Chapter 2

Limits of Sequences

Calculus Student: $\lim_{n\to\infty} s_n = 0$ means the s_n are getting closer and closer to zero but never gets there.

Instructor: *ARGHHHHH!*

Exercise 2.1 Think of a better response for the instructor. In particular, provide a counterexample: find a sequence of numbers that 'are getting closer and closer to zero' but aren't really getting close at all. What about the 'never gets there' part? Should it be necessary that sequence values are never equal to its limit?

2.1 Definition and examples

We are going to discuss what it means for a sequence to converge in three stages:

First, we define what it means for a sequence to converge to zero

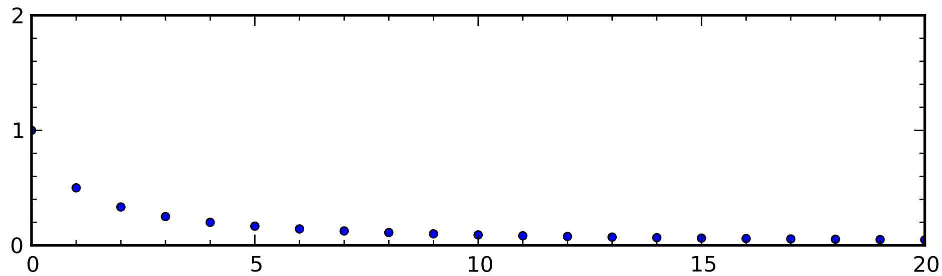
Then we define what it means for sequence to converge to an arbitrary real number.

Finally, we discuss the various ways a sequence may diverge (not converge).

In between we will apply what we learn to further our understanding of real numbers and to develop tools that are useful for proving the important theorems of Calculus.

Recall that a sequence is a function whose domain is $\mathbb{Z}^+$ or $\mathbb{Z}^{\geq}$. A sequence is most usually denoted with subscript notation rather than standard function notation, that is we write s_n rather than $s(n)$. See Section 0.3.2 for more about definitions and notations used in describing sequences.

Figure 2.1: $s_n = \dfrac{1}{n}$.



2.1.1 Sequences converging to zero.

Definition We say that the sequence s_n *converges to* 0 whenever the following hold:

For all $\epsilon > 0$, there exists a real number, N, such that

$$n > N \implies |s_n| < \epsilon.$$

Notation To state that s_n converges to 0 we write $\lim_{n\to\infty} s_n = 0$ or $s_n \to 0$.

Example 2.1 $\lim_{n\to\infty} \dfrac{1}{n} = 0$. See the graph in Figure 2.1.

Proof. Given any $\epsilon > 0$, use Archimedes Principle, *Theorem* 1.51, to find an N, such that $\frac{1}{N} < \epsilon$. Note that, if $n > N$, then $\frac{1}{n} < \frac{1}{N}$ (Exercise 1.10 d). Now, if $n > N$, we have

$$|s_n| = \frac{1}{n} < \frac{1}{N} < \epsilon.$$

In short:

$$n > N \implies |s_n| < \epsilon,$$

so we have shown that $\lim_{n\to\infty} \dfrac{1}{n} = 0$. □

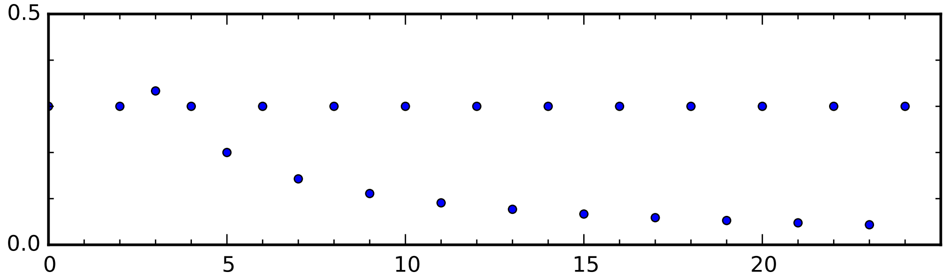
Example 2.2 If $s_n = 0$, for all n, then $\lim_{n\to\infty} s_n = 0$

Proof. Given any $\epsilon > 0$, let N be any number. Then we have

$$n > N \implies |s_n| = 0 < \epsilon,$$

because that's true for any n. □

Figure 2.2: Some values approach 0, but others don't.



Example 2.3 Why isn't the following a good definition?

" $\lim_{n\to\infty} s_n = 0$ means

For all $\epsilon > 0$*, there exists a positive integer,* N*, such that* $|s_N| < \epsilon$."

The problem is we want the sequence to get arbitrarily close to zero and to stay close. Consider the sequence:

$$s_n = \begin{cases} \frac{1}{n}, & \text{if } n \text{ is odd} \\ 0.3, & \text{otherwise.} \end{cases}$$

For any ϵ there is always an odd n with s_n less than ϵ but there there are also many even n's with values far from zero. The '$n > N$'example is an important part of the definition. See the graph in Figure 2.2.

Exercise 2.2 Prove that $\lim_{n\to\infty} \dfrac{3}{n} = 0$

Exercise 2.3 Prove that $\lim_{n\to\infty} \dfrac{1}{n^2} = 0$

Exercise 2.4 Prove that $\lim_{n\to\infty} \dfrac{(-1)^n}{n} = 0$ See Figure 2.3.

Exercise 2.5 Prove that $\lim_{n\to\infty} \dfrac{1}{n(n-1)} = 0.$

It is good to understand examples when the definition of converging to zero does not apply, as in the following example.

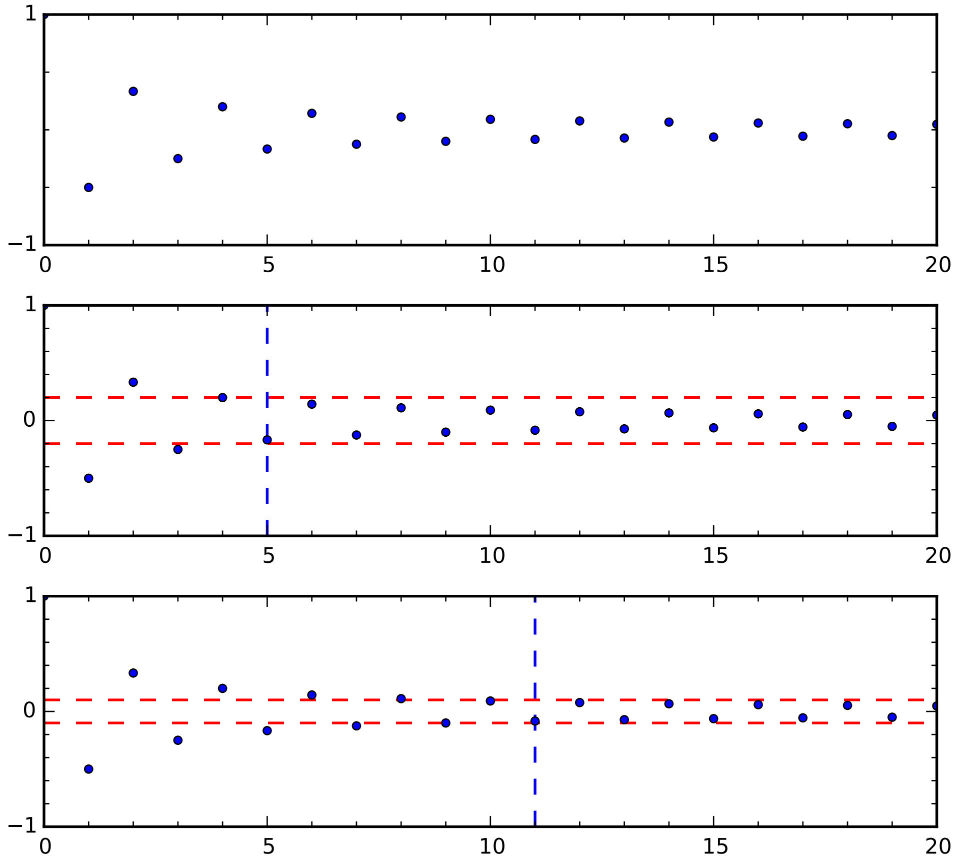
Example 2.4 Prove that the sequence, $s_n = \dfrac{n+1}{n+2}$ does not converge to 0.

Proof. We must show that there exists a positive real number, ϵ, such that for all real numbers, N, it's possible to have $n > N$ and $|s_n| > \epsilon$. $\epsilon = 0.5$ will do. We can see that

$$\frac{n+1}{n+2} = 1 - \frac{1}{n+2} > 1 - \frac{1}{2} \geq \frac{1}{2}.$$

So, in fact, any $n > N$ works for any N to give that $|s_n| > \epsilon$. □

Figure 2.3: Picking N for smaller and smaller ϵ for the sequence $s_n = \frac{(-1)^n}{n}$.



The above are good exercises but problems like these will be easier to prove – that is, no epsilons nor multiple quantifiers will be needed – once we have some theorems. For example:

Exercise 2.6 Use the following theorem to provide another proof of *Exercise* 2.4.

Theorem 2.1 *For any real-valued sequence, s_n:*

$$s_n \to 0 \iff |s_n| \to 0 \iff -s_n \to 0$$

Proof. Every implications follows because $|s_n| = ||s_n|| = |-s_n|$ □

Theorem 2.2 *If $\lim_{n\to\infty} a_n = 0$, then the sequence, a_n, is bounded. That is, there exists a real number, $M > 0$ such that $|a_n| < M$ for all n.*

Proof. Since $a_n \to 0$, there exists $N \in \mathbb{R}^+$ such that $n > N \implies |a_n| < 1$. Here we use the definition of converging to 0 with $\epsilon = 1$. (NOTE: We could use any positive number in place of 1.) Let B be a bound for the finite set $\{a_n : n \leq N\}$. This set is bounded by Theorem 1.41. Let $M = \max\{B, 1\}$ Hence any a_n is bounded by M because it is either in the finite set ($n \leq N$) and bounded by B or it is bounded by 1, because $n > N$. □

Theorem 2.3 ALGEBRAIC PROPERTIES OF LIMITS 1

Given three sequences, $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$ and a real number, c, then:

1. $\lim_{n\to\infty} a_n + b_n = 0$
2. $\lim_{n\to\infty} c \cdot a_n = 0$.
3. $\lim_{n\to\infty} a_n \cdot b_n = 0$.

Proof. 1. Let $\epsilon > 0$ be given. Because the sequences converge to 0, we can find N_1 such that

$$n > N_1 \implies |a_n| < \frac{\epsilon}{2}$$

and we can find N_2 such that

$$n > N_2 \implies |b_n| < \frac{\epsilon}{2}$$

Note that $|a_n + b_n| \leq |a_n| + |b_n|$ by the THE TRIANGLE INEQUALITY, Theorem 1.2.4. Let $N = \max\{N_1, N_2\}$, so that any $n > N$ is larger than both N_1 and N_2. Then

$$n > N \implies |a_n + b_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

so we have shown that $\lim_{n\to\infty} a_n + b_n = 0$ *NOTE: The method of finding the common N from two others is often shortcut with the following words:* Find N sufficiently large so that both $|a_n| < \frac{\epsilon}{2}$ and $|b_n| < \frac{\epsilon}{2}$. *It is assumed the reader understands the process.*

2. If $c = 0$, then $c \cdot a_n = 0$ for all n and converges to 0. So assume $c \neq 0$. Let $\epsilon > 0$ be given. Because $a_n \to 0$, we can find N such that

$$n > N \implies |a_n| < \frac{\epsilon}{|c|}$$

Note that $|c \cdot a_n| = |c| \cdot |a_n| < |c| \cdot \frac{\epsilon}{|c|} = \epsilon$. So we have shown that $c \cdot a_n \to 0$.

3. HINT: Make use of the fact that a_n is bounded and mimic the previous proof.

□

Exercise 2.7 Prove the following theorem:

Theorem 2.4 *If c_n is bounded and $a_n \to 0$, then $c_n \cdot a_n \to 0$*

The following theorem is the first in a series of 'squeeze' theorems, among the most useful tools we have at our disposal.

Theorem 2.5 SQUEEZE THEOREM *If $a_n \to 0$ and $b_n \to 0$ and $a_n \leq c_n \leq b_n$, for all $n \in \mathbb{Z}^+$, then $\lim_{n\to\infty} c_n = 0$.*

Proof. Given $\epsilon > 0$, let N be large enough so that whenever $n > N$, then both $|b_n| < \epsilon$ and $|a_n| < \epsilon$. Now, for any $n > N$, if $c_n > 0$, we have $|c_n| \leq |b_n| < \epsilon$. or if $c_n < 0$, then $|c_n| = -c_n \leq -a_n = |a_n| < \epsilon$. So, for all $n > N$ we have $|c_n| < \epsilon$. We have shown that $c_n \to 0$. □

True or False 10

Which of the following statements are true? If false, modify the hypothesis to make a true statement. In either case, prove the true statement.

a) $\lim_{n\to\infty} \frac{n^2 + n}{n^3} \to 0$

b) For all $r \in \mathbb{R}$, $\lim_{n\to\infty} \frac{1}{n + r} \to 0$.

c) For any integer, m, $\lim_{n\to\infty} \frac{1}{n^m} = 0$

d) For $r \in R$, $r^n \to 0$.

Exercise 2.8 One way to modify the last **True or False**, part d), is given in the following theorem. Use BERNOULLI'S INEQUALITY Theorem 1.27 to prove the theorem.

Theorem 2.6 *If* $0 \leq r < 1$, *then* $r^n \to 0$

Proof. EFS □

2.1.2 Sequences that converge to arbitrary limit

Definition We say that s_n converges whenever there exists a real number, s, such that $|s - s_n| \to 0$. In this case, we say that s_n converges to s, and write

$$\lim_{n\to\infty} s_n = s \text{ or } s_n \to s$$

Example 2.5 $\lim_{n\to\infty} \frac{n+1}{n+2} = 1$, because $1 - \frac{n+1}{n+2} = 1 - (1 - \frac{1}{n+2}) = \frac{1}{n+2} \to 0$, as shown in **True or False**.

Exercise 2.9 Show that $s_n \to 0$ means the same thing for both definitions: converging to 0 and converging to an arbitrary limit that happens to be 0.

Theorem 2.7 UNIQUENESS OF LIMIT *If* $a_n \to a$ *and* $a_n \to b$, *then* $a = b$.

Proof. Use the triangle inequality to see that $0 \leq |a - b| = |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b|$. Apply THE SQUEEZE THEOREM (Theorem 2.5.): the left-most term is the constant sequence, 0, the right-most term is the sum of two sequences that converge to 0, so also converges to 0, by ALGEBRAIC PROPERTIES OF LIMITS, Theorem 2.3. Hence the middle term (which is a constant sequence) also converges to 0. So $|a - b| = 0 \implies a = b$. □

Exercise 2.10 Prove: If $a_n = c$, for all n, then $\lim_{n\to\infty} a_n = c$

Theorem 2.8 *If* $\lim_{n\to\infty} a_n = a$, *then the sequence,* a_n, *is bounded.*

Proof. EFS Consider using Theorem 2.2. □

Theorem 2.9 *If* $\lim_{n\to\infty} a_n = a$ *and if* $a_n \neq 0$ *and* $a \neq 0$, *then the sequence,* a_n, *is bounded away from* 0. *That is, there exists a positive number* B, *such that* $|a_n| > B$, *for all* n.

Proof. *(Draw a numberline picture to help see this proof.)*To find such a bound, B, first note that there is $N > 0$ such that $|a_n - a| < |\frac{a}{2}|$ for all positive integers $n > N$. (Using $\epsilon = |\frac{a}{2}|$ in the definition of limit.) For those n,

$$|a_n| \geq |a| - |a_n - a| > |a| - |\frac{a}{2}| = |\frac{a}{2}|.$$

Now let $\overline{B} = \min\{|a_n| : n \leq N\}$. This set has a minimum value because it is a finite set. (Theorem 1.41) Of course, $\overline{B} > 0$ because none of the $a_n = 0$. Finally, let $B = \min\{\overline{B}, |\frac{a}{2}|\}$. So $|a_n| > B$, for all n. □

Theorem 2.10 ALGEBRAIC PROPERTIES OF LIMITS 2

Given two sequences, $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$*, then:*

1. $\lim_{n\to\infty} a_n + b_n = a + b$
2. $\lim_{n\to\infty} a_n \cdot b_n = a \cdot b$
3. *If* $a_n, a \neq 0$*, then* $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$

Proofs. For all the proofs make use of all the theorems we have about sequences that converge to zero.

1. EFS
2. HINT: Use this trick $|a_n \cdot b_n - a \cdot b| = |a_n \cdot b_n - a_n \cdot b + a_n \cdot b - a \cdot b|$, the triangle inequality and the boundedness of a converging sequence.
3. Theorem 2.9 applies to this sequence, let B be that positive number such that $|a_n| > B$, for all n. Consider the inequality

$$0 < |\frac{1}{a_n} - \frac{1}{a}| = \frac{|a - a_n|}{|a_n \cdot a|} < \frac{|a_n - a|}{B \cdot |a|}$$

Since $|a_n - a| \to 0$, we can apply ALGEBRAIC PROPERTIES OF LIMITS 1,Theorem 2.3 and THE SQUEEZE THEOREM, Theorem 2.5, to conclude that $|\frac{1}{a_n} - \frac{1}{a}| \to 0$ and hence $|\frac{1}{a_n} \to \frac{1}{a}|$.

□

LIMITS OF RATIOS An important concern of calculus is what happens to the ratio of two limits when both the numerator and denominator converge to 0. If the denominator converges to zero, but the numerator is bounded away from zero, then the ratio will be unbounded and not converge. See more in Section 2.1.5.

Exercise 2.11 Give examples of two sequences, $a_n \to 0$ and $b_n \to 0$, such that

a. $\frac{a_n}{b_n} \to 0$

b. $\dfrac{a_n}{b_n} \to c$, where c is a positive real number.

c. $\dfrac{a_n}{b_n}$ does not converge.

Theorem 2.11 ORDER PROPERTIES OF LIMITS

For real sequences, a_n, b_n, c_n and real numbers, a and c.

1. *If $a_n > c$ for all $n \in \mathbb{Z}^+$ and $a_n \to a$, then $a \geq c$*
2. *If $a_n \leq c \leq b_n$ for all n and $|a_n - b_n| \to 0$, then $a_n \to c$ and $b_n \to c$.*
3. THE SQUEEZE THEOREM *If $a_n \to c$ and $b_n \to c$ and $a_n \leq c_n \leq b_n$ for all $n \in \mathbb{Z}^+$, then $c_n \to c$*

Proof. Confirm each statement and explain how it proves the corresponding part of the theorem. Drawing a numberline picture of the situation may help.

1. Suppose $c - a > 0$, find N such that $n > N \implies |a - a_n| < c - a$.
2. For all n, $0 \leq |a_n - c| \leq |b_n - a_n|$
3. $0 \leq |c - c_n| \leq |c - a_n| + |a_n - c_n| \leq |c - a_n| + |a_n - b_n|$

□

Exercise 2.12 Counterexample for Theorem 2.11 2. Find two sequences a_n and b_n such that $|a_n - b_n| \to 0$, but neither sequence converges. Is it possible that one sequence could converge but the other does not?

Theorem 2.12 THE TAILEND THEOREM *If $a_n \to a$ and if $b_n = a_{n+m}$ for some fixed, positive integer m, then*

$$b_n \to a.$$

(A tailend of a sequence is a special case of a subsequence, see Section 2.1.4.)

Proof. EFS □

Exercise 2.13 Prove: If $a_n \to c$ and $b_n \to c$, then $|a_n - b_n| \to 0$

Exercise 2.14 Conjecture what the limit might be and prove your result.

$$s_n = \frac{3n^2 + 2n + 1}{n^2 + 1}$$

Exercise 2.15 Prove that, if $c \neq 0$, then

$$\lim_{n\to\infty} \frac{a \cdot n + b}{c \cdot n + d} = \frac{a}{c}.$$

Exercise 2.16 Prove: If $a_n \to a$, $b_n \to b$, and $a_n < b_n$ for all n, then $a \leq b$.

Exercise 2.17 Prove: If S is a bounded set, then there exists a sequence of points, $s_n \in S$ such that $s_n \to \sup S$.

Exercise 2.18 Prove: If $a_n \to a$ then $a_n^2 \to a^2$

Monotone sequences

Definition We say a sequence is *monotone* whenever it is an increasing sequence or a decreasing sequence.

Theorem 2.13 MONOTONE CONVERGENCE THEOREM *Every bounded, monotone sequence converges to a real number.*

Proof. Let s_n be a bounded, increasing sequence. Let $s = \sup\{s_n\}$ which exists because s_n is bounded above. We claim that $s_n \to s$. Given $\epsilon > 0$, use Theorem 1.46 to find $x = s_N \in \{s_n\}$ such that $s - \epsilon < s_N$. Now if $n > N$, we know $s_n > s_N$ because the sequence is increasing, so

$$|s_n - s| = s - s_n < s - s_N < \epsilon.$$

We conclude that $s_n \to s$.

If t_n is a bounded, decreasing sequence, then $s_n = -t_n$ is bounded and increasing. Since $s_n \to s$, for some s, we known that $t_n = -s$. □

Best Nested Interval Theorem

Theorem 2.14 BEST NESTED INTERVAL THEOREM *There exists one and only one real number, x, in the intersection of a sequence of non-empty, closed, nested intervals if the lengths of the intervals converge to 0. Furthermore, the sequence of right endpoints and the sequence of left endpoints both converge to x.*

Proof. Denote the intervals by $[a_n, b_n]$. Because they are nested we know that a_n is increasing and b_n is decreasing so, by MONOTONE CONVERGENCE THEOREM, there are real numbers a and b such that $a_n \to a$ and $b_n \to b$. Let c be any real number in the intersection of all the intervals. Then $a_n \leq c \leq b_n$ and since $|b_n - a_n| \to 0$ we have by the ORDER PROPERTIES OF LIMITS (Theorem 2.11 *2.*), $c = a = b$. □

Rational Approximations to Real Numbers

We have not yet shown that there are real numbers other than rational numbers. However, if there is one, the following method indicates that you can approximate it by rational numbers; that is, there is a sequence of rational numbers that converge it. The method of bisection used here is a well-used tool of analysis.

Example 2.6 Let r be any non-rational real number. Find a sequence of rational numbers that converge to r.

Using the method of bisection. There are other ways to show the existence of such a sequence. The advantage to this method is that it gives a way to construct approximations of the given real number.

First, find rational numbers a_0 and b_0 such that $a_0 < r < b_0$. By Theorem 1.60, they can be consecutive integers, so $|b_0 - a_0| = 1$. Recursively define a sequence of non-empty, closed, nested intervals $[a_n, b_n]$ such that each interval contains r and $|b_n - a_n| = \frac{1}{2_n}$. We already have the base case, $[a_0, b_0]$. Assume $[a_n, b_n]$ has been defined as required. Let m be the midpoint of $[a_n, b_n]$. Since m is a rational number(why?), $m \neq r$, so there are two cases to consider:

1. If $m < r$, let $a_{n+1} = m$ and $b_{n+1} = b_n$.
2. If $r < m$, let $a_{n+1} = a_n$ and $b_{n+1} = m$.

In either case $r \in [a_{n+1}, b_{n+1}]$ and $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$. The length of this interval is half the length of the previous interval $= \frac{1}{2} \cdot \frac{1}{2^n} = \frac{1}{2^{n+1}}$, so the lengths of the intervals converge to 0. So we have the intervals as required and they satisfy the BEST NESTED INTERVAL THEOREM (Theorem 2.14). That theorem tells us that both a_n and b_n converge to the unique number that is in all the intervals. But r is in all the intervals. So both $a_n \to r$ and $b_n \to r$. Either sequence will do to prove the theorem. □

2.1.3 Application: Existence of square roots

The proof given here of the existence of square roots is by construction. There are other ways to prove the existence of square roots – the advantage to this method is that it gives a way to calculate approximations to $\sqrt{2}$.

Example 2.7 There exists a unique positive real number, s, such that $s^2 = 2$.

Proof by bisection. We will show that there is a nested sequence of closed intervals, $I_n = [a_n, b_n]$, such that ${a_n}^2 \leq 2 \leq {b_n}^2$ and $|b_n - a_n| = \frac{1}{2^n}$. By the BEST NESTED INTERVAL THEOREM, there is a unique number, s, in all of the intervals. We will show that $s^2 = 2$

using a squeeze argument.
We define the intervals inductively: Base case: Let $a_0 = 1$ and $b_0 = 2$, so

$$a_0{}^2 = 1 \leq 2 \leq 4 = b_0{}^2, \text{ and } |b_0 - a_0| = 1 = \frac{1}{2^0}.$$

Assume a_n and b_n have been defined as desired, that is

$$a_n{}^2 \leq 2 \leq b_n{}^2, \text{ and } |b_n - a_n| = \frac{1}{2^n}.$$

Proceed inductively to define the next interval, $I_{n+1} = [a_{n+1}, b_{n+1}]$. Let m be the midpoint of the interval $[a_n, b_n]$. We know that $m^2 \neq 2$ because we know the square root can not be a rational number. (Why do we claim m is rational?) So there are only two cases to consider:

1. If $m^2 < 2$, let $a_{n+1} = m$ and $b_{n+1} = b_n$.
2. If $2 < m^2$, let $a_{n+1} = a_n$ and $b_{n+1} = m$.

Since m is the midpoint of the previous interval the length of I_{n+1} is half the length of I_n, so $|b_{n+1} - a_{n+1}| = \frac{1}{2} \cdot \frac{1}{2^n} = \frac{1}{2^{n+1}}$
Let s be the unique number in all the intervals, $[a_n, b_n]$. We know $a_n \to s$ and $b_n \to s$. So we also know that $a_n^2 \to s^2$, $b_n^2 \to s^2$. (See Exercise 2.18) Now consider the image, under the squaring function of all those intervals. The intervals $[a_n^2, b_n^2]$ are closed, non-empty ($a_n^2 < b_n^2$ because $a_n < b_n$); and nested (a_n^2 is increasing because a_n is increasing, and b_n^2 is decreasing because b_n is decreasing). These all follow because the function $x \to x^2$ is increasing on positive numbers, see Exercise 1.12. Furthermore, $|b_n^2 - a_n^2| \to 0$. The BEST NESTED INTERVAL THEOREM applies and we conclude there is a unique real number in all the intervals and that the endpoints converge to that number. 2 is in all the intervals and $a_n \to s^2$, so $s^2 = 2$, by uniqueness of limits.
To see why there is only one positive solution to the equation $x^2 = 2$, we will use Theorem 1.10, THERE ARE NO ZERO DIVISORS. Let $\sqrt{2}$ be the positive solution found above. Note, through the distributive law and some simplification, that

$$(x - \sqrt{2})(x + \sqrt{2}) = x^2 + x\sqrt{2} - x\sqrt{2} - (\sqrt{2})^2 = x^2 - 2$$

Now suppose $x^2 = 2$, then

$$\begin{aligned} x^2 - 2 &= 0 && \text{by subtracting 2} \\ (x - \sqrt{2})(x + \sqrt{2}) &= 0 && \text{as seen in the note} \end{aligned}$$

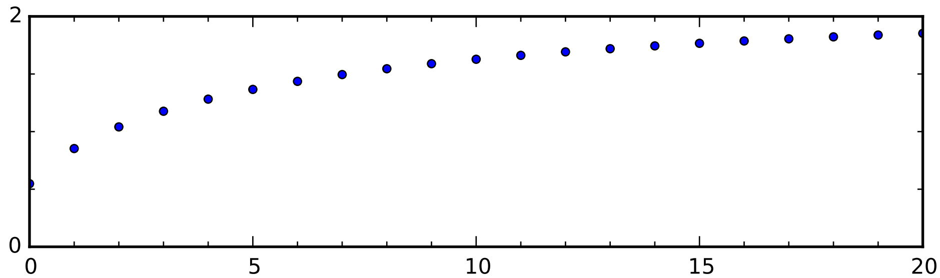
We conclude by Theorem 1.10 that $x = \sqrt{2}$ or $x = -\sqrt{2}$ are the only two solutions to $x^2 = 2$. Only $\sqrt{2}$ is positive. □

Theorem 2.15 *For all $A > 0$, there exists a unique positive real number, s, such that $s^2 = A$.*

Proof. HINT: What in the proof for Example 2.7 depends on the choice $A = 2$? □

Definition We use the symbol $\sqrt{A}$ to denote the unique number such that $(\sqrt{A})^2 = A$.

Figure 2.4: $s_n = \dfrac{\sqrt{5n+3}}{\sqrt{n+10}}$



Now that we know that every positive number has a unique positive square root we are free to use square roots in our other work.

Exercise 2.19 Conjecture what the limit might be and prove your result.

$$s_n = \frac{\sqrt{5n+3}}{\sqrt{n+10}}$$

Exercise 2.20 Use Theorem 2.13 to show that $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0$.

2.1.4 Subsequences

Definition We call sequence, s_{n_k}, whose values are a subset of the values of s_n, a *subsequence* of s_n, whenever the sequence n_k is strictly increasing. (We will assume that k is indexed on $\mathbb{Z}^{\geq}$, i.e. n_0 is the first value.)

Example 2.8 If $n_k = 2k$, then the subsequence is every other element, starting at 0, of the sequence.

Exercise 2.21 Prove: If s_{n_k} is a subsequence of s_n, then $n_k \geq k$. Note that this is true for any increasing sequence of positive integers n_k.

Exercise 2.22 Show how a 'tailend' of a sequence, as discussed in Theorem 2.12, is a subsequence of that sequence.

Theorem 2.16 *If $s_n \to s$, then any subsequence of s_n also converges to s.*

Proof. Let s_{n_k} be a subsequence of s_n. Given $\epsilon > 0$, find N so that $n > N \implies |s_n - s| < \epsilon$. Such N exists because $s_n \to s$. Now consider the subsequence: If $k > N$, then $n_k > N$ (by Exercise 2.21), so $|s_{n_k} - s| < \epsilon$. □

Example 2.9 The sequence is Exercise 2.23 has many subsequences, each of which converges to one of five different numbers. For example, $s_{5k+2} \to \sin \frac{4\pi}{5}$.

2.1.5 Divergent Sequences

Definition A sequence is said to *diverge* if there is no real number L such that the sequence converges to L.

To show that a sequence, s_n, converges we would first conjecture a possible limit, L, and then prove $s_n \to L$. To show that the sequence does not converge is perhaps harder because we have to show it doesn't converge for all possible values L. And we would need to prove the negation of the statement, $s_n \to L$, for all values of L. Here are both statements:

$s_n \to L$, means

$$\text{For all } \epsilon > 0, \text{ there exists } N > 0, \text{ such that}$$
$$n > N \implies |s_n - L| < \epsilon.$$

$s_n \not\to L$, means

$$\text{There exists an } \epsilon > 0, \text{ such that for all } N > 0$$
$$\text{there is an } n > N \text{ with } |s_n - L| \geq \epsilon.$$

Fortunately there is an easier way to show that a sequence diverges by observing subsequence behavior, using Theorem 2.16.

Example 2.10 The sequence,

$$s_n = \begin{cases} 1, & n, \text{ odd} \\ -1, & n, \text{ even} \end{cases}$$

diverges, i.e. does not converge to any real number L.

Proof. The subsequence given by $n_k = 2k + 1$ is a constant sequence: $s_{n_k} = 1$ for all k. This subsequence converges to 1. The subsequence given by $n_k = 2k$ converges to -1. If the sequence converged, both subsequences would have to converge to the same number, by Theorem 2.16. So the sequence does not converge. □

Exercise 2.23 Υ Explain why the following sequence, $s_n = \sin(\frac{2\pi n)}{5}$, diverges. The graph of this sequence is shown in Figure 2.5.

Example 2.11 Another Υ example of a *divergent* sequence is $s_n = \sin(n)$. That this sequence does not converge seems a correct conclusion considering the graph, shown in Figure 2.6. Proving that goes beyond the scope of our present discussion.

Another way a sequence may fail to converge is if it is unbounded. We consider separately the case when the limit appears to be infinite as in the sequence, $s_n = n$.

Figure 2.5: $s_n = \sin(\frac{2\pi n}{5})$.

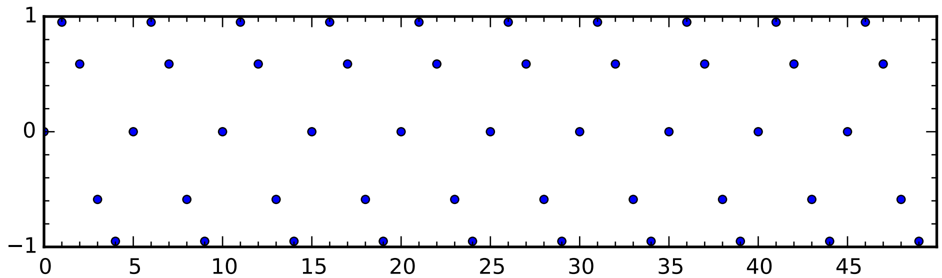
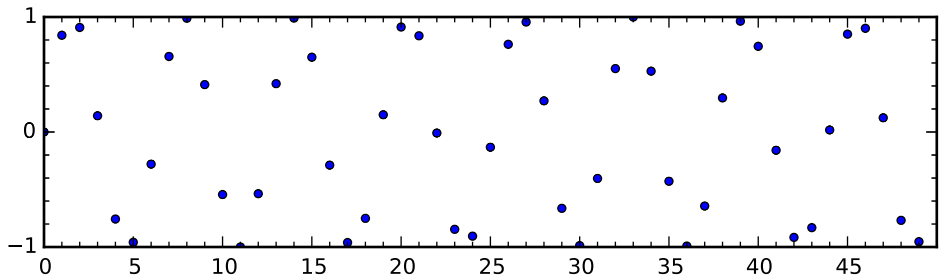


Figure 2.6: $s_n = \sin(n)$.



Definition We say that s_n *diverges to infinity* and we write, $\lim_{n\to\infty} s_n = +\infty$, whenever, for all $M > 0$, there exists $N > 0$ such that

$$n > N \implies s_n \geq M.$$

Example 2.12 $\lim_{n\to\infty} n^2 = +\infty$: If M is any positive real number, let $N = M$. Then, if $n > N$, we have that $n^2 > n > N = M$ or simply that $n^2 > M$. (EFS: Explain the step $n^2 > n$)

Exercise 2.24 Give a sequence that is unbounded but does not diverge to $+\infty$

Theorem 2.17 *If $s_n \to +\infty$, then any subsequence of s_n also diverges to $+\infty$.*

Proof. EFS □

Exercise 2.25 Prove the following theorem.

Theorem 2.18 ALGEBRAIC PROPERTIES OF DIVERGENT LIMITS

1. *$a_n \to +\infty$ and $b_n \to b$ or $b_n \to +\infty \implies (a_n + b_n) \to +\infty$.*
2. *$a_n \to +\infty$ and $b_n \to b > 0$ or $b_n \to +\infty \implies (a_n \cdot b_n) \to +\infty$.*
3. *$a_n \neq 0$, $a_n \to +\infty \implies \dfrac{1}{a_n} \to 0$*

4. *If* $a_n > 0$ *for all* n, *then* $a_n \to 0 \implies \frac{1}{a_n} \to +\infty$

Exercise 2.26 Prove the following theorem.

Theorem 2.19 *If* $r > 1$, *then* $r^n \to +\infty$

Proof. □

Exercise 2.27 LIMITS OF RATIOS Give examples of two sequences, $a_n \to +\infty$ and $b_n \to +\infty$, such that

a. $\dfrac{a_n}{b_n} \to +\infty$

b. $\dfrac{a_n}{b_n} \to 0$

c. $\dfrac{a_n}{b_n} \to c$, where c is a positive real number.

Exercise 2.28 Provide a definition and theorems about diverging to $-\infty$

a. What would it mean for the limit of a sequence to be $-\infty$?

b. If it is not part of your definition prove: $\lim_{n\to\infty} s_n = -\infty \iff \lim_{n\to\infty} -s_n = +\infty$?

c. If it is not part of your definition prove: If $\lim_{n\to\infty} s_n = -\infty$ then, for all $M < 0$, there exists $N > 0$ such that

$$n > N \implies s_n \leq M.$$

d. Give an example of a sequence that diverges to $-\infty$ and prove that it does.

e. State and prove statements of ALGEBRAIC PROPERTIES OF DIVERGENT LIMITS for sequences that diverge to $-\infty$

f. If $s_n \to -\infty$, then any subsequence of s_n also diverges to $-\infty$.

Exercise 2.29 Give a sequence $s_n \to 0$ where $\dfrac{1}{s_n}$ does not diverge $+\infty$ and does not diverge to $-\infty$.

Theorem 2.20 *If* p *is a polynomial that is not constant, then either*

$$\lim_{n\to\infty} p(n) = +\infty \text{ or } \lim_{n\to\infty} p(n) = -\infty$$

Proof. HINT: Use induction on the degree of the polynomial. □

2.2 Limits and Sets

2.2.1 Limit Points and Boundary Points

We have already seen that there is a sequence in a set S that converges to $\inf S$ and another that converges to $\sup S$. In this section, we investigate other characteristics of sets and points that would guarantee the existence of a sequence of elements within the set that converge to the point.

Definition We say the a point, p, is a *limit point of a set*, S, whenever every open interval about p contains an infinite number of points in S. In particular, it contains a point in S that is not equal to p.

NOTE: The point p need not be in S.

Definition We say the a point, p, is a *limit point of a sequence*, s_n, whenever every open interval about p contains an infinite number of s_n.

Example 2.13 The limit points of the image of s_n may be different than the limit points of s_n. Consider $s_n = (-1)^n$. The image of s_n is $\{, -1, 1\}$, a set that has no limit points. However, both -1 and 1 are limit points of the sequence because they each appear an infinite number of times in the sequence.

Theorem 2.21 *A real number p is a limit point of a set S, if and only if there exists sequence of points in $S \setminus \{p\}$ that converge to p.*

Proof. $\Longrightarrow$ Suppose p is a limit point of S. For each $n \in \mathbb{Z}^+$, let

$$s_n \in S \cap (p - \frac{1}{n}, p + \frac{1}{n}) \setminus \{p\}.$$

Such a point exists because p is a limit point. We claim that $s_n \to p$. For we know, for all n,

$$0 < |p - s_n| < \frac{1}{n}$$

From the squeeze theorem, we conclude that $|p - s_n| \to 0$ or $s_n \to p$.

$\Longleftarrow$Let $s_n \in S \setminus \{p\}$ converge to p. Let (a, b) be an open interval containing p. Consider any positive $\epsilon < \min(b - p, p - a)$, so that $p \in (p - e, p + \epsilon) \subset (a, b)$. By the convergences of s_n, there exists N such that $n > N \implies |p - s_n| < \epsilon$. These s_n's are an infinite number of values of the sequences that are in $(a, b) \setminus \{p\}$. If there were only a finite number of numbers from S in this sequence, we would have a subsequence of s_n that converges to some other number which cannot happen. So the s_n for $n > N$ are an infinite number of points in (a, b), as required to show that p is a limit point of the set S. ☐

Theorem 2.22 *A real number s is a limit point of a sequence s_n if and only if there exists a subsequence of s_n that converge to s.*

Proof. EFS □

Definition We say the a point, p, is an *boundary point* of a set, S, whenever every open interval containing p contains points in both S and $\mathbb{R} \setminus S$.

Example 2.14 Every point in a finite set is a boundary point of the set. Every point in a finite set is a boundary point of the complement of set.

True or False 11

Which of the following statements are true? If false, modify the statement to be true. Explain.

a) The endpoints of an interval are boundary points of the interval.

b) Every point in an interval is a boundary point of the interval.

c) Every point of an interval is a limit point of the interval.

d) The inf S is a limit point of S.

e) The inf S is a boundary point of S.

f) The maximum value of a S is a boundary point of S.

Example 2.15 Give an example of each of the following and explain.

a) A set and a point that is a boundary point but not a limit point of the set.

b) A set and a point that is a limit point but not a boundary point.

c) A set and a point that is neither a limit point nor a boundary point of the set.

d) A set and a point that is both a boundary point and a limit point of the set.

2.2.2 Open and Closed Sets

Definition We say a set is *open* whenever it contains none of its boundary points.

Definition We say a set is *closed* whenever it contains all of its boundary points.

Example 2.16 Open intervals are open sets because the only boundary points of an interval are the endpoints and neither are contained in the open interval.

Example 2.17 Closed intervals are closed sets be because the only boundary points of an interval are the endpoints and both are contained in the closed interval.

Theorem 2.23 *The following are equivalent*

1. *S is an open set*
2. *Every $s \in S$ is contained in an open interval that is completely contained in S.*
3. *$\mathbb{R} \setminus S$ is closed.*

Proof. □

Theorem 2.24 *The following are equivalent*

1. *S is an closed set*
2. *S contains all of its limit points.*
3. *$\mathbb{R} \setminus S$ is open.*

Proof. □

Exercise 2.30 Is $\mathbb{R}$ open or closed? Is $\varnothing$ open or closed?

Exercise 2.31 $\{x : x^2 \leq 57\}$ is a closed set.

True or False 12

Which of the following statements are true? If false, modify the statement to be true. Explain.

a) An open set never contains a maximum.

b) A closed set always contains a maximum.

Theorem 2.25 *Union and Intersection properties*

1. *The intersection of a collection of closed sets is closed.*
2. *The union of a collection of open sets is open.*
3. *The intersection of a finite collection of open sets is open.*
4. *The union of a finite collection of closed sets is closed.*

Proof. HINTS:

1. Use the 'contains all limit points' criteria for closed sets.
2. Use the 'there is an open interval about any point' criteria for open sets.

□

2.2.3 Optional – Connected sets

Suppose that there were no real number, s, such that $s^2 = 56$. Consider the two sets $U = \{s : s^2 < 56\}$ and $V = \{s : s^2 > 56\}$. Then an interval like $[7, 8]$ could be divided into two distinct parts by the disjoint sets, U and V. There would be a 'hole' in the numberline. This leads to the definition of connected that says a connected set cannot be covered by two distinct open sets. That intervals are connected is a way of understanding the completeness axiom and investigating sets that may have more complicated structure than intervals.

Definition We say a set, S, is *connected* if it *is not* contained in the union of two *disjoint* non-empty, open sets.

Example 2.18 Finite sets are not connected

Here we present another nice application of the Nested Interval Theorem.

Theorem 2.26 *A connected set is a, possibly infinite, interval.*

Proof. Hint: this will be easiest to handle using the criteria for intervals investigated in Section 1.3.6. Let C be a connected subset of $\mathbb{R}$. Suppose $a, b \in C$ and $a < b$. Suppose there is an x such that $a < x < b$. If $x \notin C$, then let $U = (-\infty, x)$ and $V = (x, \infty)$. U and V are open intervals and together they cover C if x is not in C. This contradicts that C is connected. So we conclude $x \in C$. WE have shown that C satisfies the condition 1.1 that defines an interval. So C is an interval. □

The more interesting part is that every interval is connected.

Theorem 2.27 *Any interval is connected.*

Proof. Again, we will use the criteria 1.1. Let I be an interval and assume it is not connected. Let U and V be two disjoint open sets such that $I \in U \cup V$. Since both $\mathbb{R} \setminus U$ and $\mathbb{R} \setminus V$ are closed this means that I is also covered by two distinct closed sets. We will use this and the fact that closed sets contain all of their limit points later in the proof.

Since neither U nor V are empty, pick $u_0 \in U$ and $v_0 \in V$ and assume, without loss of generality, that $u_0 < v_0$. Define inductively a sequence of closed, nested intervals, $[u_n, v_n]$ with $u_n \in I \cap U$ and $v_n \in I \cap V$ and length, $|v_n - u_n| = \frac{|v_0 - u_0|}{2^n}$. The base case, $[u_0, v_0]$, satisfies the conditions. Assume $[u_n, v_n]$ has been defined. Let m be the midpoint of the interval $[u_n, v_n]$. So,

$$u_0 < m < v_0$$

Now $u_n, v_n \in I$ and so $m \in I$ by 1.1. There are two cases:

1. If $m \in U$, let $u_{n+1} = m$ and $v_{n+1} = v_n$.

2. If $m \in V$, let $u_{n+1} = u_n$ and $v_{n+1} = m$.

In either case, then $u_{n+1} \in I \cap U$ and $v_{n+1} \in I \cap V$. and $[u_{n+1}, v_{n+1}] \subset [u_n, v_n]$. So we have a closed interval of the required form. Since m is the midpoint, we known that

$$|v_{n+1} - u_{n+1}| = \frac{1}{2}|v_n - u_n| = \frac{1}{2} \cdot \frac{v_0 - u_0}{2^n} = \frac{v_0 - u_0}{2^{n+1}}$$

These intervals are nested, closed and non-empty so we can applied the Best Nested Interval Theorem to say there is a point, $x \in [u_n, v_n]$, for all n, and that $u_n \to x$ and $v_n \to x$. Since x is in all the intervals, it is between two points of I and so is in I.

Now x is a limit point of $\mathbb{R} \setminus U$, a closed set, so it must be in $\mathbb{R} \setminus U$. That is, x is not in U. But x is also a limit point of $\mathbb{R} \setminus V$, another closed set, so x is not in V. x is in neither V nor U, but it is a point in I so U and V cannot cover the interval as originally supposed. Therefore, the interval is connected. □

2.3 The Bolzano-Weierstrass Theorem and Cauchy Sequences

Theorem 2.28 THE BOLZANO-WEIERSTRASS THEOREM *Every bounded sequence has a converging subsequence.*

Outline of proof: Name the sequence s and let M be a bound for the sequence. That is, for all n, $|s_n| < M$. We will construct, using a bisection method, a sequence of non-empty, closed nested intervals whose lengths converge to 0 and a subsequence of s, such that k^{th} element of the subsequence is contained in the k^{th} interval. Necessarily, this subsequence converges to the common point of the intervals. More precisely, we define inductively a sequence of non-empty, closed, nested intervals, $[u_k, v_k]$, such that each interval contains an infinite number of sequence values and such that each interval is half the length of the previous interval. Along the way, we define the required subsequence, $s_{n_k} \in [u_k, v_k]$.

The base case is $u_0 = -M$ and $v_0 = M]$. Since all sequence points are in this interval, pick one, s_{n_0}. Assume $[u_k, v_k]$ has been defined. Let m be the midpoint of the interval $[u_k, v_k]$. There are two cases. They are not mutually exclusive, so pick the first if both are true.

1. If there is an infinite number of $s_n \in [m, v_k]$ let $u_{k+1} = m$ and $v_{k+1} = v_k$.

2. If there is an infinite number of $s_n \in [u_k, m]$ let $u_{k+1} = u_k$ and $v_{k+1} = m$.

In either case, then $[u_{k+1}, v_{k+1}]$ contains an infinite number of s_n and hence will contain one whose index value, n_{k+1} is greater than n_k. (There are only a finite number with index value less than n_k.) So, inductively, we have the required sequence of intervals. We also have constructed a subsequence, s_{n_k}. We know there is a unique common point, s, of all the intervals, $[u_k, v_k]$ and that $u_k \to s$ and $v_k \to s$. Because $u_k \leq s_{n_k} \leq v_k$, we conclude by the squeeze theorem that $s_{n_k} \to s$. We have constructed a converging subsequence for the sequence. □

Note on Bolzano-Weirstrass: The limit point, s, is by no means unique as we have seen sequences may have even an infinite number of limit points. The proof is also not deterministic in the sense that it does not really help us to construct a limit point. This is because it offers no procedure for determining which interval contains an infinite number of points. Still, it helps to know that a convergent subsequence exists. The theorem is also useful for understanding the following condition that guarantees that a sequence converges without explicitly finding the limit.

Definition We say that the sequence s_n *is Cauchy* whenever the following hold:

For all $\epsilon > 0$, there exists a real number, N, such that

$$n,\, m > N \implies |s_n - s_m| < \epsilon.$$

Example 2.19 The sequence $a_n = n$ is not Cauchy because $|a_n - a_{n+1}| = 1$.

Example 2.20 The sequence, $b_n = \frac{1}{n}$ is Cauchy because for all positive integers, $m > n$, we have

$$|\frac{1}{n} - \frac{1}{m}| = \frac{|m-n|}{mn} < \frac{m}{mn} = \frac{1}{n}$$

The following two lemmas provide all the tools needed to prove our main theorem, 2.31.

Lemma 2.29 *A Cauchy sequence is bounded.*

Proof. HINT: Mimic the proof of Theorem 2.2. □

Lemma 2.30 *If any subsequence of a Cauchy sequence converges, then the sequence itself converges.*

Proof. HINT: Consider: $|s_n - s| \leq |s_n - s_{n_k}| + |s_{n_k} - s|$. The first term can be made small by the Cauchy criteria and second because the subsequence converges to s. □

Theorem 2.31 *A sequence converges if and only if it is a Cauchy sequence*

Outline of proof: $\Longrightarrow$ We will show that if there is a sequence, $s_n \to s$, then the sequence is Cauchy. So, let $\epsilon > 0$ be given. By convergence of s_n find N so that $n > N \implies |s_n - s| < \frac{\epsilon}{2}$. Then we have,

$$|s_n - s_m| \leq |s_n - s| + |s - s_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So the sequence is Cauchy.

$\Longleftarrow$ By Lemma 2.29 the sequence is bounded. By THE BOLZANO-WEIERSTRASS THEOREM the sequence has a converging subsequence. By Lemma 2.30 the sequence also converges. □

The following theorem generalizes the nested interval theorem to closed sets.

Theorem 2.32 *The countable intersection of nested, non-empty bounded closed sets is not empty.*

Proof. Let C_n be the closed sets. Since the sets are nested there is a common bounded for all of them. since each is non-empty pick $s_n \in C_n$. This sequence is bounded so there is a subsequence that converges to some real number s. This point is a limit point for the intersection. Since the intersection is closed s is contained in it. □

2.4 Series and Power series

Series are special kinds of sequences where one keeps a running sum of a sequence, a_n to create a new sequence, s_n:

Definition The number

$$s_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \cdots + a_{n-1} + a_n$$

is called the n^{th} partial sum of the *generating sequence*, a_n. If the sequence, s_n, converges to a point, s, we say the the series converges to s and we write

$$s = \sum_{k=0}^{\infty} a_k$$

Definition A *power series* is a special kind of series where the generating sequence is of the form $a_n = c_n \cdot r^n$. If the c_n's are constant, $c_n = c$ we call it a *geometric series*.

Theorem 2.33 ALGEBRAIC PROPERTIES OF SERIES *If* $s = \sum_{k=0}^{\infty} a_k$ *and* $t = \sum_{k=0}^{\infty} b_k$, *then*

1. $s + t = \sum_{k=0}^{\infty}(a_k + b_k)$.

2. *If* $c \in \mathbb{R}$, *then* $c \cdot s = \sum_{k=0}^{\infty} c \cdot a_k$.

Exercise 2.32 Give examples of each of the following:

a. A series that diverges to $+\infty$.

b. A series whose partial sums oscillate between positive and negative numbers.

Exercise 2.33 If $a_n \geq 0$ for all n, then $s_n = \sum_{k=0}^{n} a_k$ is an increasing sequence.

Exercise 2.34 If $s = \sum_{k=0}^{\infty} a_k$ and $t = \sum_{k=0}^{\infty} b_k$, write a possible formula for the terms of a series that might be $s \cdot t$. Prove that your series converges and is equal to $s \cdot t$.

Exercise 2.35 Show that $\sum_{k=0}^{\infty} \frac{1}{k}$ diverges.

2.4.1 Convergence of Geometric series

Lemma 2.34 *For all* $r \neq 1$,

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

Proof.

Let $s_n = \sum_{k=0}^{n} r^k$	then we see that
$r \cdot s_n = \sum_{k=0}^{n} r^{k+1}$	distributive law across sums
$= \sum_{k=1}^{n+1} r^k$	adjusting indices
$s_n - r \cdot s_n = 1 - r^{n+1}$	the two sums have the same terms except for the first in $s_n(r^0 = 1)$ and last in $r \cdot s_n(r^{n+1})$.

Solving for s_n gives the result. □

Theorem 2.35 *If* $0 \le |r| < 1$, *then*

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

Proof. EFS □

2.4.2 Decimals

Definition A *decimal representation* looks like $a_0.a_1a_2a_3\cdots a_n\cdots$, where a_0 is an integer and a_n are integers between 0 and 9, inclusively. It represents the number which is given by the power series where $r = \frac{1}{10}$. We write

$$a_0.a_1a_2a_3\cdots a_n\cdots = a_0 + a_1\cdot(\frac{1}{10})^1 + a_2\cdot(\frac{1}{10})^2 + a_3\cdot(\frac{1}{10})^3 + \cdots = \sum_{n=0}^{\infty} a_n\cdot(\frac{1}{10})^n$$

Exercise 2.36 Prove: $0.999999... = 1$.

Theorem 2.36 *Every decimal representation is a convergent power series and hence every decimal representation is a real number.*

Proof. The partial sums are bounded by $a_0 + \sum_{k=1}^{n} 9\cdot(\frac{1}{10})^k < a_0 + 1$. The partial sums are increasing. Every bounded increasing sequence converges by MONOTONE CONVERGENCE THEOREM 2.13. □

As well, every real number has a decimal representation that converges to it, more precisely

Theorem 2.37 *Given any* $r \in \mathbb{R}$, *there exists a sequence of integers* a_n, *where* $0 \le a_k \le 10$ *for all* $k \ge 1$ *such that*

$$\sum_{n=0}^{\infty} a_n\cdot(\frac{1}{10})^n = r$$

Proof. Use Theorem 1.59. and THE BEST NESTED INTERVAL THEOREM. □

Notice that nothing is said about the representation of a real number being unique. In fact any rational number that has a representation that ends with an infinite string of 0's has another representation that ends in a string of $9's$. And vice versa.

Exercise 2.37 What changes must be made to the procedure used to find the decimal representation of a real number, as described in the proof of Theorem 2.37, to produce a ending string of 0's instead of 9's?

Exercise 2.38 Make a flowchart of theorems and connective arguments that take us from the Completeness Axiom to the representation of real numbers as decimals.

2.4.3 B-ary representation of numbers in $[0, 1]$

There is really nothing special about the choice of 10 for representing real numbers as power series. If instead, we picked 2 we would get binary representation which is of course of great value in computer science. If you follow the procedure for this choice of base, you'll notice the procedure for finding the representation of a real number uses the bisection method. Octal representation, base 8, and hexidecimal representation, base 16, are also extensively used in computer science. In the next chapter, we will have occasion to use tertiary, or base 3, expansions.

Right now, we restate the theorems about decimal representation using a generic base B, where B is any positive integer greater than 1. We call these *B-ary representations*.

Exercise 2.39 Pick a single digit number for B to use to complete all the exercises and proofs in this section.

Definition A *B-ary representation* looks like $a_0.a_1a_2a_3\cdots a_n\cdots$, where a_0 is an integer and a_n are integers between 0 and $B-1$, inclusively. It represents the number which is given by the power series where $r = \frac{1}{B}$. We write

$$a_0.a_1a_2a_3\cdots a_n\cdots = a_0 + a_1\cdot(\frac{1}{B})^1 + a_2\cdot(\frac{1}{B})^2 + a_3\cdot(\frac{1}{B})^3 + \cdots = \sum_{n=0}^{\infty} a_n\cdot(\frac{1}{B})^n$$

Exercise 2.40 If a is the symbol that represents the $B-1$, prove: $0.aaaaaaa... = 1$.

Theorem 2.38 *Every B-ary representation is a convergent power series and hence every B-ary representation is a real number.*

Proof. The partial sums are bounded by $a_0 + \sum_{k=1}^{n}(B-1)\cdot(\frac{1}{B})^k < a_0 + 1$. The partial sums are increasing. Every bounded increasing sequence converges. □

As well, every real number has a B-ary representation that converges to it, more precisely

Theorem 2.39 *Given any $r \in \mathbb{R}$, there exists a sequence of integers a_n, where $0 \leq a_k \leq B$ for all $k \geq 1$ such that*

$$\sum_{n=0}^{\infty} a_n\cdot(\frac{1}{B})^n = r$$

Proof. Use Theorem 1.59. and THE BEST NESTED INTERVAL THEOREM. □

Notice that nothing is said about the representation of a real number being unique. In fact any rational number that has a representation that ends with an infinite string of 0's has another representation that ends in a string of a's. And vice versa. (a is still the symbol for $B - 1$.)

Exercise 2.41 What changes must be made to the procedure used to find the B-ary representation of a real number, as described in the proof of Theorem 2.39, to produce a ending string of 0's instead of a's?