_3).$$

3. $\chi_1 = 1$ and $\chi_2 = 1$. It is easily seen that

$$\lim_{s\to 0} W(\chi(s), f_s, \hat{a}) = \frac{1}{6}(A(f) + B(f)r + C(f)r^2 + D(f)r^3),$$

for certain functionals that appear as coefficients. We know from the open cell that for certain f the functional $A(f)$ is nonzero. Computations using MAPLE V show that:

$$\begin{aligned} D &= \text{a nonzero contant } .P_1(1),\\ C &= 18(1-q^{-1})(32-49q^{-1}+89q^{-2})P_1(1) + 144(1-q^{-1})^3P_1'(1)\\ &\quad - 144(1-q^{-1})P_2'(1), \end{aligned}$$

and finally

$$\begin{aligned} B = {} & 6(1-q^{-1})(88-13q^{-1}+233q^{-2})P_1(1)\\ & - 72(1-q^{-1})^2(-4+9q^{-1})P_1'(1)\\ & + 288(1-q^{-1})(-1+2q^{-1})P_2'(1)\\ & + 18(1-q^{-1})P_1''(1)\\ & - 36(1-q^{-1})P_2''(1) + 18(1-q^{-1})P_3''(1). \end{aligned}$$

It now follows from the proposition above that these coefficients are linearly independent, if nonzero. D is nonzero, because it is the Whittaker functional of the induction through the Siegel parabolic subgroup. Suppose C is identically zero. Then

$$\begin{aligned} C(\chi(\hat{a})f) &= \alpha P_1(1,\chi(\hat{a})f) + \beta P_1^{'}(1,\chi(\hat{a})f) + \gamma P_2^{'}(1,\chi(\hat{a})f) \\ &= C(f) + -3\beta \log_q |a| P_1(1,f) - r\gamma \log_q |a| P_2(1,f) \\ &= -3\beta \log_q |a| P_1(1,f) - r\gamma \log_q |a| P_2(1,f). \end{aligned}$$

Now a straightforward computation shows that this can never happen. The same argument proves the nonvanishing of B. This proves that in this case

$$L(s, 1 \times 1 \rtimes \chi_3) = L(s,\chi_3)^4.$$

This finishes the proof of the first part of the theorem. Parts (b) through (f) of the theorem follow from the classification lemmas, Lemma 4.2, and similar computations as above. □

5. Local computations for the Siegel parabolic. Let σ be a supercuspidal representation of $GL(2)$, and χ a quasi-character of $GL(1)$. We would like to prove the following theorem:

THEOREM 5.1.

(a) *If $\sigma \rtimes \chi$ is irreducible,*

$$L(s, \sigma \rtimes \chi) = L(s,\chi).L(s,\chi.\omega_\sigma). \tag{10}$$

(b) *If $\sigma = \sigma_0.\upsilon^{1/2}$, and $\omega_{\sigma_0} = 1$, then the unique irreducible quotient of $\sigma \rtimes \chi$ admits $L(s,\chi.\omega_\sigma)$ as its L-functions.*

(c) *If $\sigma = \sigma_0.\upsilon^{-1/2}$, and $\omega_{\sigma_0} = 1$, then the unique irreducible sub-representation of $\sigma \rtimes \chi$ admits $L(s,\chi)$ as its L-functions.*

The conditions in parts (b) and (c) of the theorem come from Shahidi's classification of representations supported in the Siegel parabolic subgroup [23] .

Proof. Let $\pi = \sigma \rtimes \chi$. The proof is divided into two steps.

Step 1. We know from [22] that the Whittaker functional for the representation π is given by the following:

$$\lambda_\pi(f) = \int_N f\left[\begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix} n; e\right] \theta^{-1}(n)\, dn.$$

The integral is over the unipotent radical of the Siegel parabolic subgroup. When $\omega_\sigma \neq 1$ this in particular implies that:

$$\lambda_\pi\left(\pi\begin{pmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1\end{pmatrix} f\right) = \chi(a)|a|^{3/2}\lambda_\pi^1(f) + \chi(a)\omega_\sigma(a)|a|^{3/2}\lambda_\pi^2(f),$$

for certain functionals λ^1 and λ^2, when $|a| < \delta(f)$. Also it can be seen that:

$$\lambda_\pi^1(f) = \int_N f(w^{-1}n; e)\, dn,$$

when support of f is in the open cell. By applying the long intertwining operator to the above identity we get the following:

$$W(\pi, f, \hat{a}) = \lambda_\pi^1(f)\chi(a)|a|^{3/2} + C(\pi, w)\lambda_{\tilde{w}\pi}^1(A(\pi, w)f)\chi(a)\omega_\sigma(a)|a|^{3/2},$$

where $\hat{a}$ is the obvious matrix element. Now define a new functional by the following identity:

$$\tilde{\lambda_\pi}(f) = f(e; e).$$

This functional satisfies

$$\tilde{\lambda_\pi}(\pi(\hat{a})f) = \chi(a)\omega_\sigma(a)|a|^{3/2}\tilde{\lambda_\pi}(f).$$

It follows that there is a number D depending only on π such that

$$\lambda_{\tilde{w}\pi}^1(A(\pi, w)f) = D\tilde{\lambda_\pi}(f).$$

This is because the functional λ_π^1 has a different asymptotic behavior, and that dim $\pi^*_{N,\bar{\theta}} \leq 2$.

Claim 1. $D = C(\pi, w)^{-1}C(\tilde{w}\pi, w^{-1})^{-1}$

Proof of the claim. Look at the open cell! □

This gives the following identity:

$$W(\pi, f, \hat{a}) = \lambda_\pi^1(f)\chi(a)|a|^{3/2} + C(\tilde{w}\pi, w^{-1})^{-1}f(e; e)\chi(a)\omega_\sigma(a)|a|^{3/2}$$

for $|a| < \delta(f)$. As λ_π^1 and $\tilde{\lambda}_\pi$ are linearly independent, this gives the result for the cases when $\omega_\sigma \neq 1$.

Step 2. Now let $\omega_\sigma = 1$. Let π_s denote the representation $\sigma.\nu^s \rtimes \chi.\nu^{-s}$. For $f \in Ind(\sigma \times \chi | P \cap K, K)$ define the function f_s by the following:

$$f_s(pk) = \delta_P(p)^{\frac{s}{3}+\frac{1}{2}}(\sigma \otimes \chi)(p)f(k).$$

It's now obvious that f_s is a well-defined function on G and it belongs to the space of π_s. Now we have the following identity:

$$W(\pi_s, f_s, \hat{a}) = \lambda^1_{\pi_s}(f_s)\chi(a)|a|^{\frac{3}{2}-s} + C(\tilde{w}\pi_s, w^{-1})^{-1}f(e;e)\chi(a)|a|^{\frac{3}{2}+s}$$

for $|a| < \delta(f, s)$. We showed in Proposition 3.9 that this expansion is uniform in s, i.e. $\delta(f, s) = \delta(f)$. From [24] and the note at the end of the Appendix:

$$C(\tilde{w}\pi_s, w^{-1})^{-1} = Aq^{-ns}\frac{1 - q^{-(1+2s)}}{1 - q^{2s}}.$$

We also know from [22] that the left-hand side of the equation above has an analytic continuation to an entire function on the whole complex plane. Thus $\lambda^1_{\pi_s}(f_s)$ must have a pole of order 1. Let $X = q^{-s}$, and write the power series expansion of $\lambda^1_{\pi_s}(f_s)$ as the following:

$$\lambda^1_{\pi_s}(f_s) = \frac{a_{-1}}{X - 1} + a_0 + a_1(X - 1) + a_2(X - 1)^2 + \cdots.$$

Then we have the following for $|a| < \delta(f)$:

(11) $\lim_{s\to 0^+} W(\pi_s, f_s, \hat{a})$

(12) $$= |a|^{3/2}\chi(a)\left[2a_{-1}\log_q |a| + a_0 + \frac{-3 + 7q^{-1} + 2n - 2nq^{-1}}{2(1 - q^{-1})}a_{-1}\right].$$

This is because we want the poles to cancel out. This is guaranteed by

$$a_{-1}(f) = -\frac{1}{2}A(1 - q^{-1})f(e;e).$$

This in particular implies that a_{-1} is not identically zero. Now we have the following lemma which proves that a_0 and a_1 are linearly independent.

LEMMA 5.2. *a_0 and a_{-1} satisfy*

$$a_{-1}(\pi(\hat{a})f) = \chi(a)|a|^{\frac{3}{2}}a_{-1}(f),$$
$$a_0(\pi(\hat{a})f) = \chi(a)|a|^{\frac{3}{2}}a_0(f) + \log_q |a|\chi(a)|a|^{\frac{3}{2}}a_{-1}(f).$$

Proof. This follows from the invariance equation for λ_π and the binomial theorem. □

This finishes that proof of the theorem. □

6. Appendix: computation of local coefficients. Here we will present explicit computations of the local coefficients that appeared in Propositions 4.3 above. We refer the reader to Section 3 of [22] for preliminaries on local coefficients. Our computations are motivated by those of [14] and [15]. More general results can be found in [23] and [24]. Recall the definition of local coefficients:

$$\lambda_\pi = C(\pi, w)\lambda_{\tilde{w}\pi} A(\pi, w).$$

Here we have assumed that π is induced, and $A(\pi, w)$ is an intertwining integral. We know that for every representation π and Weyl elements w and w'

$$C(\pi, ww') = C(\pi, w')C(w'\pi, w),$$

provided $l(w)+l(w') = l(ww')$. In our special case, this implies that we only need to compute the local coefficients for w_1 and w_2. Define a function $\gamma\colon G \longrightarrow \mathbb{C}$ by

$$\gamma(g) = \begin{cases} \delta_B^{1/2}(b)(\chi_1 \otimes \chi_2 \otimes \chi_3)(b)\mathrm{ch}_{N(\mathcal{O})}(n) & \text{if } g \in Bw_l^{-1}N, \text{ and } g = bw_l^{-1}n, \\ 0 & \text{Otherwise.} \end{cases}$$

It is obvious that $\gamma \in \chi$. Also $\lambda_\chi(\gamma) = 1$. This implies that

$$C(\chi, w)^{-1} = \lambda_{w\chi} \circ A(\chi, w)(\gamma).$$

We will now explicitly compute the right-hand side. For a quasi-character η of $F^\times$, define a function Φ_η on F by the following:

$$\Phi_\eta(x) = \int_{F^\times} \eta(y)\mathrm{ch}_{\mathcal{O}}(y+x)\frac{dy}{|y|}.$$

We let

$$\delta(\eta) = \begin{cases} 1 & \eta \text{ unramified,} \\ 0 & \text{otherwise.} \end{cases}$$

Also for unramified η

$$L(\eta) = \frac{1}{1-\eta(\pi)}.$$

A straightforward computation shows that

$$\Phi_\eta(x) = \begin{cases} (1-\frac{1}{q})\delta(\eta)L(\eta) & \text{if } x \in \mathcal{O}, \\ 0 & \text{if } -c(\eta) < \operatorname{ord} x < 0, \\ \eta(-x)|x|^{-1} & \text{otherwise.} \end{cases} \tag{13}$$

Here $c(\eta)$ is the conductor of the quasi-character η.

LEMMA 6.1.

$$\lim_{k\to\infty} \int_{\mathfrak{p}^{-k}} \Phi_\eta(x)\theta(-x)\,dx = \begin{cases} \frac{1-q^{-1}\eta(\pi^{-1})}{1-\eta(\pi)} & \eta \text{ unramified,} \\ \eta(-\pi^{c(\eta)}) \int_{\mathcal{U}} \psi(-\pi^{-c(\eta)}u)\chi(u)\,du & \eta \text{ ramified.} \end{cases}$$

Proof. The assertion follows from (13) above and Lemma 1 of [21]. □

Note 6.2. The limit is equal to Tate's invariant factor! (cf. p. 291 of [21], also [24] and [15])

I denote the limit appearing in the lemma by $C(\eta)$. Now we can compute the local coefficients.

1. w_1.

$$\begin{aligned}
&\lambda_{w_1\chi} \circ A(\chi, w_1)(\gamma) \\
&= \lim_{k\to\infty} \int_{N(\mathfrak{p}^{-k})} A(\chi, w_1)(\gamma)(w_l^{-1}n)\theta^{-1}(n)\,dn \\
&= \lim_{k\to\infty} \int_{N(\mathfrak{p}^{-k})} \int_F \gamma\left(w_1^{-1} \begin{pmatrix} 1 & y & & \\ & 1 & & \\ & & 1 & -y \\ & & & 1 \end{pmatrix} w_l^{-1} n \right) \theta^{-1}(n)\,dy\,dn \\
&= \lim_{k\to\infty} \int_{N(\mathfrak{p}^{-k})} \int_F \gamma\left(\begin{pmatrix} -y^{-1} & 1 & & \\ & y & & \\ & & y^{-1} & 1 \\ & & & -y \end{pmatrix} w_l^{-1} \begin{pmatrix} 1 & y^{-1} & & \\ & 1 & & \\ & & 1 & -y^{-1} \\ & & & 1 \end{pmatrix} n \right) \theta^{-1}(n)\,dy\,dn \\
&= \chi_1(-1) \lim_{k\to\infty} \int_{\mathfrak{p}^{-k}} \Phi_{\chi_1\chi_2^{-1}}(x)\theta(-x)\,dx \\
&= \chi_1(-1) C(\chi_1\chi_2^{-1}).
\end{aligned}$$

2. w_2. Similar to the case of w_1 we get the following:

$$\begin{aligned}\lambda_{w_1\chi} \circ A(\chi, w_1)(\gamma) &= \lim_{k\to\infty} \int_{\mathfrak{p}^{-k}} \Phi_{\chi_2}(x)\theta(-x)\,dx \\ &= C(\chi_2).\end{aligned}$$

Now the following proposition is immediate:

PROPOSITION 6.3.

$$C(w_2\chi, w_2)^{-1} = C(\chi_2^{-1}),$$
$$C(w_2w_1\chi, w_1w_2)^{-1} = \chi_2(-1)C(\chi_1^{-1})C(\chi_1^{-1}\chi_2),$$

and

$$C(w_2w_1w_2\chi, w_2w_1w_2)^{-1} = \chi_2(-1)C(\chi_1^{-1})C(\chi_2^{-1})C(\chi_1^{-1}\chi_2^{-1}) \square$$

Note 6.4 It is proved in [22] with notations therein that for unitary σ

$$C(\tilde{w}\upsilon, \tilde{w}\sigma, w^{-1}) = \overline{C(-\bar{\upsilon}, \sigma, w)}.$$

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