University of Illinois at Chicago

Interactive Notes For
Real Analysis

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Preface

These notes are all about the Real Numbers and Calculus. We start from scratch with definitions and a set of nine axioms. Then, using basic notions of sets and logical reasoning, we derive what we need to know about real numbers in order to advance through a rigorous development of the theorems of Calculus.

In Chapter 0 we review the basic ideas of mathematics and logical reasoning needed to complete the study. Like Euclid's Basic Notions, these are the things about sets and logic that we hold to be self-evident and natural for gluing together formal arguments of proof. This chapter can be covered separately at the beginning of a course or referred to throughout on an 'as needed' basis. It contains all the common definitions and notation that will be used throughout the course.

Students already think about real numbers in different ways: decimal representation, number line, fractions and solutions to equations, like square roots. They are familiar with special real numbers, with infinite, non-repeating decimals, like π and e. All these ways of representing real numbers will be investigated throughout this axiomatic approach to the development of real numbers. The Axioms for Real Numbers come in three parts:

The Field Axioms (Section 1.1) postulate basic algebraic properties of number: commutative and associative properties, the existence of identities and inverses.

The Order Axioms (Section 1.2) postulate the existence of positive numbers. Consequences of include the existence of integers and rational numbers.

The Completeness Axiom (Section 1.3) postulates the existence of least upper bound for bounded sets of real numbers. Consequences of completeness include infinite decimals are real numbers and that there are no 'gaps' in the number line.

The completeness of the real numbers paves the way for develop the concept of limit, Chapter 2, which in turn allows us to establish the foundational theorems of calculus establishing function properties of continuity, differentiation and integration, Chapters 4 and 5.

Goals

1. Prove the Fundamental Theorem of Calculus starting from just nine axioms that describe the real numbers.

2. Become proficient with reading and writing the types of proofs used in the development of Calculus, in particular proofs that use multiple quantifiers.

3. Read and repeat proofs of the important theorems of Real Analysis:

   - *The Nested Interval Theorem*
   - *The Bolzano-Weierstrass Theorem*
   - *The Intermediate Value Theorem*
   - *The Mean Value Theorem*
   - *The Fundamental Theorem of Calculus*

4. Develop a library of the examples of functions, sequences and sets to help explain the fundamental concepts of analysis.

Two Exercises to get started.

True or False 1

Which of the following statements are true? Explain your answer.

a) $0.\overline{9} > 1$

b) $0.\overline{9} < 1$

c) $0.\overline{9} = 1$

Calculation 1

Using your calculator only for addition, subtraction, multiplication and division, approximate $\sqrt{56}$. Make your answer accurate to within 0.001 of the exact answer. Write a procedure and explain why it works.

Contents

Chapter 0

Basic Notions

0.0 Getting Started

Biggles to secretary: Now, when I've got these antlers on - when I've got these antlers on I am dictating and when I take them off (takes them off) I am not dictating.
– from "Biggles Dictates a Letter," Monty Python's Flying Circus.

About moose antlers Υ: Many things in this book are already understood (or maybe we just think we understand them) and we don't want to forget them completely. At the same time we develop the real number system with a minimal set of concepts to guide us, we want to be able to use our intuition and ideas already mastered to guide our way and help us understand. We do want to keep straight where we are in this game. That's where moose antlers come in. This is how it works: when moose antlers are on, we can use what we already know to think about examples and proofs. When they are off we only think about the axioms and theorems that we have proven so far.

Look for the Υ moose antlers throughout the book. At those points feel free to use what mathematical knowledge and intuition you have to answer the questions. Otherwise, what you have at your disposal is the nine axioms and any previous theorems we have derived from those axioms using the basic notions of sets and logic that are summarized in this chapter.

0.1 Sets

0.1.1 Common Sets

Υ Some of all of these sets will be familiar to you from previous mathematical experiences. Throughout this book, we will be starting from scratch and defining each of them. References are provided below. They are all listed here to establish common notation. You may have used different notation for some of these sets and you may have other common sets you'd like to include. Do not hesitate to make your concerns known!

$\mathbb{R}$ represents the set of all real numbers. This set is the main interest and star of this course. And as in all good books the character will be developed slowly and carefully throughout the course. In the beginning, we assume a few things about how the elements in this set behave under the operations of addition and multiplication. This is quite abstract – we don't have any idea what the elements (which we will call numbers) of this set really are or even if such a set of things exists in any "real" (You can decide if this pun is intended or not) sense. From the axioms we will derive enough information to set up the familiar models for real numbers are – principally, decimal representation and the number line. We will also be able to conclude that any other system that satisfies the same axioms is essentially the same as the real number system we describe.

$\mathbb{R}^+$ represents the set of positive real numbers. Defining characteristics of this set will be established in Section 1.2

$\mathbb{R}^{\geq} = \mathbb{R}^+ \cup \{0\}$ represents the set of non-negative real numbers.

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the set of ordered pairs of real numbers - also called the Cartesian plane. In this book it is mostly used in reference to functions that map $\mathbb{R}$ to $\mathbb{R}$. In subsequent study of real analysis, $\mathbb{R}^n$ - ordered n-tuples of real numbers - take more central roles.

$\mathbb{N}$ and $\mathbb{Z}^+$ both represent the set of positive integers. It is a subset of the real numbers and we will later establish the characteristics of this set from the axioms of $\mathbb{R}$. Also called the set of Natural numbers. Very often the characteristics of these sets are establish by The Peano Axioms. The real numbers are then constructed from the integers. This is not the approach in this book. See Section 1.2.2.

$\mathbb{N}_0$ or $\mathbb{Z}^{\geq}$ both denote the set of non-negative integers.

$\mathbb{Z}$ represents the set of all integers. From our Υantler-less point of view we know nothing about this set. We will establish defining characteristics that will agree school-based ideas of what integers are.

$\mathbb{Q}$ represents the set of all rational numbers. They can be defined after we clarify the notion of devision and have defined the integers.

$\mathbb{Q}^+$ represents the set of all positive rational numbers.

$\varnothing$ represents the empty set.

0.1.2 Set Notation

Υ Moose antlers are tricky in this section. The book will assume you know (antlers on or not) about sets and functions and that you understand the set notation described in this chapter. It's part of the basic notions you'll need to proceed. However, all the examples of sets below require Υ moose antlers to understand. And you'll want the antlers on to come up with other examples.

We discuss three different ways to denote a set.

1. *By list.* This works perfectly for small finite sets, like $\{3, 36, 17\}$. It is also used for infinite sets that can be listed. For example, $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$ or $\{1, 6, 11, 16, \cdots\}$. Describing a set this way requires that everyone knows what rule is being used to generate the numbers.

   Example 0.1 The set $\{2, 3, 5, \cdots\}$ might be the prime numbers or it might be the Fibonacci numbers or it might be the integers of the form $2^n + 1$.

2. *By condition.*

   Example 0.2 $\{x : x \text{ is a prime number}\}$, read "the set of all x such that x is a prime number."

3. *Constructively by giving a formula that describes the elements of the set.* For example, $\{n^2 : n \in \mathbb{Z}\}$, the set of perfect square numbers. Note that you need to describe the set of all possible values for each variable in the formula. Note that an element of the set may be described more than once but this does not change the set. That is, $\{n^2 : n \in \mathbb{Z}\} = \{n^2 : n \in \mathbb{Z}^{\geq}\}$.

   Example 0.3 $\{\sin \frac{n\pi}{2} : \text{n is an integer}\} = \{0, 1, -1\}$

Exercise 0.1 Υ Is there a simpler description of $\{x^2 : x \in \mathbb{R}\}$?

Exercise 0.2 Υ Let S be the set of all odd positive integers. Describe this set in each of the three ways listed above.

Exercise 0.3 Υ Find numbers a, b, c so that the formula, $an^2 + bn + c$, for $n \in \mathbb{Z}^{\geq}$, describes a set like the one indicated in Example 0.1

0.1.3 Operations on Sets

We can also construct sets from other sets. This book assumes you are familiar with the union and intersection of a collection of sets and the complement of set with respect to another one. We write the complement of X with respect to Y as $Y \setminus X = \{y \in Y : y \notin X\}$.

0.2 Logic

0.2.1 Logical Statements

Throughout this book we will be proving theorems about real numbers. Theorems are statements that are either true or false and are stated in the form 'If p, then q' and notated $p \implies q$, where p and q are also statements. p is called the hypothesis (or antecedent) and q is the conclusion (or consequence) of the theorem. The proof of the theorem proceeds from the assumed fact that p is true and goes through a series of logically valid statements until one can conclude q.

Sometimes it is easier to show that the negation of the statement of the theorem is false in order to prove the theorem true. Therefore it is good to understand how to negate a statement. $\sim p$ denotes the negation of p. Please keep in mind that either p is true or $\sim p$ is true, but not both. Here is a table of different related statements and their negations. We will use these names for related statements throughout the book.

	statement	negation of statement
	$p \implies q$	p and $\sim q$
converse	$q \implies p$	q and $\sim p$
contrapositive	$\sim q \implies \sim p$	$\sim q$ and p

A statement and its contrapositive always have the same truth value. A statement and its converse may have different values.

Exercise 0.4 How do the following two statements fit into the table.

a) $\sim p$ or q

b) p or $\sim q$

Sometimes the hypothesis of a theorem is not explicitly stated but it is always there – if for no other purpose than to establish what sets the variables in the statement belong to. Throughout the book you are encouraged to explicitly state the hypothesis and the conclusion of theorems.

True or False 2

Which of the following statements is true? Explain. Modify the false statement to make a true statement.

a) If $x \in [2, 4]$, then $x^2 \in [4, 16]$.

b) If $x^2 \in [4, 16]$, then $x \in [2, 4]$.

0.2.2 Quantifiers

Our logical statements will almost always contain variables and those variables may be 'quantified.' One way to think about it is that the statement is describing a set by a condition. The quantifiers tell you 'how many' numbers are in the set. There are two different quantifiers:

1. FOR ALL Consider the following two statements and convince yourself that they mean the same thing:

$$\text{For all } x \in \mathbb{R}, x^2 \geq 0 \tag{0.1}$$
$$\{x \in \mathbb{R} : x^2 \geq 0\} = \mathbb{R} \tag{0.2}$$

2. THERE EXISTS Consider the following two statements and convince yourself that they mean the same thing:

$$\text{There exists } x \in \mathbb{R}, x^2 = 9 \tag{0.3}$$
$$\{x \in \mathbb{R} : x^2 = 9\} \neq \varnothing \tag{0.4}$$

Exercise 0.5 Write the negation of each of the above statements.

How to prove statements that contain quantifiers is a main concern in later discussions about real numbers.

0.3 Functions

No Υ mooseantlers needed for this section. Everything is defined in terms of sets and using basic logic!

0.3.1 Definitions, Notation and Examples

Definition A *function, f,* is a set of order pairs in $\mathbb{R} \times \mathbb{R}$ with the property that for each first coordinate x, there is a unique second coordinate, y, such that (x, y) is in the function. Because the y is unique for a given x, we can unambiguously write $y = f(x)$, the customary functional notation. The *domain of f* is the set of all first coordinates in f and the *target of f* is a set that contains all second coordinates in f. In these notes the target will always be a subset $\mathbb{R}$. Thus the function is the set, $\{(x, f(x)) : x \in D\}$. Sometimes this set is called the *graph* of f, but the graph is not a separate object from the function, as we have defined it.

Here is a long list of definitions and notation conventions that we will use throughout the book. It is assumed they are mostly familiar to the reader:

Notation If D is the domain of a function f, we write $f : D \to \mathbb{R}$

Definition The following definitions apply to the function, $f : D \to \mathbb{R}$. These notions should be familiar as they are fundamental to any study of mathematics.

- For any $S \subset D$, we say the *image* of S under f, and write $f(S)$, to mean $\{f(x) : x \in S\}$. The *image* of f is $f(D)$.
- For any $T \subset f(D)$, we say the *pre-image* of T under f, and write $f^{-1}(T)$, to mean $\{x : f(x) \in T\}$.
- We say that f is $1 - 1$ or f is an *injection,* if for all $x, y \in D$
$$f(x) = f(y) \implies x = y,$$
- If $E \subset \mathbb{R}$ is the target of f and if $f(D) = E$, we say that f is *onto* $E \subset \mathbb{R}$, In this case, we may also say that f is a surjection.
- We say that f is *bijection*, if it is both an injection and surjection. In this case, we also say that f shows a $1 - 1$ correspondence between D and E.

Example 0.4 The pre-image of $[.25, 1]$ under the function $x \to x^2$ is $[-1, -.5] \cup [.5, 1]$

Exercise 0.6 Give an example of each of the following. Include the domain as part of the description. Sketch the graph of the function.

a) a function that is not $1 - 1$ but is *onto*

b) a function whose pre-image of some set T is $\mathbb{R}^+$

c) a function whose image is $\mathbb{R}^+$

Figure 0.1: Graph of $s_n = \dfrac{1}{n}$

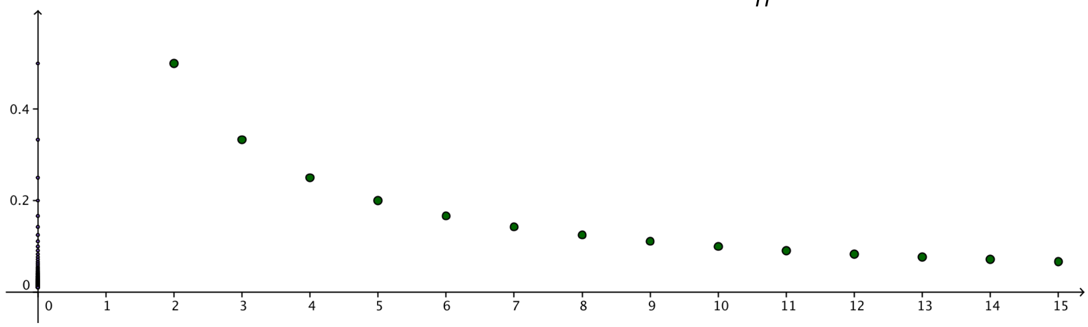
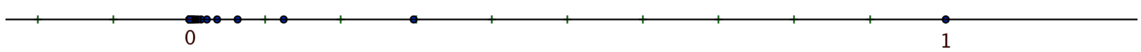


Figure 0.2: The image of $s_n = \dfrac{8}{n^3}$.



0.3.2 Sequences are functions $\mathbb{Z}^+ \to \mathbb{R}$ or $\mathbb{Z}^{\geq} \to \mathbb{R}$

Definition A function whose domain is $\mathbb{Z}^+$ or $\mathbb{Z}^{\geq}$ is called a *sequence*.

Notation A sequence is most usually denoted with subscript notation rather than standard function notation, that is we write s_n rather than $s(n)$.

Example 0.5 The graph in Figure 0.1 shows part of the graph of a sequence that maps $\mathbb{Z}^+ \to \mathbb{R}$ and is given by the formula, $s_n = \dfrac{1}{n}$. In addition, the first 100 numbers in the image of the sequence on the $y - axis$.

Example 0.6 Another way to picture a sequence is to plot the image on a number line, as shown in Figure 0.2. The downside is that the order of the sequence is not explicitly given. Here the image of the sequence, $s_n = \dfrac{8}{n^3}$, is shown on a horizontal number line. The order of the sequence values is not shown on this picture. You need to see the formula, as well, to understand that the values are being listed in order from right to left. The values in the image bunch up at zero to become indistinguishable from each other and from 0. The picture is insightful, but imprecise.

Notation Because of the ordering of the natural numbers, a sequence can be given by listing the first few values without reference to the domain or a formula, as in

$$3, \frac{3}{4}, \frac{1}{3}, \frac{3}{25}, \frac{1}{12} \cdots \tag{0.5}$$

This type of notation can be convenient but it never tells the whole story. How does the sequence continue past the values given? The finite sequence may suggest a pattern but one can't be sure without more information. We don't know whether the domain starts at 0 or 1, but a formula could be adjusted to fit either case. NOTE: It would be wrong to include braces $\{\}$ around the sequence because that would indicate a *set*. It would be how to denote the *image* of the sequence.

Exercise 0.7 Find a possible formula that would generate the sequence in Example 0.5. Sketch the graph of this sequence.

0.4 True or False

The following True or False problems explore different logical statements using quantifiers in a variety of ways.

True or False 3

Which of the following statements are true? Explain. Change $\mathbb{Z}$ to $\mathbb{R}$ and redo.

a) There exists an $x \in \mathbb{Z}$, such that x is odd.

b) For all $x \in \mathbb{Z}$, x is even.

c) There exists an $x \in \mathbb{Z}$, such that $2x$ is odd.

d) For all $x \in \mathbb{Z}$, $2x$ is even.

True or False 4

(From Morgan) Which of the following statements are true? Explain.

a) For all $x \in \mathbb{R}$, there exists a $y \in \mathbb{R}$ such that $y > x^2$.

b) There exists an $y \in \mathbb{R}$, such that for all $x \in \mathbb{R}$, $y > x^2$.

c) There exists an $y \in \mathbb{R}$, such that for all $x \in \mathbb{R}$, $y < x^2$.

d) For all $a, b, c \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $ax^2 + bx + c = 0$.

True or False 5

Which of the following statements are true? Explain.

a) There exists a real number, x, such that $x^2 = 9$.

b) There exists a unique real number, x, such that $x^2 = 9$.

c) There exists a unique positive real number, x, such that $x^2 = 9$.

The following True or False problem concerns the notion of pre-image.

True or False 6

Which of the following statements true? Prove or give a counterexample. Consider conditions of f that would make the statements True.

a) $f(X \cap Y) = f(X) \cap f(Y)$

b) $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$

c) $f(f^{-1}(Y)) = Y$.

d) $f^{-1}(f(Y)) = Y$.

Exercise 0.8 In some cases, it may be easier to determine if the negation of a statement is true or false. If you haven't already, write the negation of each statement in the True or False problems.

Chapter 1

The Real Number System

We begin by supposing the existence of a set, $\mathbb{R}$, whose elements we call *real numbers* We suppose we know a few things about numbers. We know when two of them are equal, section 1.0.1. We know that we can add or multiple two of them and get an unique number, section 1.0.2. We know some conventions for how to write expressions involving adding and multiplying, section 1.0.3. Most importantly we postulate nine *Axioms*, the defining characteristics of real numbers. These include the *Field Axioms*, Section 1.1; the *Order Axioms*, Section 1.2; and the *Completeness Axiom*, Section 1.3.

We do not assume that we can represent real numbers as decimals. Nor how to represent real numbers on a number line. However, our Υ intuition using these two models for real numbers can guide our thinking.

1.0 Definitions and Basic Notions from Algebra

1.0.1 Equality

Basic Notion 1 EQUALITY OF REAL NUMBERS IS AN EQUIVALENCE RELATION. *The following properties apply for all real number x and y:*

Reflexive $x = x$
Symmetric *if* $x = y$*, then* $y = x$
Transitive *if* $x = y$ *and* $y = z$*, then* $x = z$

1.0.2 Addition and Multiplication

We assume the existence of a set of two *binary* operations on the 'numbers' in this set. Basic notions about equality apply. Both addition and multiplication produce a unique, answer, meaning that adding a number a and multiplying by a number m are both functions.

Basic Notion 2 UNIQUENESS OF ADDITION *For all* $a, x, y \in \mathbb{R}$.

$$x = y \implies a + x = a + y.$$

Basic Notion 3 UNIQUENESS OF MULTIPLICATION *For all* $m, x, y \in \mathbb{R}$,

$$x = y \implies m \cdot x = m \cdot y.$$

It is often useful, and some people prefer, to consider addition and multiplication as functions. That is, for every real number a, there is a function, $s_a : \mathbb{R} \to \mathbb{R}$, given by

$$s_a(x) = x + a,$$

and, for every real number m, there is a function, $t_m : \mathbb{R} \to \mathbb{R}$, given by

$$t_m(x) = m \cdot x.$$

The UNIQUENESS OF ADDITION AND MULTIPLICATION says that these functions are indeed functions, i.e. there is only one value for each element in the domain.

The uniqueness of these operations is used in our preliminary work when doing things like adding the same number to both sides of an equation.

1.0.3 Expressions

Binary means that the operation works on only two numbers at a time, so expressions like $a + b + c$ aren't meaningful until we know more about what rules apply. However, we can include parentheses in expressions and so legitimately know what to do. The expression, $a + (b + c)$, is meaningful: first add b to c, then add a to the result. This use of parentheses is assumed familiar to the student of this book. (Υ Once we know the associative and commutative rules, $a + b + c$ is not an ambiguous expression.)

Basic to working with equations with variables and real numbers is being able to 'substitute' equal expressions for each other. This is how we can build more and more complicated, and thus interesting, expressions. This is however, not an easy concept to formalize. Here is a stab at it:

Basic Notion 4 SUBSTITUTION *If u is a real number such that $u = E(x, y, z, \cdots)$, where $E(x, y, z, \cdots)$ is any legitimate expression for a real number involving other real numbers $x, y, z, \cdots$, then u may be interchanged for $E(x, y, z, \cdots)$ in any other expression without changing the value of that expression. NOTE: transitivity, from Basic Notion 1, is an elementary example of substitution.*

Υ Eventually, we want the substitution principle to apply to more sophisticated expressions like $\sin(a + b)$ or re^{-3c} or $\lim_{x\to p} f(x)$, so we use the (admittedly) imprecise language, 'legitimate expression'. Once one learns that these expressions, or others, represent real numbers, we are free to use substitution on such expressions: If $u = \sin(x)$, then $\sin^2(x) + 2\sin(x) + 1 = u^2 + 2u + 1$

1.1 The Field Axioms

There is a set $\mathbb{R}$ of *real numbers* with an addition ($+$) and a multiplication ($\cdot$) operator, that satisfy the following properties:

Axiom 1 COMMUTATIVE LAWS. *For all real numbers, a and b,*

FOR ADDITION $a+b=b+a$
FOR MULTIPLICATION $a\cdot b=b\cdot a$

Axiom 2 ASSOCIATIVE LAWS. *For all real numbers, a, b and c,*

FOR ADDITION $a+(b+c)=(a+b)+c$
FOR MULTIPLICATION $a\cdot(b\cdot c)=(a\cdot b)\cdot c$

Axiom 3 DISTRIBUTIVE LAW. *For all real numbers, a, b and c,*

$$a\cdot(b+c)=a\cdot b+a\cdot c$$

Axiom 4 EXISTENCE OF IDENTITY ELEMENTS.

FOR ADDITION *There is a real number, 0, such that, $a+0=a$, for all $a\in\mathbb{R}$*
FOR MULTIPLICATION *There is a real number, 1, such that, $1\cdot a=a$, for all $a\in\mathbb{R}$*
FURTHERMORE, $0\neq 1$

Axiom 5 EXISTENCE OF INVERSES.

FOR ADDITION *For all $a\in\mathbb{R}$, there is an $x\in\mathbb{R}$, such that $a+x=0$.*
FOR MULTIPLICATION *For all $a\in\mathbb{R}$, $a\neq 0$, there is an $x\in\mathbb{R}$, such that $a\cdot x=1$.*

1.1.0 Consequences of the Field Axioms

In this section we state and prove many facts about real numbers. We call these facts theorems and, occasionally, corollaries or lemmas. Theorems are not always stated explicitly in the form $p \implies q$, but they can be stated that way. Often the only hypothesis is that the elements being discussed are real numbers, i.e. we are always assuming that there is a set of real numbers that satisfy the basic axioms as well as all theorems we prove.

Exercise 1.1 Provide proofs for Theorems 1.0 - 1.15. Some of the proofs are provided.

Theorem 1.0 *For all* $a \in \mathbb{R}$, $0 + a = a$ *and* $1 \cdot a = a$

Theorem 1.1 CANCELLATION LAW FOR ADDITION *If* $a + b = a + c$, *then* $b = c$.

Proof of Theorem. Let e be a real number from **Axiom 5** such that $a + e = 0$. By **Axiom 1**, it is also true that $e + a = 0$. *Note: here we use what is sometimes called a two column proof: the left side is a valid conclusion following from the previous statements and the right side is the justification or* warrant *for that conclusion.*

$a + b = a + c$	given, the hypothesis
$e + (a + b) = e + (a + c)$	uniqueness of addition, add e to both sides
$(e + a) + b = (e + a) + c$	associative law for addition, **Axiom 2**
$0 + b = 0 + c$	by substitution, as stated above: $e + a = 0$
$b = c$	Theorem 1.0

□

Theorem 1.2 *The number,* 0*, the additive identity of* **Axiom 4***, is unique.*

Proof. Note: the strategy used to show that a number is unique is to assume there are two numbers that satisfy the given condition and then show that they are equal. Assume there exists another real number $0'$ such that $a + 0' = a$ for all $a \in \mathbb{R}$, then for all $a \in \mathbb{R}$, we have that $a + 0 = a + 0'$. By the CANCELLATION LAW FOR ADDITION, Theorem 1.1, we can cancel the a's to get, $0 = 0'$. This shows that 0 is unique. □

Theorem 1.3 EXISTENCE AND UNIQUENESS OF SUBTRACTION *For all* $a, b \in \mathbb{R}$, *there is a unique solution,* $x \in \mathbb{R}$, *to the equation* $a + x = b$.

Notation The unique solution to the equation, $a + x = b$, is denoted by $b - a$. The unique solution to the equation, $a + x = 0$, $0 - a$, is written simply $-a$. We call it the *additive inverse* of a or the *negative* of a.

Proof. Let e be a real number from **Axiom 5** such that $a + e = 0$ and let $x = e + b$. Now consider

$a + x = a + (e + b)$	substituting our definition of x
$= (a + e) + b$	associative law for addition, **Axiom 2**
$= 0 + b$	substituting 0 for $a + e$
$= b$	Theorem 1.0

To show uniqueness, assume that there is another real number, y, such that $a+y = b$. Then $a+x = a+y$. So by the CANCELLATION LAW FOR ADDITION, Theorem 1.1, $x = y$. □

Theorem 1.4 *For all* $a \in \mathbb{R}$,

$$-(-a) = a.$$

Proof. Given any number $a \in \mathbb{R}$,

$-a + a = a + (-a)$	commutative law for addition, **Axiom 1**
$a + (-a) = 0$	definition of $-a$ from Theorem 1.3.
$-a + a = 0$	transitive property of equality

So a is the negative of $-a$. The negative of $-a$ is written, $-(-a)$, as seen in the note after Theorem 1.3. □

Theorem 1.5 ADDITION DISTRIBUTES ACROSS SUBTRACTION *For all* $a, b, c \in \mathbb{R}$,

$$a(b - c) = ab - ac.$$

Proof.

$a(b - c) + ac = a((b - c) + c)$	distributive law, **Axiom 3**
$a((b - c) + c) = ab$	definition $b - c$ from Theorem 1.3
$a(b - c) + ac = ab$	transitive property of equality
$a(b - c) = ab - bc$	EXISTENCE AND UNIQUENESS OF SUBTRACTION, Theorem 1.3

□

Theorem 1.6 *For all* $a \in \mathbb{R}$,

$$0 \cdot a = a \cdot 0 = 0.$$

Proof. This is a special case of Theorem 1.5, where $b = c$. □

Theorem 1.7 CANCELLATION LAW FOR MULTIPLICATION *For all* $a, b, c \in \mathbb{R}$,

$$ab = ac \text{ and } a \neq 0 \implies b = c.$$

Proof. EFS □

Theorem 1.8 *The number,* 1, *of* **Axiom 4** *is unique.*

Proof. EFS □

Theorem 1.9 EXISTENCE AND UNIQUENESS OF DIVISION *For all* $a,\ b \in \mathbb{R}$, $a \neq 0$, *there is a unique solution,* $x \in \mathbb{R}$, *to the equation* $a \cdot x = b$.

Notation The unique solution to the equation $a \cdot x = b$ is written $\frac{b}{a}$. The unique solution to the equation $a \cdot x = 1$, $\frac{1}{a}$, is also written as a^{-1}. It is the multiplicative inverse of a and also callied the *reciprocal* of a.

Proof. EFS □

Theorem 1.10 THERE ARE NO ZERO DIVISORS

$$a \cdot b = 0 \implies a = 0 \text{ or } b = 0.$$

Proof. EFS □

Example 1.1 Modular arithmetic is an example where there are zero divisors. Because $2 \cdot 4 \equiv 0 \mod 8$ and neither is $\equiv 0 \mod 8$ we call both 2 and 4 *zero divisors mod* 8.

Theorem 1.11 *For all* $a \in \mathbb{R}$, $a \neq 0$,

$$(a^{-1})^{-1} = a \text{ or } \frac{1}{\frac{1}{a}} = a.$$

Proof. EFS □

Theorem 1.12 *For all* $a, b \in \mathbb{R}$, $b \neq 0$,

$$\frac{a}{b} = a \cdot b^{-1}.$$

Proof. EFS □

Theorem 1.13 ADDITION OF FRACTIONS *For all* $a, b, c, d \in \mathbb{R}$, $b \neq 0$ *and* $d \neq 0$,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Proof. EFS □

Theorem 1.14 MULTIPLICATION OF FRACTIONS *For all $a, b, c, d \in \mathbb{R}$, $b \neq 0$ and $d \neq 0$,*

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

Proof. EFS □

Corollary 1.14.1 *For all $a, b \in \mathbb{R}$, $a \neq 0$ and $b \neq 0$,*

$$\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b} \text{ or } (a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$$

Proof. EFS □

NOTE: Sometimes corollary 1.14.1 is proven first as a lemma and is then used to prove theorem 1.14.

Theorem 1.15 EXISTENCE AND UNIQUENESS OF SOLUTION TO LINEAR EQUATIONS *For all a, b, $c \in \mathbb{R}$, $a \neq 0$, there is a unique solution, $x \in \mathbb{R}$, to the equation*

$$a \cdot x + b = c.$$

Proof. EFS □

Exercise 1.2 Prove using the Axioms 1 through 5 and the Theorems 1.0 - 1.15.

For all $a, b, c, d \in \mathbb{R}$.

a) $-0 = 0$; $1^{-1} = 1$

b) $-(a + b) = -a - b$

c) $-(a - b) = -a + b$

d) $(a - b) + (b - c) = a - c$

e) $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$

f) $-(\frac{a}{b}) = \frac{-a}{b} = \frac{a}{-b}$

1.1.1 Comments on the Field Axioms

Addition vs multiplication Except for the Distributive Law, **Axiom 3**, and the part of **Axiom 5** where 0 is excluded from having a multiplicative inverse, the axioms are symmetric in addition and multiplication. **Axiom 3** says that 'multiplication distributes across addition.' What would happen if the opposite were true that 'addition distributes across multiplication?'

Exercise 1.3 Write down a rule that would say that addition distributes across multiplication. Prove that it cannot be true if the Field Axioms are true.

Exercise 1.4 Something to contemplate: Why is it necessary to exclude a multiplicative inverse for 0?

Associativity and commutativity Without parentheses, we do not know how to resolve an expression like $a + b + c$ or $a \cdot b \cdot c$ The Associative Law, **Axiom 2**, tells us that the two ways to add parentheses will give you the same answer. This is worth pondering with specific numbers (Υ required). Consider the expression, $3+4+5$. We can resolve to either $7 + 5$ or $3 + 9$. We get the same thing, 12, either way.

Exercise 1.5 How many ways can you add parentheses to $2+3+4+5$ to get a different way to sum the numbers? Let c_n = the number of ways to put parentheses on an addition string with n numbers. What is c_n? (These are well-known as the Carmichael Numbers.)

1.1.2 Examples of Fields

A set, together with a well-defined addition and multiplication, is called a *Field* if the Field Axioms (Section 1.1) are all satisfied.

Example 1.2 Υ Think of some other number systems, such as

a) $\mathbb{Q}$, the rational numbers.

b) $\mathbb{R}^2$, the plane of ordered pairs of real numbers.

c) $\mathbb{C}$, the complex numbers.

d) $\mathbb{Z}/\mathbb{Z}n$, integers $\mod n$

e) $\mathbb{Q}[x]$, polynomials with rational coefficients

f) $\mathbb{Q}(x)$, rational functions (ratios of functions $\in \mathbb{Q}[x]$).

Does each system have a well-defined addition and multiplication? Which satisfy the Field Axioms?

1.2 The Order Axioms

ϒSome geometric concerns One useful way to represent real numbers is on a number line. To do this we need rules to decide how to place the numbers. When is one number to the right or to the left of another number? Put another way, when is one real number 'larger' than another? Or even, what does 'larger' mean? The standard approach is to first decide which ones are 'larger' than 0, section 1.2. So we can declare that 'b is larger than a' whenever '$b - a$ is larger than 0.' The Order Axioms, then, are closely related to notions of distance and length, section 1.2.4. More surprisingly, perhaps, is that the Order Axioms allow us to think about integers, section 1.2.2. Then rational numbers can be defined, section 1.2.3. In this section we see how these concepts are developed from axioms. There exists a subset, $\mathbb{R}^+ \subset \mathbb{R}$, with the following properties:

Axiom 6 *If a and b are in $\mathbb{R}^+$, then $a + b \in \mathbb{R}^+$ and $a \cdot b \in \mathbb{R}^+$.*

Axiom 7 *If $a \neq 0$, then either $a \in \mathbb{R}^+$ or $-a \in \mathbb{R}^+$ but not both.*

Axiom 8 $0 \notin \mathbb{R}^+$

1.2.1 Consequences of the Order Axioms

Definition We say that a real number, x, is a *positive* number whenever $x \in \mathbb{R}^+$. We say that a real number, x, is a *negative* number whenever $-x \in \mathbb{R}^+$.

Notation If $b - a$ is a positive number, we write $a < b$ or $b > a$. In this case, we say 'a is less than b' or 'b is greater than a.'

An immediate and important consequence of the order axioms is:

Theorem 1.16 *1 is a positive number.*

Proof. By **Axiom 7** if $1 \notin \mathbb{R}^+$, then $-1 \in \mathbb{R}^+$ which would mean that

$$(-1) \cdot (-1) = 1 \in \mathbb{R}^+.$$

Which shows not only that $1 \in \mathbb{R}^+$ but also that $-1 \notin \mathbb{R}^+$. □

Exercise 1.6 Provide proofs for Theorems 1.17 - 1.21. Some of the proofs are provided.

Theorem 1.17 LAW OF TRICHOTOMY *For all $a, b, c \in \mathbb{R}$, exactly one of the following are true:*

$$a = b,\ a < b,\ b < a$$

NOTE: In the special case, when one of a and b is zero the Law of Trichotomy says that a real number is exclusively positive, negative, or zero.

Proof. EFS □

Theorem 1.18 TRANSITIVITY *For all* $a, b, c \in \mathbb{R}$,

$$a < b \text{ and } b < c \implies a < c$$

Proof. EFS □

NOTE: 'Less than' ($<$) forms a relation between numbers. We have just shown that it is a transitive relation. However, it is neither symmetric nor reflective so it is not an equivalence relation like '$=$'. See **Basic Notion 1**

Exercise 1.7 There are three variations of TRANSITIVITY when '$\leq$' replaces '$<$' in one or the other or both of spots in the hypothesis. State each one, providing the strongest conclusion in each case. Prove at least one of your statements. Use TRANSITIVITY , Theorem 1.18 rather than repeating proofs.

Exercise 1.8

Notation There are many different varieties of *intervals.* Write each one of the following using set notation:

$(a, b) = \{x \in \mathbb{R} : a < x \text{ and } x < b\}$. Sometimes abbreviated: $a < x < b$

$[a, b]$

$(a, b]$

$[a, b)$

(a, ∞)

$[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$.

(∞, b)

$(\infty, b]$

Exercise 1.9 How would you notate the set, $\{x : a < x \text{ or } x < b\}$? How does it vary with whether or not $a < b$ or $b < a$?

Theorem 1.19 ADDITION PRESERVES ORDER *If a is a real number, then*

$$x < y \implies a + x < a + y$$

Proof. EFS □

Theorem 1.20 *If $a < b$ and $c < d$, then $a + c < b + d$*

Proof. EFS □

Multiplication preserves order only when the multiplication factor is positive:

Theorem 1.21 *If $m > 0$, then multiplication by m preserves order, that is*

$$x < y \implies m \cdot x < m \cdot y.$$

If $m < 0$, then multiplication by m reverses order, that is

$$x < y \implies m \cdot x > m \cdot y.$$

Proof. EFS □

Exercise 1.10 Here are some more basic facts about order. You may want to prove them in a different 'order.'

a) For all $a \in \mathbb{R}$, if $a \neq 0$, then $a^2 > 0$.

b) If $a < b$, then $-b < -a$.

c) If $a > 0$, then $a^{-1} > 0$.

d) If $0 < a < b$, then $a^{-1} > b^{-1} > 0$.

e) The sum of two negative numbers is negative.

f) The product of two negative numbers is positive.

g) The product of a negative number and a positive number is negative.

h) For all $a, b \in \mathbb{R}$, $a^2 + b^2 = 0 \iff a = b = 0$.

True or False 7

If true, prove the statement. If false, restate to make a true fact and prove it.

a) $x^2 > x$.

b) If w_1, $w_2 > 0$ and $w_1 + w_2 = 1$, then

$$a < b \implies a < w_1 a + w_2 b < b$$

c) If $a < b$ and $c < d$, then $ac < bd$.

Definition We say a function, $f : D \to \mathbb{R}$, is *increasing* whenever, for all $x, y \in D$,

$$x \leq y \implies f(x) \leq f(y).$$

We say a function is *strictly increasing* whenever, for all $x, y \in D$,

$$x < y \implies f(x) < f(y).$$

We say a function is *decreasing* whenever, for all $x, y \in D$,

$$x \leq y \implies f(y) \leq f(x).$$

We say a function is *strictly decreasing* whenever, for all $x, y \in D$,

$$x < y \implies f(y) < f(x).$$

Example 1.3 The function $s_a(x) = x + a$ is increasing by Theorem 1.19.

Exercise 1.11 When is multiplication by m an increasing function? When is it decreasing?

Exercise 1.12 Prove that the function, $f(x) = x^2$, is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$.

Exercise 1.13 Prove that the function, $f(x) = \frac{1}{x}$, is decreasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

Exercise 1.14 Prove that the function, $f(x) = \frac{x+1}{x-1}$, is decreasing on $(1, \infty)$.

1.2.2 Integers

ΥMathematical induction. We have mentioned earlier that another way to develop the real number system is to start with integers and the Peano Axioms then construct the rational and irrational numbers. With our current endeavor, however, we still have nothing to say about integers. We do know 1 and so could define integers: $2 = 1 + 1$ and $3 = 2 + 1$ and so on. Of course, it is the 'so on' that leaves us with less precision than we like for a usable definition. But we have the idea of an inductive process and, in this section, we see how to make induction part of our development. We start with a definition:

Definition We say that a subset of $\mathbb{R}$ is an *Inductive Set* whenever both of the following conditions hold:

- $1 \in S$
- If $n \in S$, then $n + 1 \in S$

Example 1.4 The set of positive real numbers is an inductive set. $1 \in \mathbb{R}^+$ by Theorem 1.16 . The second condition follows because the sum of two positive real numbers is a positive real number by **Axiom 6.**

Exercise 1.15 What is the largest inductive set you can think of? What is the smallest?

Definition We say that a real number is a *positive integer* if it is contained in every inductive set.

Notation We denote the set of all positive integers as $\mathbb{Z}^+$.

$\mathbb{Z}^+$ is the *smallest inductive set* in the sense that it is contained in every other one.

Theorem 1.22 *The positive integers are positive real numbers.*

Proof. This is because the set of positive real numbers is an inductive set, so every positive integer is contained in it. □

Definition The *negative integers* are $\{-n : n \in \mathbb{Z}^+\}$, the negative positive integers, denoted by $\mathbb{Z}^-$. The *integers* are the positive integers together with the negative integers and 0, denoted by $\mathbb{Z} = \mathbb{Z}^+ \cup \{0\} \cup \mathbb{Z}^-$.

Exercise 1.16 Prove: There is no integer in the open interval $(0, 1)$.

Mathematical Induction.

To establish the algebraic structure of the integers there is some work to do and Mathematical Induction will be a major tool. The following theorem establishes the legitimacy of the induction procedure.

Theorem 1.23 MATHEMATICAL INDUCTION *Let S be a set of positive integers that is an inductive set, i.e. satisfies the following two conditions:*

- $1 \in S$
- *If $n \in S$, then $n + 1 \in S$*

then $S = \mathbb{Z}^+$.

Proof. By definition, S is an induction set so $\mathbb{Z}^+ \subset S$. By hypothesis, $S \subset \mathbb{Z}^+$. So $S = \mathbb{Z}^+$. □

This theorem is the basis for proof by mathematical induction: To prove a fact by mathematical induction, first restate the fact as a statement about a subset of positive integers. For example, define a set of positive integers, S, such that $n \in S$ if and only if some property, $P(n)$, is true. Then show that S is an inductive set by

- Showing a 'base case,' $P(1)$ is true ($1 \in S$)
- Showing an inductive step: If $P(n)$ is true ($n \in S$), then $P(n+1)$ is true ($n+1 \in S$).

Finally, apply Theorem 1.23 to conclude that S is all positive integers, so $P(n)$ is true for all $n \in \mathbb{Z}^+$. Υ By all means use methods you have used before and are comfortable with, but do understand how the process fits into the grand scheme of things.

The first theorems we will prove with induction establish the algebraic structure of the integers.

The integers form a commutative ring

Since the integers are a subset of the real numbers, they satisfy of all the field axioms, except for the existence of multiplicative inverses, which is not ring axiom. So we only need to show closure:

Theorem 1.24 ALGEBRAIC PROPERTIES OF INTEGERS

1. *The sum of two integers is an integer.*
2. *The product of two integers is an integer.*
3. *The negative of an integer is an integer.*

Outline of proof for sums. First, fix a positive integer m. Use induction to show that $\{n \in \mathbb{Z}^+ : m+n \in \mathbb{Z}\} = \mathbb{Z}^+$ and $\{n \in \mathbb{Z}^+ : -n+m \in \mathbb{Z}\} = \mathbb{Z}^+$. Finally, show that the sum of two negative integers is an integer without another induction proof. $\square$

Exercise 1.17 Prove: The only two integers that have a multiplicative inverse are 1 and -1.

The Well-Ordering Principle

When establishing integers from axioms, MATHEMATICAL INDUCTION is sometimes used as an axiom. Sometimes the following theorem is used instead and this theorem will be useful for us later. In any case, it can be proved with mathematical induction.

Theorem 1.25 THE WELL-ORDERING PRINCIPLE *Every non-empty set of positive integers contains a smallest integer.*

Proof. Let W be a subset of the positive integers that does not contain a smallest element. We will show that $W = \varnothing$. Let $S = \{k : [1, k] \cap W = \varnothing\}$. We will show that S is an inductive set.

- $1 \in S$ because if it were not in S it would be in W and it would be the smallest element in W.
- Assume $n \in S$. This means that no $k \leq n$ are in W. Now if $n+1 \in W$ it would be the smallest integer in W, but W does not have a smallest element so $[1, n+1] \cap W = \varnothing$. In other words, $n+1 \in S$.

Therefore S is an inductive set of positive integers, so must be all of them. In other words W is empty. We conclude that any non-empty set of positive integers must contain a smallest element. □

NOTE: Do not confuse the above theorem with THE WELL-ORDERING THEOREM, a theorem dependent on the Axiom of Choice. The Axiom of Choice is often included in the axioms for set theory despite certain bizarre behavior such as the Banach-Tarski Paradox. These considerations become of more interest in the the study of Lebesgue integration and will not come up for us in this course.

Exercise 1.18 Use the THE WELL-ORDERING PRINCIPLE to prove that there is no positive integer M such that $2^k < M$ for all $k \in \mathbb{Z}^+$.

Exercise 1.19 You may have noticed that our proof of THE WELL-ORDERING PRINCIPLE, Theorem 1.25, could be simplified by using *Strong Induction.* Find a good statement of strong induction and prove it using MATHEMATICAL INDUCTION, Theorem 1.23. Proceed to rewrite the proof of THE WELL-ORDERING PRINCIPLE, Theorem 1.25, using strong induction.

Practice with Induction.

Exercise 1.20 For practice with mathematical induction, prove the following two theorems. We will need both of them later in the course.

Theorem 1.26 *For all positive integers n,* $\sum_{k=0}^{n} k^2 = \frac{1}{6} \cdot n \cdot (n+1) \cdot (2n+1)$

Proof. EFS □

Theorem 1.27 BERNOULLI'S INEQUALITY *For any positive real number, x, and for all positive integers n,* $(1+x)^n \geq 1 + n \cdot x$

Proof. EFS □

Exercise 1.21 Bernoulli's Inequality is useful for many things. Use it to prove that

$$\frac{1}{2^n} < \frac{1}{n}, \text{ for all } n > 0.$$

Inductive Definitions

This way to define integers is an example of using an *inductive definition*. Defining a sequence *by recursion* is another. The first few elements in the sequence are given and the rest are defined in terms of the previous ones. For example we define a sequence, s_n by recursion (or inductively) by first defining $s_0 = 1$ and then, once we have defined s_k for $k < n$, define $s_n = n + 4$. Now we understand the sequence to be $1, 5, 9, 13 \ldots$. A direct or, closed form, description of the sequence is simply $s_n = 1 + 4 \cdot n$. Later, we will see examples of defining sequences of sets recursively when using the NESTED INTERVAL THEOREM.

Exercise 1.22 List the first 10 numbers in the following sequence, given by a recursive formula.

a) $x_0 = 1$ and $x_n = x_{n-1} + 5$.

b) $x_0 = 1$, $x_1 = 1$ and $x_n = x_{n-2} + x_{n-1}$.

Exercise 1.23 Identify a pattern for the following sequences. Write a recursive formula and a closed formula to describe each one.

a) $6, 18, 54, 162, \cdots$

b) $2.0, 0.2, 0.02, 0.002 \cdots$

Exercise 1.24 Write a recursive formula to capture and continue this sequence:
$\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}},\ \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}},\ \cdots$

Exercise 1.25 ϒMake a formal inductive definition that captures the procedure you established for approximating $\sqrt{56}$, accurate to within $(0.1)^n$.

1.2.3 Rational Numbers

Definition We say that a real number, r, is a *rational number* whenever there exist integers n and m such that

$$r = \frac{n}{m}$$

We denote the set of all rational numbers by $\mathbb{Q}$.

Exercise 1.26 Prove the following two theorems to establish the structure of the rational numbers.

Theorem 1.28 ALGEBRAIC PROPERTIES OF $\mathbb{Q}$

1. *0 and 1 are rational numbers.*
2. *The sum of two rational numbers is a rational number.*
3. *The product of two rational numbers is a rational number.*
4. *The negative of a rational number is a rational number.*
5. *The multiplicative inverse a rational number is a rational number.*

Proof. EFS □

Theorem 1.29 $\mathbb{Q}^+ = \{r \in \mathbb{Q} : r > 0\}$ *satisfies the Order Axioms.*

Proof. EFS □

Exercise 1.27 Use the previous two theorems to support the claim: the rational numbers form an Ordered Field.

Exercise 1.28 The following exercise about even and odd integers gives enough ammunition to show that the square root of 2 cannot be rational. Think of a good definition for an even integer and an accompanying good definition for an odd integer. From your definitions, prove the following things (not necessarily in this order; find the order that works well for your definitions):

a) The sum of two even integers is even; the sum of two odd integers is even; the sum of an even integer and an odd integer is odd.

b) Zero is an even; 1 is an odd; every integer is either odd or even, but not both.

c) x is odd iff $-x$ is odd; x is even iff $-x$ is even.

d) x is odd iff $x + 1$ is even; x is even iff $x + 1$ is odd.

e) Every integer (positive or negative or zero) is either odd or even.

f) The product of two odd integers is odd; the product of an even integer and any integer is even.

g) For any integer n, $2n$ is an even integer and $2n + 1$ is an odd integer.

h) If n^2 is an even integer, then n is also an even integer.

i) Every integer can be expressed as a power of 2 times an odd integer.

j) Every rational number can be written as $\frac{m}{n}$ where both m and n are integers and at least one of them is odd.

If you cannot prove these things from your definitions, you will need to change your definitions so that you can. We can now prove the following theorem.

Theorem 1.30 *There is no rational number, s, such that $s^2 = 2$.*

Proof. Assume not, so there is a rational number $s = \frac{n}{m}$ and $s^2 = 2$. By Exercise 1.28 j) we can assume that one of n or m is odd. Now $s^2 = 2 = \frac{n^2}{m^2}$ or $2m^2 = n^2$. n^2 is even so n is even by Exercise 1.28 h). Let k be the integer such that $n = 2k$. Then $2m^2 = 4k^2$ and $m^2 = 2k^2$. Now Exercise 1.28 j) says m is even, in contradiction to the assumption that one of m and n is odd. So there can be no rational number whose square is 2. □

NOTE: Nothing is being said about whether or not the square root of 2 actually exists, only that, if it does exist, it cannot be a rational number. Because the rational numbers satisfy all of our axioms so far, we know we need more to be able to say that there is a real number, s, such that $s^2 = 2$. The final axiom is THE COMPLETENESS AXIOM which provides what is needed and we will get there soon.

1.2.4 Distance, absolute value and the Triangle Inequality

Discussion of the number line Υ

It's easy to figure out a way to mark real numbers on a line using compass and straightedge constructions from Euclidean geometry. First, label one point on the line 0 and another 1. By convention, 1 is to the right of 0. Set the compass to the length of the segment determined by these two points. Use this setting to mark off 2, 3 and so on. By going the other way, mark off $-1, -2, -3, \cdots$ Rational numbers can be included by employing similar triangles. Addition is defined by copying a segment next to another segment. Multiplication can be defined by similar triangle constructions. But this does not get all of the real numbers because we know we can construct $\sqrt{2}$ – it is the diagonal of a square which has sides of length 1. However, there are many other real numbers that can not be constructed. Eventually we will be able to locate numbers by using THE COMPLETENESS AXIOM. We do not intend to development the number line rigorously but we do use it to enhance our intuition. In particular, we are about to introduce the idea of the distance between to real numbers.

Absolute Value and Distance

Definition The *absolute value of* x, written as $|x|$, defines a function. The value of the function is given in parts:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0. \end{cases}$$

Theorem 1.31 *Four obvious facts about any real number, x:*

1. *If $x > 0$, then $x = |x|$*
2. *If $x < 0$, then $x = -|x|$*
3. $|x| = 0 \iff x = 0$.
4. *If $x \neq 0$, then $|x| > 0$.*

Proof. EFS □

Definition We say that the *distance between a and b* is $|b - a|$. We say that the *length* of an interval is the distance between its endpoints.

It can be useful to think about absolute value in the more intuitive ideas of distance. For example, Theorem 1.31 can be restated: The distance between any real number and zero is positive, unless that number is zero - in which case, the distance is zero.

Theorem 1.32 *The distance between two numbers is zero if and only if the two numbers are equal.*

Proof. $|a - b| = 0 \iff a - b = 0 \iff a = b$. □

The following, while obvious, is useful because it can eliminate cases when proving the THE TRIANGLE INEQUALITY, Theorem 1.2.4, which in turn is very useful throughout analysis.

Lemma 1.33 $-|x| \leq x \leq |x|$

Proof. If x is positive, it is the right endpoint of the interval $[-|x|, |x|\,]$. If x is negative, it is the left endpoint. If $x = 0$, then $-|x| = x = |x|$. In any case, $-|x| \leq x \leq |x|$. □

Exercise 1.29 Provide proofs for Theorems 1.34 - 1.36. State the theorems in terms of distances when possible. Some of the proofs are provided.

Theorem 1.34 $|a| < |b| \iff -|b| < a < |b|$

Proof. (Distance formulation of theorem: if a is closer to zero than b, then a is in the interval, $[-|b|, |b|]$) Using Lemma 1.33 and order facts from Exercise 1.10, we have

$$-|b| < -|a| \leq a \leq |a| < |b|$$

□

Theorem 1.35 $|a| \geq |b| \iff a \leq -|b|$ *or* $a \geq |b|$

Proof. The statement of this theorem is the contrapositive of Theorem 1.34 and hence true. □

The Triangle Inequality

THE TRIANGLE INEQUALITY and related facts will be used repeated when we discuss limits.

Theorem 1.36 THE TRIANGLE INEQUALITY *in four versions.*

The first version is the most commonly used in analysis:

1. THE TRIANGLE INEQUALITY *For any two real number,* x *&* y*,*

$$|x + y| \leq |x| + |y|$$

2. DISTANCE FORM OF THE TRIANGLE INEQUALITY *For any three real numbers,* x*,* y *&* z*,*

$$|x - y| \leq |x - z| + |z - y|$$

Distance formulation: the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides.

3. BACKWARDS TRIANGLE INEQUALITY *For any two real number,* x *&* y*,*

$$|x - y| \geq |x| - |y|$$

Distance formulation: the distance between two numbers is greater than the difference between the absolute values of the numbers.

4. *For any two real numbers,* x *&* y*,*

$$|x - y| \geq ||x| - |y||$$

Distance formulation: the distance between two points is greater than or equal to the distance between the absolute values of the numbers.

Proof. The last three versions follow from the first by judicious choice of variables.

1. The following makes use of the fact that $|x| + |y| = ||x| + |y||$ when using Theorem 1.34.

$$\begin{aligned} -|x| \leq x \leq |x| & \quad \text{by Lemma 1.33} \\ -|y| \leq y \leq |y| & \quad \text{by Lemma 1.33} \\ -(|x| + |y|) \leq x+y \leq |x| + |y| & \quad \text{add the two inequalities} \\ |x+y| \leq |x| + |y| & \quad \text{by Theorem 1.34} \end{aligned}$$

2. Let $\overline{x} = x - z$ and $\overline{y} = z - y$ and apply 1. to $\overline{x} + \overline{y}$.

3. EFS

4. EFS

□

Exercise 1.30 Prove: If $a < c < d < b$ then the distance between c and d is less than the distance between a and b.

Exercise 1.31 Use THE TRIANGLE INEQUALITY, Theorem 1.36, to prove these two other versions.

a) $|x - y| \leq |x| + |y|$

b) $|x + y| \geq |x| - |y|$

Exercise 1.32 Here are some more basic facts about absolute value. Prove them directly from the definition.

a) $|-x| = |x|$

b) $|y - x| = |x - y|$

c) $|x^2| = |x|^2$

d) If $x^2 = c$, then $|x|^2 = c$

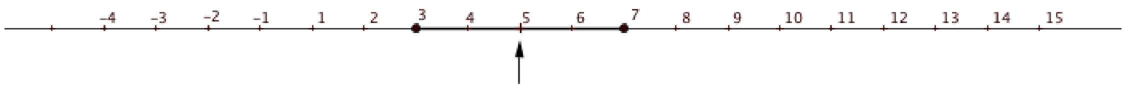
e) $|x \cdot y| = |x| \cdot |y|$

f) $|x^{-1}| = |x|^{-1}$

True or False 8

Which of the following statements are true? Try stating and graphing each one as a fact about distances.

a) $x < 5 \implies |x| < 5$



b) $|x| < 5 \implies x < 5$

c) $|x - 5| < 2 \implies 3 < x < 7$

d) $|1 + 3x| \leq 1 \implies x \geq -\frac{2}{3}$

e) There are no real numbers, x, such that $|x - 1| = |x - 2|$

f) For every $x > 0$, there is a $y > 0$ such that $|2x + y| = 5$

g) $|a - x| < \epsilon \iff x \in (a - \epsilon, a + \epsilon)$

Example 1.5 Using absolute value is often a convenient way to define intervals. Confirm that

$$[3, 7] = \{x : |x - 5| \leq 2|\}$$

In general, if $a \leq b$, then

$$[a, b] = \{x : |x - m| \leq \frac{d}{2}\}, \text{ where } m = \frac{a + b}{2} \text{ and } d = b - a.$$

Exercise 1.33 Υ Graph the set determined by each inequality on a number line. Explain your conclusion.

a) $|2x - 4| < 5$

b) $|2x - 4| \geq 5$

Exercise 1.34 Prove: If x is in the interval (a, b) then the distance between x and the midpoint of the interval is less than half the length of the interval.

Exercise 1.35 If the distance between two integers is less than 1, the integers are equal.

1.2.5 Bounded and unbounded sets

Definition We say a set $S \subset \mathbb{R}$ is *bounded* whenever there exists a positive real number, M, such that $|s| \leq M$ for all $s \in S$.

Definition We say a set $S \subset \mathbb{R}$ is *bounded above* whenever there exists a real number M such that $s \leq M$ for all $s \in S$.

Definition We say a set $S \subset \mathbb{R}$ is *bounded below* whenever there exists a real number m such that $s \geq m$ for all $s \in S$.

Both of the following theorems describe common techniques used in proofs about boundedness.

Theorem 1.37 *A set $S \subset \mathbb{R}$ is bounded if and only if it is bounded above and bounded below.*

Proof. EFS □

Theorem 1.38 *A $S \subset \mathbb{R}$ is bounded above if and only if the set,*

$$-S = \{x \in \mathbb{R} : -x \in S\},$$

is bounded below.

Proof. EFS □

Theorem 1.39 *The real numbers are not bounded*

Proof. hint: State the negation of the definition of boundedness. □

Exercise 1.36 ϒ Give three examples of sets that are bounded and three examples of sets that are not bounded. Which of your assertions can you prove to be true?

Exercise 1.37 ϒ Give three examples of sets that are bounded above but not below. Which of your assertions can you prove to be true?

Definition We say that $\mathcal{L}$ is a *least upper bound* of a set $S \subset \mathbb{R}$ whenever both of the following conditions hold:

1. $\mathcal{L}$ is an upper bound of S
2. if u is an upper bound of S, then $\mathcal{L} \leq u$.

If $\mathcal{L}$ is a *least upper bound* of a set S and if $\mathcal{L} \in S$, we call $\mathcal{L}$ the *maximum* of S.

Definition Write out a definition for *greatest lower bound*, $\mathcal{G}$, and *minimum* of S.

Theorem 1.40 *Any two least upper bounds for a non-empty set, S, are equal. Any two greatest lower bounds for a non-empty set, S, are equal.*

Proof. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ both be least upper bounds for S. Without loss of generality, assume that $\mathcal{L}_1 \leq \mathcal{L}_2$ Since $\mathcal{L}_2$ is a least upper bound, it must be less than or equal to $\mathcal{L}_1$, which is a upper bound of S. Hence, $\mathcal{L}_2 \leq \mathcal{L}_1 \leq \mathcal{L}_2$. By trichotomy, $\mathcal{L}_1 = \mathcal{L}_2$. A similar proof works for the greatest lower bounds. □

Notation As usual uniqueness allows us to name the least upper bound and the greatest lower bound of a set, should they exist. We use the abbreviation $\sup S$ and say, *supremum of* S, for the least upper bound of set S. Similarly we use the abbreviation $\inf S$ and say, *infinum of* S, for the greatest lower bound of S. In the case when $\sup S \in S$, we also call it the *maximum* of S. If $\inf S \in S$, we call it the *minimum* of S

Example 1.6 The greatest lower bound of the open interval, $(5, 10)$, is 5

Proof. There are two steps in the proof of this fact:

1. 5 is a lower bound: By definition of the open interval, $5 < x$ for all $x \in (5, 10)$.

2. 5 is greater than any other lower bound: Suppose h is a lower bound greater than 5, so $10 > h > 5$. Consider $m = \frac{5+h}{2}$, the average of 5 and h. We know $5 < m < h < 10$. Since $m \in (5, 10)$, h is not a lower bound for $(5, 10)$. Since any number greater than 5 is not a lower bound, 5 must be the greatest one.

□

Exercise 1.38 For any two real numbers, $a < b$, the least upper bound of the interval (a, b) is b.

Exercise 1.39 Let N be an integer and let $S = \{s \in \mathbb{R} : s^2 \leq N\}$. Find a rational number that is a upper bound of S. Prove your assertion.

Definition We say that a set is *finite* whenever there exists a $1-1$ correspondence between the set and the set of all positive integers less than or equal to n, for some positive integer n. The *order* of the set, or the number of elements in the set, is n. [A $1-1$ correspondence is a bijection.]

Theorem 1.41 *A finite set has a maximum and a minimum element.*

Proof. HINT: Use induction on the number of elements in the set. □

Example 1.7 Υ What is the $\inf\{\frac{1}{n} : n \in \mathbb{Z}^+\}$?

Example 1.8 Υ Does $\mathbb{Z}^+$ have an upper bound?

Definition We say a function is *bounded (bounded above) (bounded below)* whenever the image of the function is bounded (bounded above) (bounded below). The definition includes the possibility that the function is a sequence.

Exercise 1.40 Give three examples of functions that are bounded.

Exercise 1.41 Give three examples of functions that are not bounded.

1.3 The Completeness Axiom

True or False 9

Which of the following statements are true? Explain.

a) There exists a least positive real number.

b) For all positive numbers ϵ, there exists a positive integer, N, such that $\frac{1}{N} < \epsilon$.

The answers to the last question is 'Yes' (see Theorem 1.48), but we can't prove it yet. We need another axiom:

Axiom 9 THE COMPLETENESS AXIOM *A non-empty set of real numbers that is bounded above has a least upper bound.*

1.3.1 Consequences of the Completeness Axiom

Clearly, there is an analogous fact for lower bounds, but it need not be stated as part of the axiom. Instead it can be proved from the axiom. The technique is a standard good trick to know.

Theorem 1.42 EXISTENCE OF GREATEST LOWER BOUND *A non-empty set of real numbers that is bounded below has a greatest lower bound.*

Outline of Proof: If S is a non-empty set that is bounded below, let $T = \{x : -x \in S\}$. Show that T is bounded above, apply **Axiom 9**, and make conclusions about the original set S. Draw number line pictures to help explain the strategy of the proof. ☐

A word about ϵ The greek letter, 'epsilon,' ϵ, is often used in situations where the interesting part is numbers getting arbitrary small. What we mean by *arbitrarily small* is that the inf of the set of positive ϵ's we are considering is 0. We use ϵ to stand in for 'error,' which we like to be small.

Theorem 1.43 *Let S be a non-empty set of real numbers that is bounded above.*

$$\textit{For all } \epsilon > 0, \textit{ there exists } x \in S \textit{ such that } \sup S - \epsilon < x.$$

NOTE: This is what we mean when we say that there are numbers in S get *arbitrarily close to* $\sup S$.

Outline of Proof. If $\epsilon > 0$, $\sup S - \epsilon$ cannot be a upper bound. Draw a numberline picture to help explain the situation. □

Exercise 1.42 State and prove the analogous theorem for the greatest lower bound of a set that is bounded below.

Theorem 1.44 *Let S be a non-empty set of real numbers that is bounded above. Let U be the set of all upper bounds for S, that is,*

$$U = \{u \in \mathbb{R} : u \geq s \text{ for all } s \in S\},$$

then $\inf U = \sup S$.

Outline of Proof. Any $u \in U$ is an upper bound for S, so $u \geq \sup S$, the least upper bound. So $\sup S$ is a lower bound for U. Any number greater than $\sup S$, is an upper bound for S and hence $\in U$. So $\sup S$ is the greatest lower bound. Draw a numberline picture to help explain the situation. □

Exercise 1.43 State and prove an analogous theorem for a set of real numbers that is bounded below.

The following theorem is a forerunner to THE NESTED INTERVAL THEOREM which we will be using extensively fro the rest of the course.

Theorem 1.45 *Given two non-empty subsets, $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$, such that every element in A is a lower bound for B and every element in B is an upper bound for A, there exists a real number between the two sets. That is, there exists a real number, r, such that for all $a \in A$ and $b \in B$, $a \leq r \leq b$ In fact,*

$$\sup A \leq r \leq \inf B$$

Outline of Proof: Apply **Axiom 9** to argue that A has a least upper bound and that B has a greatest lower bound. Show that $\sup A$ is a lower bound for B, and hence that $\sup A \leq \inf B$. Then r could be any number in between the two. Draw a number line picture to illustrate the proof. □

Exercise 1.44 Prove: If $S \subset \mathbb{R}^{\geq}$ and if S is bounded below, then $\inf\{x^2 : x \in S\} = (\inf S)^2$.

1.3.2 The Nested Interval Theorem

Theorem 1.46 THE NESTED INTERVAL THEOREM *The intersection of a sequence of non-empty, closed, nested intervals is not empty. Furthermore, if the least upper bound of the left endpoints is equal to the greatest lower bound of the right endpoints, there is only one point in the intersection.*

NOTE: Notate the sequence by $I_n = [a_n, b_n]$, for $n > 0$. Convince yourself of the following and draw a numberline picture to help explain the theorem.

1. That each interval, I_n, is not empty is equivalent to saying that for all n, $a_n \le b_n$,
2. That each interval is closed means that the endpoints are contained in the interval and that's why we used the closed brackets to denote the intervals.
3. That the sequence is *nested* means that, for all $n > 0$, $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ or that $a_n \le a_{n+1}$ and that $b_{n+1} \le b_n$.

Proof of The Nested Interval Theorem 1.46. Let $A = \{a_n : n > 0\}$ and let $B = \{b_n : n > 0\}$. Since the intervals are non-empty and nested, $a_n \le b_m$ for all positive integers n and m. Theorem 1.45 applies to the sets, A and B, so there exists a real number, r, such that $a_n \le r \le b_n$ for all n. Since r is in all the intervals, it is also in the intersection. And since it is also true that $\sup a_n \le r \le \inf b_n$, if $\sup a_n = \inf b_n$, then any point in the intersection must be equal to both. □

1.3.3 Archimedes Principle

The integers are not bounded

Theorem 1.47 *The set of positive integers is not bounded above.*

Proof. We will prove this theorem by contradiction: assume the set of positive integers is bounded. By **Axiom 9** there would be a least upper bound. Let L be this least upper bound. Then $L - 1$, being less than L, is not an upper bound for the positive integers. Let N be a positive integer greater than $L - 1$. We have $L - 1 \le N \implies L \le N + 1$. Since L is an upper bound for the set of positive integers and $N + 1$ is a positive integer, we also have that $L \ge N + 1$. Together this means that $L = N + 1$. But then $N + 2$ is an integer great than L so L couldn't be a upper bound. □

Exercise 1.45 Use Theorem 1.47 to show that the integers are not bounded.

Theorem 1.48 ARCHIMEDES PRINCIPLE *For all real numbers $\epsilon > 0$, there exists a positive integer N such that $\frac{1}{N} < \epsilon$.*

Proof. The negation of this statement is that there exists a positive real number ϵ, such that for all positive integers, N, $\frac{1}{N} \ge \epsilon$. But this says that $N \le \frac{1}{\epsilon}$ for all N, or that the integers are bounded by $\frac{1}{\epsilon}$. This is false so the Archimedes Principle must be true. □

This easy restatement of Archimedes Principle is the first of many squeeze theorems.

Theorem 1.49 SQUEEZE THEOREM 1 *If* $0 \leq h < \frac{1}{n}$ *for all* $n \in \mathbb{Z}^+$*, then* $h = 0$.

Proof. Assume h satisfies the hypotheses but is not 0. By ARCHIMEDES PRINCIPLE, Theorem 1.48 , there exist $n \in \mathbb{Z}^+$ such that $\frac{1}{n} < h$. □

Theorem 1.50 GENERALIZED ARCHIMEDES PRINCIPLE *For all real numbers* x *and* $d > 0$*, there exists a positive integer* N *such that* $x < N \cdot d$.

Proof. Use Archimedes Principle to prove this theorem. Note that Archimedes Principle, in turn, follows from this theorem. □

Exercise 1.46 Show that Archimedes Principle holds for rational numbers. That is, prove the following theorem without using the Archimedes Principle.

Theorem 1.51 *For all rational numbers,* $r > 0$*, there exists a positive integer,* N*, such that* $0 < \frac{1}{N} < r$.

Proof. □

Exercise 1.47 The following theorem follows from Archimedes Principle using Exercise 1.21. We use it in the next (optional) section and in later work. Prove it.

Theorem 1.52 *For all* $B, h > 0$*, there exists an integer* $n > 0$ *such that* $\dfrac{B}{2^n} < h$

Proof. EFS □

1.3.4 Optional – Nested Interval Theorem and Archimedes prove the Completeness Axiom

Theorem 1.53 *The Nested Interval Theorem and Archimedes Principle imply the Completeness Axiom*

Proof. Let S be a non-empty set with an upper bound. Construct a sequence of nested closed intervals, $[a_n, b_n]$, by bisection, so that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$, $a_n \in S$, b_n an upper bound for S, and $|b_n - a_n| \leq \frac{b_0 - a_0}{2^n}$.

- Let a_0 be a point in S (S is non-empty!) and let b_0 be a upper bound for S If $a_0 = b_0$ this is the least upper bound and we are done, so assume $a_0 < b_0$.
- Assume $[a_n, b_n]$ is defined as required. Let m be the mid-point of $[a_n, b_n]$. There are two cases:
  1. If m is an upper bound for S, let $a_{n+1} = a_n$ and $b_{n+1} = m$.

2. If m is not an upper bound for S then there exists $a_{n+1} \in S$ that is greater than or equal to m. Note that $a_{n+1} \leq b_n$ because b_n is an upper bound for S. Let $b_{n+1} = b_n$.

If $a_{n+1} = b_{n+1}$ this is the least upper bound and we are done, so assume $a_{n+1} < b_{n+1}$. Notice that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ and $|b_{n+1} - a_{n+1}| \leq \frac{1}{2}|b_n - a_n| \leq \frac{1}{2}\frac{b_0 - a_0}{2^n} = \frac{b_0 - a_0}{2^{n+1}}$, so the new interval satisfies the requirements.

By THE NESTED INTERVAL THEOREM, all these intervals contain a common point, b. By ARCHIMEDES PRINCIPLE, b is the only common point: For, if a is another one then $0 < |b - a| < \frac{b_0 - a_0}{2^n} < \frac{1}{n}$ for all n, this implies $b = a$ by SQUEEZE THEOREM 1.

Claim: b is an upper bound for S. Proof: Suppose not. then there is some $a \in S$ with $a > b$. By ARCHIMEDES PRINCIPLE, in the form of Theorem 1.52, for some n, $|a - b| > \frac{|b_0 - a_0|}{2^n} = |b_n - a_n|$ so (removing parentheses) $a - b > b_n - a_n > b_n - b$ ($a_n < b$) or $a > b_n$. This shows that a must be an upper bound. a must be the least upper bound since it is in S. So $a \leq b_n$ all n. a is in all the intervals so $a = b$.

Supposes b is not the least upper bound, there there exists an upper bound for S, a, with $a < b \leq b_n$, for all n. Since a is an upper bound for S, $a_n \leq a$ for all n so a is in all the intervals. $a = b$ again.

So b is the least upper bound for S. □

1.3.5 Rational numbers are dense in $\mathbb{R}$

Definition We say that a subset $D \subset \mathbb{R}$ is *dense in* $\mathbb{R}$, whenever every open interval of $\mathbb{R}$ contains an element of D.

Exercise 1.48 The following sequence of theorems can be used to prove that the rational numbers are dense in $\mathbb{R}$. Prove Theorems 1.55 - 1.59.

Theorem 1.54 *Any non-empty set of integers that is bounded below has a minimum number.*

Proof. EFS □

Theorem 1.55 *For every real number, r, there exists a unique integer, n, such that $r < n \leq r + 1$.*

Proof. HINT: For $r > 0$, use well-ordering to find smallest positive integer greater than or equal to r. This will be n. □

Theorem 1.56 *For every real number, r, there exists a unique integer, m, such that $m \leq r < m + 1$.*

Proof. EFS □

Theorem 1.57 *If $b > a + 1$, then the open interval, (a, b) contains an integers.*

Proof. EFS □

Theorem 1.58 $\mathbb{Q}$ IS DENSE IN $\mathbb{R}$ *Every non-empty open interval contains a rational number.*

Proof. EFS □

Theorem 1.59 *Given any real number, α, there exists a rational number arbitrarily close to α. That is, given $\epsilon > 0$, there exists a rational number, r, such that $|\alpha - r| < \epsilon$*

Proof. Let r be a rational number in the interval, $(\alpha - \epsilon, \alpha + \epsilon)$. □

1.3.6 Optional – An Alternative Definition of Interval

This section is a mini-lesson on how making good, mathematical definitions can simplify understanding and proving. The problem with our current definition of interval is that there are too many parts to it. We have to worry about open and closed endpoints as well as unbounded intervals. A simplified definition may make it easier to prove things about intervals. The following two theorems exploit a condition that is simple to use. Reminder: our current definition says that an interval is one of the following:

$$(a, b) \text{ or } [a, b] \text{ or } (a, b] \text{ or } [a, b) \text{ or } (a, \infty) \text{ or } [a, \infty) \text{ or } (\infty, b) \text{ or } (\infty, b]$$

Exercise 1.49 Write all cases for the proof of the following Theorem, 1.60. The proof is straight forward but tedious because each type of interval must be dealt with separately.

Theorem 1.60 *If S is an interval, then*

$$a, b \in S \text{ and } a < c < b \implies c \in S. \tag{1.1}$$

Proof. EFS □

What if we used the condition 1.1 from the theorem as the definition of interval? Would we get the same sets are intervals?

Theorem 1.61 *If a set $S \subset \mathbb{R}$ satisfies condition 1.1, then S is an interval.*

Proof. First, suppose S is bounded. Let z be any point strictly between $a = \inf S$ and $b = \sup S$. There is a point, $x \in S$ greater than z (otherwise z would be a upper bound, less than b, the least upper bound) and a point $y \in S$ less than z (otherwise z would be a lower bound, greater than a, the greatest lower bound). By the condition, $z \in S$. Now $(a, b) \subset S \subset [a, b]$, so S is an interval by our previous definition.

Second, suppose that S is unbounded above, but not below. Let z be any point greater than $a = \inf S$. There is a point, $x \in S$ greater than z (otherwise z would be a upper bound for S.) and a point $y \in S$ less than z (otherwise z would be a lower bound, greater than a, the greatest lower bound). By the condition, $z \in S$. Now $(a, \infty) \subset S \subset [a, \infty)$ so S is an interval by our previous definition.

The other two cases are similar.

□

Notice that this proof depends on the completeness axiom. In the rational numbers, there are sets that satisfy condition (1.1) but are not intervals in our original definition.

Exercise 1.50 Show that $\{r \in \mathbb{Q} : r^2 > 2\}$ satisfies condition (1.1), but it cannot be written as $(a, +\infty)$ for any $a \in \mathbb{Q}$.