Chapter 2

Limits of Sequences

Calculus Student: $\lim_{n \to \infty} s_n = 0$ means the s_n are getting closer and closer to zero but never gets there.

Instructor: *ARGHHHHH!*

Exercise 2.1 Think of a better response for the instructor. In particular, provide a counterexample: find a sequence of numbers that 'are getting closer and closer to zero' but aren't really getting close at all. What about the 'never gets there' part? Should it be necessary that sequence values are never equal to its limit?

2.1 Definition and examples

We are going to discuss what it means for a sequence to converge in three stages:

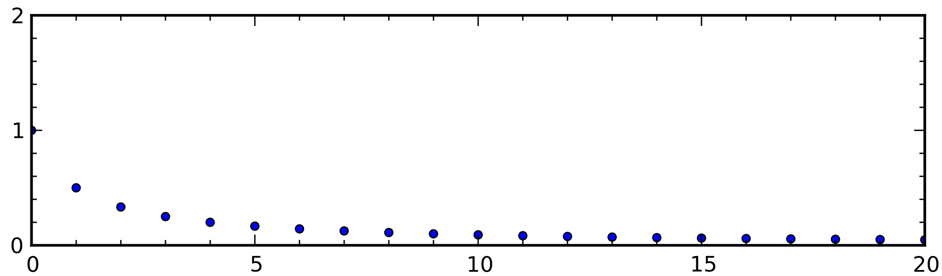
First, we define what it means for a sequence to converge to zero

Then we define what it means for sequence to converge to an arbitrary real number.

Finally, we discuss the various ways a sequence may diverge (not converge).

In between we will apply what we learn to further our understanding of real numbers and to develop tools that are useful for proving the important theorems of Calculus.

Figure 2.1: $s_n = \frac{1}{n}$.



2.1.1 Sequences converging to zero.

Definition We say that the sequence s_n *converges to* 0 whenever the following hold:

For all $\epsilon > 0$, there exists a real number, N, such that

$$n > N \implies |s_n| < \epsilon.$$

Notation To state that s_n converges to 0 we write $\lim_{n\to\infty} s_n = 0$ or $s_n \to 0$.

Example 2.1 $\lim_{n\to\infty} \frac{1}{n} = 0$. See the graph in Figure 2.1.

Proof. Given any $\epsilon > 0$, use Archimedes Principle, *Theorem* 1.48, to find an N, such that $\frac{1}{N} < \epsilon$. Note that, if $n > N$, then $\frac{1}{n} < \frac{1}{N}$ (Exercise 1.10 d)). Now, if $n > N$, we have

$$|s_n| = \frac{1}{n} < \frac{1}{N} < \epsilon.$$

In short:

$$n > N \implies |s_n| < \epsilon,$$

so we have shown that $\lim_{n\to\infty} \frac{1}{n} = 0$. □

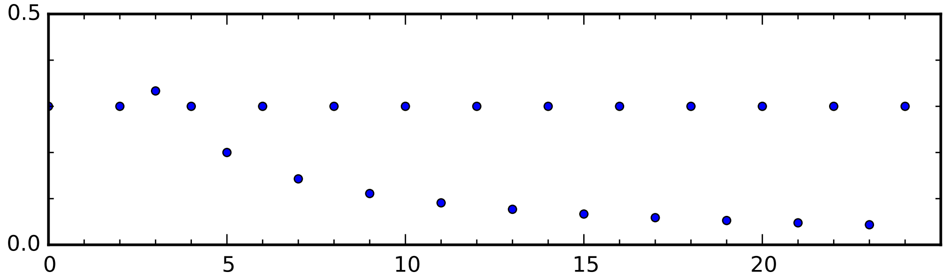
Example 2.2 If $s_n = 0$, for all n, then $\lim_{n\to\infty} s_n = 0$

Proof. Given any $\epsilon > 0$, let N be any number. Then we have

$$n > N \implies |s_n| = 0 < \epsilon,$$

because that's true for any n. □

Figure 2.2: Some values approach 0, but others don't.



Example 2.3 Why isn't the following a good definition?

" $\lim_{n\to\infty} s_n = 0$ means

For all $\epsilon > 0$, there exists a real number, N, such that $|s_N| < \epsilon$."

The problem is we want the sequence to get arbitrarily close to zero and to stay there. Consider the sequence:

$$s_n = \begin{cases} \frac{1}{n}, & \text{if } n \text{ is odd} \\ 0, & \text{otherwise.} \end{cases}$$

There is always an odd N to be less than any ϵ but there there are many even n's with values far from zero. The $n > N$ is an important part of the definition. See the graph in Figure 2.2.

Exercise 2.2 Sketch the sequence and prove that $\lim_{n\to\infty} \dfrac{1}{n^2} = 0$

Exercise 2.3 Sketch the sequence and prove that $\lim_{n\to\infty} \dfrac{(-1)^n}{n} = 0$ See Figure 2.3.

Exercise 2.4 Sketch the sequence and prove that $\lim_{n\to\infty} \dfrac{1}{n(n-1)} = 0.$

Exercise 2.5 Prove the following theorem:

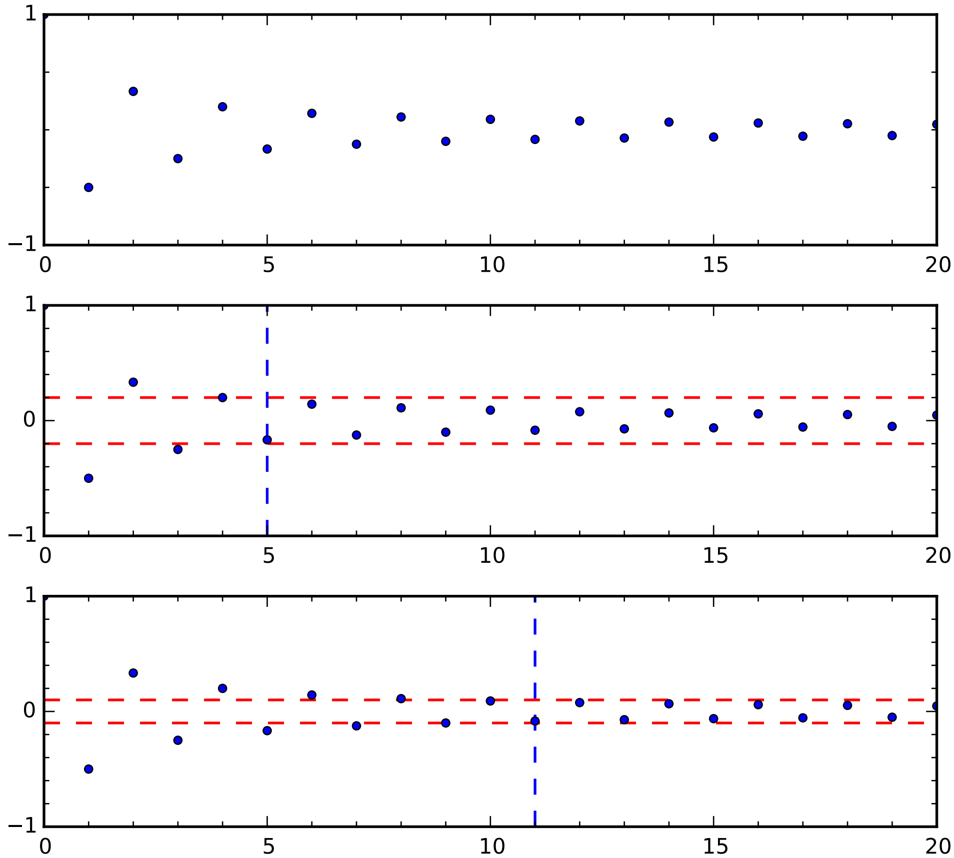
Theorem 2.1 $s_n \to 0 \iff |s_n| \to 0 \iff -s_n \to 0$

Proof. EFS □

The above are good exercises but problems like these will be easier to prove – that is, no epsilons will be needed – once we have some theorems. First, a useful fact:

Theorem 2.2 *If $\lim_{n\to\infty} a_n = 0$, then the sequence, a_n, is bounded. That is, there exists a real number, $M > 0$ such that $|a_n| < M$ for all n.*

Figure 2.3: Picking N for smaller and smaller ϵ for the sequence $s_n = \frac{(-1)^n}{n}$.



Proof. Since $a_n \to 0$, there exists $N \in \mathbb{Z}^+$ such that $n > N \implies |a_n| < 1$. Here we use the definition of converging to 0 with $\epsilon = 1$. (NOTE: We could use any positive number in place of 1.) Let B be a bound for the finite set $\{a_n : n \leq N\}$. This set is bounded by Theorem 1.41. Let $M = \max\{B, 1\}$ Hence any a_n is bounded by M because it is either in the finite set ($n \leq N$) and bounded by B or it is bounded by 1, because $n > N$. □

Theorem 2.3 ALGEBRAIC PROPERTIES OF LIMITS 1

Given three sequences, $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$ *and a real number,* c, *then:*

1. $\lim_{n\to\infty} a_n + b_n = 0$
2. $\lim_{n\to\infty} c \cdot a_n = 0.$
3. $\lim_{n\to\infty} a_n \cdot b_n = 0.$

Proof. EFS □

Exercise 2.6 Prove: If c_n is bounded and $a_n \to 0$, then $c_n \cdot a_n \to 0$

Theorem 2.4 SQUEEZE THEOREM *If* $a_n \to 0$ *and* $b_n \to 0$ *and* $a_n \leq c_n \leq b_n$, *for all* $n \in \mathbb{Z}^+$, *then* $\lim_{n\to\infty} c_n = 0$.

Proof. SQUEEZE THEOREM Given $\epsilon > 0$, let N be large enough so that whenever $n > N$, then both $|b_n| < \epsilon$ and $|a_n| < \epsilon$. Now, for any $n > N$, if $c_n > 0$, we have $|c_n| \leq |b_n| < \epsilon$. or if $c_n < 0$, then $|c_n| = -c_n \leq -a_n = |a_n| < \epsilon$. So, for all $n > N$ we have $|c_n| < \epsilon$. We have shown that $c_n \to 0$. □

Exercise 2.7 Prove that $\lim_{n\to\infty} \frac{n^2 + n}{n^3} \to 0$

Exercise 2.8 Prove: For any positive integer, m, $\lim_{n\to\infty} \frac{1}{n^m} = 0$

Exercise 2.9 Use BERNOULLI'S INEQUALITY Theorem 1.27 to prove the following theorem.

Theorem 2.5 *If* $0 \leq r < 1$, *then* $r^n \to 0$

Proof. □

2.1.2 Sequences that converge to arbitrary limit

Definition We say that s_n converges whenever there exists a real number, s, such that $|s - s_n| \to 0$. In this case, we say that s_n converges to s, and write

$$\lim_{n\to\infty} s_n = s \text{ or } s_n \to s$$

Example 2.4 $\lim_{n\to\infty} \frac{n-1}{n} = 1$, because $1 - \frac{n-1}{n} = 1 - (1 - \frac{1}{n}) = \frac{1}{n} \to 0$

Exercise 2.10 Show that $s_n \to 0$ means the same thing for both definitions: converging to 0 and converging to an arbitrary limit that happens to be 0.

Theorem 2.6 UNIQUENESS OF LIMIT *If $a_n \to a$ and $a_n \to b$, then $a = b$.*

Proof. Use the triangle inequality to see that $0 \le |a - b| = |a - a_n + a_n - b| \le |a - a_n| + |a_n - b|$. Apply THE SQUEEZE THEOREM (Theorem 2.3 part 2.4.): the left-most term is the constant sequence, 0, the right-most term is the sum of two sequences that converge to 0, so also converges to 0. Hence the middle term also converges to 0. So $|a - b| = 0 \implies a = b$. □

Theorem 2.7 *If $\lim_{n\to\infty} a_n = a$, then the sequence, a_n, is bounded.*

Proof. EFS Consider using Theorem 2.2. □

Exercise 2.11 Prove: If $a_n = c$, for all n, then $\lim_{n\to\infty} a_n = c$

Theorem 2.8 ALGEBRAIC PROPERTIES OF LIMITS 2

Given two sequences, $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then:

1. $\lim_{n\to\infty} a_n + b_n = a + b$
2. $\lim_{n\to\infty} a_n \cdot b_n = a \cdot b$
3. *If $a_n, a \neq 0$, then* $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$

Proofs.

1. EFS
2. EFS
3. To prove this one, we start by showing that the sequence a_n is bounded away from 0, meaning that there exists a positive number B, such that $|a_n| > B$, for all n. To find such a bound, B, first note that there is $N > 0$ such that $|a_n - a| < |\frac{a}{2}|$ for all positive integers $n > N$. (Using $\epsilon = |\frac{a}{2}|$ in the definition of limit.) For those n,

$$|a_n| \ge |a| - |a_n - a| > |a| - |\frac{a}{2}| = |\frac{a}{2}|.$$

Now let $\overline{B} = \min\{|a_n| : n \le N\}$. This set has a minimum value because it is a finite set. $\overline{B} > 0$ because none of the $a_n = 0$. Finally, let $B = \min\{\overline{B}, |\frac{a}{2}|\}$. So $|a_n| > B$, for all n.

Now, given $\epsilon > 0$, find N such $|a_n - a| < \epsilon \cdot B \cdot |a|$ then

$$|\frac{1}{a_n} - \frac{1}{a}| = \frac{|a - a_n|}{|a_n \cdot a|} < \frac{|a_n - a|}{B \cdot |a|} < \frac{\epsilon \cdot B \cdot |a|}{B \cdot |a|} = \epsilon$$

□

LIMITS OF RATIOS An important concern of calculus is what happens to the ratio of two limits when both the numerator and denominator converge to 0. If the denominator converges to zero, but the numerator is bounded away from zero, then the ratio will be unbounded and not converge.

Exercise 2.12 Give examples of two sequences, $a_n \to 0$ and $b_n \to 0$, such that

a. $\frac{a_n}{b_n} \to 0$

b. $\frac{a_n}{b_n} \to c$, where c is a positive real number.

c. $\frac{a_n}{b_n}$ does not converge.

Theorem 2.9 ORDER PROPERTIES OF LIMITS

For real sequences, a_n, b_n, c_n and real numbers, a and c.

1. *If $a_n > c$ for all $n \in \mathbb{Z}^+$ and $a_n \to a$, then $a \geq c$*
2. *If $a_n \leq c \leq b_n$ for all n and $|a_n - b_n| \to 0$, then $a_n \to c$ and $b_n \to c$.*
3. THE SQUEEZE THEOREM *If $a_n \to c$ and $b_n \to c$ and $a_n \leq c_n \leq b_n$ for all $n \in \mathbb{Z}^+$, then $c_n \to c$*

Proof. Confirm each statement and explain how it proves the corresponding part of the theorem

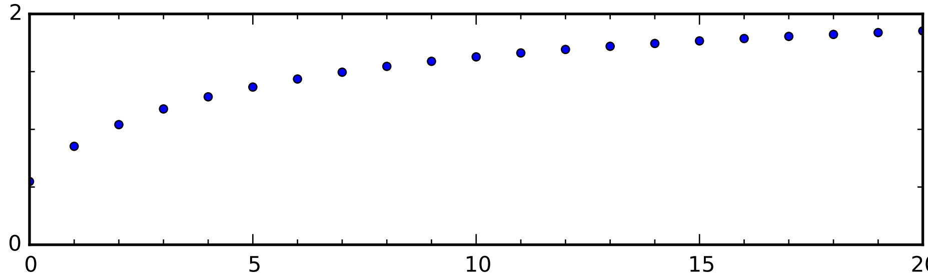
1. Suppose $c - a > 0$, find N such that

$$n > N \implies |a - a_n| < c - a$$

2. For all n, $0 \leq |a_n - c| \leq |b_n - a_n|$
3. $0 \leq |c - c_n| \leq |c - a_n| + |a_n - c_n| \leq |c - a_n| + |a_n - b_n|$

□

Figure 2.4: $s_n = \dfrac{\sqrt{5n+3}}{\sqrt{n+10}}$



Theorem 2.10 THE TAILEND THEOREM *If $a_n \to a$ and if $b_n = a_{n+m}$ for some fixed, positive integer m, then*

$$b_n \to a.$$

Proof. A tailend of a sequence is a special case of a subsequence, see Section 2.1.4. □

Exercise 2.13 Prove: If $a_n \to c$ and $b_n \to c$, then $|a_n - b_n| \to 0$

Exercise 2.14 Conjecture what the limit might be and prove your result.

$$s_n = \frac{3n^2 + 2n + 1}{n^2 + 1}$$

Exercise 2.15 Conjecture what the limit might be and prove your result.

$$s_n = \frac{\sqrt{5n+3}}{\sqrt{n+10}}$$

Exercise 2.16 Prove that, if $c \neq 0$, then

$$\lim_{n\to\infty} \frac{a \cdot n + b}{c \cdot n + d} = \frac{a}{c}.$$

Exercise 2.17 Prove: If $a_n \to a$, $b_n \to b$, and $a_n < b_n$ for all n, then $a \leq b$.

Exercise 2.18 Prove: If S is a bounded set, then there exists a sequence of points, $s_n \in S$ such that $s_n \to \sup S$.

Exercise 2.19 Prove: If $a_n \to a$ then $a_n^2 \to a^2$

Monotone sequences

Definition We say a sequence is *monotone* whenever it is an increasing sequence or a decreasing sequence.

Theorem 2.11 MONOTONE CONVERGENCE THEOREM *Every bounded, monotone sequence converges to a real number.*

Proof. Let s_n be a bounded, increasing sequence. Let $s = \sup\{s_n\}$ which exists because s_n is bounded above. We claim that $s_n \to s$. Given $\epsilon > 0$, use Theorem 1.43 to find $x = s_N \in \{s_n\}$ such that $s - \epsilon < s_N$. Now if $n > N$, we know $s_n > s_N$ because the sequence is increasing, so

$$|s_n - s| = s - s_n < s - s_N < \epsilon.$$

We conclude that $s_n \to s$. If t_n is a bounded, decreasing sequence, then $s_n = -t_n$ is bounded and increasing. Since $s_n \to s$, for some s, we known that $t_n = -s$. ☐

Best Nested Interval Theorem

Theorem 2.12 BEST NESTED INTERVAL THEOREM *There exists one and only one real number, x, in the intersection of a sequence of non-empty, closed, nested intervals if the lengths of the intervals converge to 0. Furthermore, the sequence of right endpoints and the sequence of left endpoints both converge to x.*

Proof. Denote the intervals by $[a_n, b_n]$. Because they are nested we know that a_n is increasing and b_n is decreasing so, by MONOTONE CONVERGENCE THEOREM, there are real numbers a and b such that $a_n \to a$ and $b_n \to b$. Let c be any real number in the intersection of all the intervals. Then $a_n \leq c \leq b_n$ and since $|b_n - a_n| \to 0$ we have by the ORDER PROPERTIES OF LIMITS (Theorem 2.9 *2.*), $c = a = b$. ☐

Rational Approximations to Real Numbers

We have not yet shown that there are real numbers other than rational numbers. However, if there is one, the following method indicates that you can approximate it by rational numbers; that is, there is a sequence of rational numbers that converge it. The method of bisection used here is a well-used tool of analysis.

Example 2.5 Let r be any non-rational real number. Find a sequence of rational numbers that converge to r.

Using the method of bisection. There are other ways to show the existence of such a sequence. The advantage to this method is that it gives a way to construct approximations of the given real number.

First, find rational numbers a_0 and b_0 such that $a_0 < r < b_0$. (By Theorem 1.56, they can both be integers.) Recursively define a sequence of non-empty, closed, nested intervals $[a_n, b_n]$ such that each interval contains r and is half the length of the preceding interval. We already have the base case, $[a_0, b_0]$. Assume $[a_n, b_n]$ has been defined as required. Let m be the midpoint of $[a_n, b_n]$. Since m is a rational number, $m \neq r$, so there are two cases to consider:

1. If $m < r$, let $a_{n+1} = m$ and $b_{n+1} = b_n$.
2. If $r < m$, let $a_{n+1} = a_n$ and $b_{n+1} = m$.

in either case $r \in [a_{n+1}, b_{n+1}] \subset [a_n, b_n]$. The length of this interval is half the length of the previous interval. The lengths of the intervals converge to 0 so, by the BEST NESTED INTERVAL THEOREM (Theorem 2.12), r is the only real number in all the intervals and both $a_n \to r$ and $b_n \to r$. □

2.1.3 Application: Existence of square roots

The proof given here of the existence of square roots is by construction. There are other ways to prove the existence of square roots – the advantage to this method is that it gives a way to calculate approximations to $\sqrt{2}$.

Example 2.6 There exists a unique positive real number, s, such that $s^2 = 2$.

Proof by bisection. We will show that there is a nested sequence of closed intervals, $I_n = [a_n, b_n]$, such that ${a_n}^2 \leq 2 \leq {b_n}^2$ and $|b_n - a_n| = \frac{1}{2^n}$. By the BEST NESTED INTERVAL THEOREM, there is a unique number, s, in all of the intervals. We will show that $s^2 = 2$ using a squeezing argument.
We define the intervals inductively: Base case: Let $a_0 = 1$ and $b_0 = 2$, so

$$ {a_0}^2 = 1 \leq 2 \leq 4 = {b_0}^2, \text{ and } |b_0 - a_0| = 1 = \frac{1}{2^0}. $$

Assume a_n and b_n have been defined as desired

$$ {a_n}^2 \leq 2 \leq {b_n}^2, \text{ and } |b_n - a_n| = \frac{1}{2^n}. $$

Proceed inductively to define the next interval, $I_{n+1} = [a_{n+1}, b_{n+1}]$. Let m be the midpoint of the interval $[a_n, b_n]$. We know that $m^2 \neq 2$ because we know the square root can not be a rational number. (Why do we claim m is rational?) So there are only two cases to consider:

1. If $m^2 < 2$, let $a_{n+1} = m$ and $b_{n+1} = b_n$.
2. If $2 < m^2$, let $a_{n+1} = a_n$ and $b_{n+1} = m$.

Since m is the midpoint of the previous interval the length of I_{n+1} is half the length of I_n, so $|b_{n+1} - a_{n+1}| = \frac{1}{2} \cdot \frac{1}{2^n} = \frac{1}{2^{n+1}}$
Let s be the unique number in all the intervals. We know $a_n \to s$ and $b_n \to s$. So we also know that $a_n^2 \to s^2$, $b_n^2 \to s^2$. (See Exercise 2.19) Now consider the image, under the squaring function of all those intervals. The intervals $[a_n^2, b_n^2]$ are closed, non-empty ($a_n^2 < b_n^2$ because $a_n < b_n$); and nested (a_n^2 is increasing because a_n is

increasing, and b_n^2 is decreasing because b_n is decreasing). These all follow because the function $x \to x^2$ is increasing on positive numbers, see Exercise 1.12. Furthermore, $|b_n^2 - a_n^2| \to 0$. The BEST NESTED INTERVAL THEOREM applies and we conclude there is a unique real number in all the intervals and that the endpoints converge to that number. 2 is all the intervals and $a_n \to s^2$, so $s^2 = 2$, by uniqueness of limits. □

Theorem 2.13 *For all $a > 0$, there exists a unique positive real number, s, such that $s^2 = a$.*

Proof. HINT: What in the proof for Example 2.6 depends on the choice $a = 2$? □

2.1.4 Subsequences

Definition We call sequence, s_{n_k}, whose values are a subset of the values of s_n, a *subsequence* of s_n, whenever the sequence n_k is strictly increasing. (We will assume that k is indexed on $\mathbb{Z}^{\geq}$, i.e. n_0 is the first value.)

Example 2.7 If $n_k = 2k$, then the subsequence is every other element, starting at 0, of the sequence.

Exercise 2.20 Prove: If s_{n_k} is a subsequence of s_n, then $n_k \geq k$. Note that this is true for any increasing sequence of positive integers n_k.

Theorem 2.14 *If $s_n \to s$, then any subsequence of s_n also converges to s.*

Proof. Let s_{n_k} be a subsequence of s_n. Given $\epsilon > 0$, find N so that $n > N \implies |s_n - s| < \epsilon$. Such N exists because $s_n \to s$. Now consider the subsequence: If $k > N$, then $n_k > N$ (by Exercise 2.20), so $|s_{n_k} - s| < \epsilon$. □

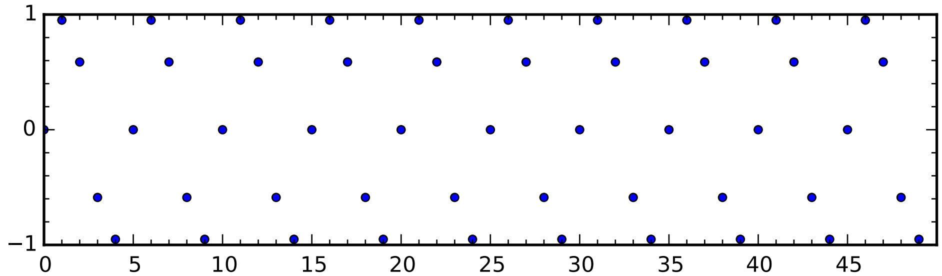
Example 2.8 The sequence is Example 2.21 has many subsequences, each of which converges to one of five different numbers. For example, $s_{5k+2} \to \sin \frac{4\pi}{5}$.

2.1.5 Divergent Sequences

Definition A sequence is said to *diverge* if there is no real number L such that the sequence converges to L.

To show that a sequence, s_n, converges we would first conjecture a possible limit, L, and then prove $s_n \to L$. To show that the sequence does not converge is perhaps harder because we have to show it doesn't converge for all possible values L. And we would need to prove the negation of the statement, $s_n \to L$, for all values of L. Here are both statements:

Figure 2.5: $s_n = \sin(\frac{2\pi n}{5})$.



$s_n \to L$, means

$$\text{For all } \epsilon > 0, \text{ there exists } N > 0, \text{ such that}$$
$$n > N \implies |s_n - L| < \epsilon.$$

$s_n \not\to L$, means

$$\text{There exists an } \epsilon > 0, \text{ such that for all } N > 0$$
$$\text{there is an } n > N \text{ with } |s_n - L| \geq \epsilon.$$

Fortunately there is an easier way to show that a sequence diverges by observing subsequence behavior, using Theorem 2.14.

Example 2.9 The sequence,

$$s_n = \begin{cases} 1, & n, \text{ odd} \\ -1, & n, \text{ even} \end{cases}$$

diverges, i.e. does not converge to any real number L.

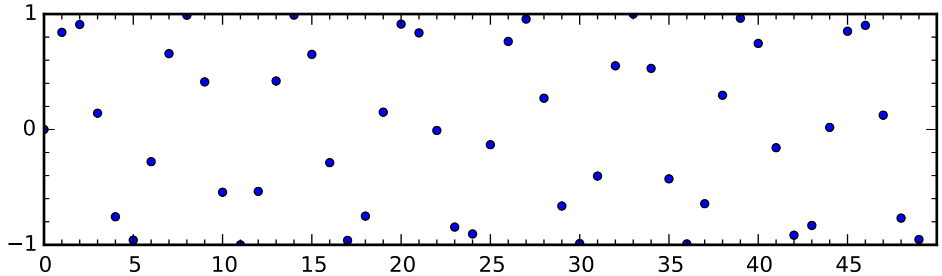
Proof. The subsequence given by $n_k = 2k + 1$ is a constant sequence: $s_{n_k} = 1$ for all k. This subsequence converges to 1. The subsequence given by $n_k = 2k$ converges to -1. If the sequence converged, both subsequences would have to converge to the same number, by Theorem 2.14. So the sequence does not converge. □

Exercise 2.21 ϒ Explain why the following sequence, $s_n = \sin(\frac{2\pi n)}{5}$, diverges. The graph of this sequence is shown in Figure 2.5.

Example 2.10 Another ϒ example of a *divergent* sequence is $s_n = \sin(n)$. That this sequence does not converge seems a correct conclusion considering the graph, shown in Figure 2.6. Proving that goes beyond the scope of our present discussion.

Another way a sequence may fail to converge is if it is unbounded. We consider separately the case when the limit appears to be infinite as in the sequence, $s_n = n$.

Figure 2.6: $s_n = \sin(n)$.



Definition We say that s_n *diverges to infinity* and we write, $\lim_{n\to\infty} s_n = +\infty$, whenever, for all $M > 0$, there exists $N > 0$ such that

$$n > N \implies s_n \geq M.$$

Example 2.11 $\lim_{n\to\infty} n^2 = +\infty$: If M is any positive real number, let $N = M$. Then, if $n > N$, we have that $n^2 > n > N = M$ or simply that $n^2 > M$. (EFS: Explain the step $n^2 > n$)

Exercise 2.22 Give a sequence that is unbounded but does not diverge to $+\infty$

Theorem 2.15 *If $s_n \to +\infty$, then any subsequence of s_n also diverges to $+\infty$.*

Exercise 2.23 Prove the following theorem.

Theorem 2.16 ALGEBRAIC PROPERTIES OF DIVERGENT LIMITS

1. $a_n \to +\infty$ *and* $b_n \to b$ *or* $b_n \to +\infty \implies (a_n + b_n) \to +\infty$.
2. $a_n \to +\infty$ *and* $b_n \to +\infty \implies (a_n \cdot b_n) \to +\infty$.
3. $a_n \to +\infty \implies \dfrac{1}{a_n} \to 0$

Exercise 2.24 Use BERNOULLI'S INEQUALITY Theorem 1.27, to prove the following theorem.

Theorem 2.17 *If $r \geq 1$, then $r^n \to +\infty$*

Proof. □

Exercise 2.25 LIMITS OF RATIOS Give examples of two sequences, $a_n \to +\infty$ and $b_n \to +\infty$, such that

a. $\dfrac{a_n}{b_n} \to +\infty$

b. $\dfrac{a_n}{b_n} \to 0$

c. $\dfrac{a_n}{b_n} \to c$, where c is a positive real number.

Exercise 2.26 Provide a definition and theorems about diverging to $-\infty$

a. What would it mean for the limit of a sequence to be $-\infty$?

b. If it is not part of your definition prove: $\lim_{n\to\infty} s_n = -\infty \iff \lim_{n\to\infty} -s_n = +\infty$?

c. If it is not part of your definition prove: If $\lim_{n\to\infty} s_n = -\infty$ then, for all $M < 0$, there exists $N > 0$ such that

$$n > N \implies s_n \leq M.$$

d. Give an example of a sequence that diverges to $-\infty$ and prove that it does.

e. State and prove statements of ALGEBRAIC PROPERTIES OF DIVERGENT LIMITS for sequences that diverge to $-\infty$

f. If $s_n \to -\infty$, then any subsequence of s_n also diverges to $-\infty$.

Exercise 2.27 Give a sequence $s_n \to 0$ where $\dfrac{1}{s_n}$ does not diverge to either $+\infty$ or $-\infty$.

Theorem 2.18 *If p is a polynomial that is not constant, then either*

$$\lim_{n\to\infty} p(n) = +\infty \text{ or } \lim_{n\to\infty} p(n) = -\infty$$

Proof. EFS □

2.2 Limits and Sets

2.2.1 Limit Points and Boundary Points

editor: Handle references to numbers as points and as elements of a set. BE CONSISTENT. editor: consider defining neighborhoods

We begin with a definitions of different types of special points of sets that are useful to think about: limit points and boundary points.

Definition We say the a point, p, is a *limit point of a set*, S, whenever every open interval about p contains an infinite number of points in S. In particular, it contains a point in S that is not equal to p.

NOTE: The point p need not be in S.

Definition We say the a point, p, is a *limit point of a sequence*, s_n, whenever every open interval about p contains an infinite number of s_n.

editor: Think a bit about the difference between these two definitions. In particular, the infinite number in the first case refers to numbers in the set,S, whereas, in the second case, infinite number refers to numbers in the domain *of s_n.* That, for sequences, the points can all be p. Whereas for sets all but at most one of the points are not p.

Example 2.12 The limit points of the image of s_n may be different than the limit points of s_n. Consider $s_n = (-1)^n$. The image of s_n is $\{, -1, 1\}$, a set that has no limit points. However, both -1 and 1 are limit points of the sequence because they each appear an infinite number of times in the sequence.

Theorem 2.19 *A real number p is a limit point of a set S, if and only if there exists sequence of points in $S \setminus \{p\}$ that converge to p.*

Proof. $\Longrightarrow$ Suppose p is a limit point of S. For each $n \in \mathbb{Z}^+$, let

$$s_n \in S \cap (p - \frac{1}{n}, p + \frac{1}{n}) \setminus \{p\}.$$

Such a point exists because p is a limit point. We claim that $s_n \to p$. For we know, for all n,

$$0 < |p - s_n| < \frac{1}{n}$$

From the squeeze theorem, we conclude that $|p - s_n| \to 0$ or $s_n \to p$.

$\Longleftarrow$Let $s_n \in S \setminus \{p\}$ converge to p. Let (a, b) be an open interval containing p. Consider any positive $\epsilon < \min(b - p, p - a)$, so that $p \in (p - e, p + \epsilon) \subset (a, b)$. By the convergences of s_n, there exists N such that $n > N \implies |p - s_n| < \epsilon$. These s_n's are an infinite number of values of the sequences that are in $(a, b) \setminus \{p\}$. If there were only a finite number of numbers from S in this sequence, we would have a subsequence of s_n that converges to some other number which cannot happen. So the s_n for $n > N$ are an infinite number of points in (a, b), as required to show that p is a limit point of the set S. □

Theorem 2.20 *If s is a limit point of a sequence s_n, then there exists a subsequence of s_n that converge to s.*

Proof. EFS □

Definition We say the a point, p, is an *boundary point* of a set, S, whenever every open interval containing p contains points in both S and $\mathbb{R} \setminus S$.

Example 2.13 Every point in a finite set is a boundary point of the set. Every point in a finite set is a boundary point of the complement of set.

True or False 10

Which of the following statements are true? If false, modify the statement to be true. Explain.

a) The endpoints of an interval are boundary points of the interval.

b) Every point in an interval is a boundary point of the interval.

c) Every point of an interval is a limit point of the interval.

d) The inf S is a limit point of S.

e) The inf S is a boundary point of S.

f) The maximum value of a S is a boundary point of S.

Example 2.14 Give an example of each of the following and explain.

a) A set and a point that is a boundary point but not a limit point of the set.

b) A set and a point that is a limit point but not a boundary point.

c) A set and a point that is neither a limit point nor a boundary point of the set.

d) A set and a point that is both a boundary point and a limit point of the set.

2.2.2 Open and Closed Sets

Definition We say a set is *open* whenever it contains none of its boundary points.

Definition We say a set is *closed* whenever it contains all of its boundary points.

Example 2.15 Open intervals are open sets because the only boundary points of an interval are the endpoints and neither are contained in the open interval.

Example 2.16 Closed intervals are closed sets be because the only boundary points of an interval are the endpoints and both are contained in the closed interval.

Theorem 2.21 *The following are equivalent*

1. *S is an open set*
2. *Every $s \in S$ is contained in an open interval that is completely contained in S.*
3. *$\mathbb{R} \setminus S$ is closed.*

Theorem 2.22 *The following are equivalent*

1. *S is an closed set*
2. *S contains all of its limit points.*
3. *$\mathbb{R} \setminus S$ is open.*

Exercise 2.28 Is $\mathbb{R}$ open or closed? Is $\varnothing$ open or closed?

Exercise 2.29 $\{x : x^2 \leq 57\}$ is a closed set.

True or False 11

Which of the following statements are true? If false, modify the statement to be true. Explain.

a) An open set never contains a maximum.

b) A closed set always contains a maximum element.

2.2.3 Optional – Connected sets

Suppose that there were no real number, s, such that $s^2 = 57$. Consider the two sets $U = \{s : s^2 < 57\}$ and $V = \{s : s^2 > 57\}$. Then an interval like $[0, 8]$ could be divided into two distinct sets, U and V. There would be a 'hole' in the numberline. This leads to the definition of connected that says a connected set cannot be covered by two distinct open sets. That intervals are not connected a way of understanding connectedness and investigating sets that may have more complicated structure than intervals. Still it is reassuring that intervals are connected/

Definition We say a set, S, is *connected* if it *is not* contained in the union of two *disjoint* non-empty, open sets.

Example 2.17 Finite sets are not connected

Here we present another nice application of the Nested Interval Theorem.

Theorem 2.23 *A connected set is a, possibly infinite, interval.*

Proof. Hint: this will be easiest to handle with an lemma: A subset of $\mathbb{R}$ is an interval whenever x, y are in the set so is every z in between. □

The more interesting part is that every interval is connected.

Theorem 2.24 *The interval* $[a, b]$ *is connected.*

Proof. Assume not. Let U and V be two disjoint open sets such that $[a, b] \in U \cup V$. Since both $\mathbb{R} \setminus U$ and $\mathbb{R} \setminus V$ are closed this means that $[a, b]$ is also covered by two distinct closed sets. We will use this and the fact that closed sets contain all of their limit points.

Since neither U nor V are empty, pick $u_0 \in U$ and $v_0 \in V$ and assume, without loss of generality, that $u_0 < v_0$, so

$$a \leq u_0 < v_0 \leq b.$$

Define inductively a sequence of closed, nested intervals, $[u_n, v_n]$ with $u_n \in U$ and $v_n \in V$ and length, $|v_n - u_n| = \frac{|v_0 - u_0|}{2^n}$. The base case, $[u_0, v_0]$, satisfies the conditions. Assume $[u_n, v_n]$ has been defined. Let m be the midpoint of the interval $[u_n, v_n]$. Now $m \in (a, b)$ so is in either U or V. There are two cases:

1. If $m \in U$, let $u_{n+1} = m$ and $v_{n+1} = v_n$.
2. If $m \in V$, let $u_{n+1} = u_n$ and $v_{n+1} = m$.

In either case, then $u_{n+1} \in U$ and $v_{n+1} \in V$. and $[u_{n+1}, v_{n+1}] \subset [u_n, v_n]$. So we have a closed interval of the required form. Since m is the midpoint, we known that

$$|v_{n+1} - u_{n+1}| = \frac{1}{2}|v_n - u_n| = \frac{1}{2} \cdot \frac{v_0 - u_0}{2^n} = \frac{v_0 - u_0}{2^{n+1}}$$

These intervals are nested, closed and non-empty so we can applied the Extended Nested Interval Theorem to say there is a point, $x \in [u_n, v_n]$, for all n, and that $u_n \to x$ and $v_n \to x$.

Now x is a limit point of $\mathbb{R} \setminus U$, a closed set, so it must be in $\mathbb{R} \setminus U$. That is, x is not in U. But x is also a limit point of $\mathbb{R} \setminus V$, another closed set, so x is not in V. x is in neither V nor U, but it is a point in $[a, b]$, so U and V cannot cover the interval as originally supposed. Therefore, the interval is connected. □

2.3 The Bolzano-Weierstrass Theorem and Cauchy Sequences

Theorem 2.25 THE BOLZANO-WEIERSTRASS THEOREM *Every bounded sequence has a converging subsequence.*

Outline of proof: Name the sequence s_n and let M be a bound for the sequence, that is For all n, $|s_n| < M$. We will construct a sequence of non-empty, closed nested intervals whose lengths converge to 0 and a subsequence of s_n such that s_{n_m} is contained in the m^{th} interval. Necessarily, this subsequence converges to the common point of the intervals. □

Definition A sequence is called a *Cauchy Sequence* whenever

$$\textit{For all } \epsilon > 0\textit{, there exists a real number, } N\textit{, such that}$$

$$\text{if both } n,\ m > N \implies |s_n - s_m| < \epsilon.$$

Theorem 2.26 *A sequence converges if and only if it is a Cauchy sequence*

Outline of proof: $\implies$ straight forward triangle inequality considerations

$\impliedby$ Mimic the proof that a convergent sequence is bounded to show that a Cauchy sequence is bounded. Apply B-W and finally use a triangle-inequality argument to show that the sequence must converge to the limit of the subsequence. □

2.4 Series and Power series

Series are special kinds of sequences where one keeps a running sum of a sequence, a_n to create a new sequence, s_n:

Definition The number

$$s_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \cdots + a_{n-1} + a_n$$

is called the n^{th} partial sum of the *generating sequence*, a_n. If the sequence, s_n, converges to a point, s, we say the the series converges to s and we write

$$s = \sum_{k=0}^{\infty} a_k$$

Definition A *power series* is a special kind of series where the generating sequence is of the form $a_n = c_n \cdot r^n$. If the c_n's are constant, $c_n = c$ we call it a *geometric series*.

Theorem 2.27 ALGEBRAIC PROPERTIES OF SERIES *If* $s = \sum_{k=0}^{\infty} a_k$ *and* $t = \sum_{k=0}^{\infty} b_k$, *then*

1. $s + t = \sum_{k=0}^{\infty}(a_k + b_k)$.
2. *If* $c \in \mathbb{R}$, *then* $c \cdot s = \sum_{k=0}^{\infty} c \cdot a_k$.

Exercise 2.30 Give examples of each of the following:

a. A series that diverges to $+\infty$.

b. A series whose partial sums oscillate between positive and negative numbers.

Exercise 2.31 If $a_n \geq 0$ for all n, then $s_n = \sum_{k=0}^{n} a_k$ is an increasing sequence.

Exercise 2.32 If $s = \sum_{k=0}^{\infty} a_k$ and $t = \sum_{k=0}^{\infty} b_k$, write a possible formula for the terms of a series that might be $s \cdot t$. Prove that your series converges and is equal to $s \cdot t$.

2.4.1 Convergence of Geometric series

Theorem 2.28 *If* $0 \leq |r| < 1$, *then*

$$\sum_{k=0}^{\infty} r^n = \frac{1}{1-r}$$

Proof. Hint: Mimic this trick from high school to see why $0.33333\cdots$ is equal to $\frac{1}{3}$ – be sure that you can justify each step:

$S = 0.33333\cdots$	given
$10 \cdot S = 3.33333\cdots$	multiplication of decimals
$9 \cdot S = 3$	Subtracting the first line from the second line
$S = \frac{3}{9}$	divide by 9
$S = \frac{1}{3}$	reduce fraction

□

2.4.2 Decimals

Definition A *decimal representation* looks like $a_0.a_1a_2a_3\cdots a_n\cdots$, where a_0 is an integer and a_n are integers between 0 and 9, inclusively. It represents the number which is given by the power series where $r = \frac{1}{10}$. We write

$$a_0.a_1a_2a_3\cdots a_n\cdots = a_0 + a_1\cdot\frac{1}{10} + a_2\cdot\frac{1}{10}^2 + a_3\cdot\frac{1}{10}^3 + \cdots = \sum_{n=0}^{\infty} a_n\cdot\frac{1}{10}^n$$

Exercise 2.33 Prove: 0.999999... = 1.

Theorem 2.29 *Every decimal representation is a convergent power series and hence every decimal representation is a real number.*

Proof. The partial sums are bounded by $a_0 + \sum_{k=1}^{n} 9\cdot\frac{1}{10}^k \leq a_0 + 1$. The partial sums are increasing. Every bounded increasing sequence converges. □

As well, every real number has a decimal representation that converges to it, more precisely

Theorem 2.30 *Given any $r \in \mathbb{R}$, there exists a sequence of integers a_n, where $0 \leq a_k \leq 10$ for all $k \geq 1$ such that*

$$\sum_{n=0}^{\infty} a_n\cdot\frac{1}{10}^n = r$$

Proof. Use Theorem 1.55. and THE EXTENDED NESTED INTERVAL THEOREM. □

Notice that nothing is said about the representation of a real number being unique. In fact any rational number that has a representation that ends with an infinite string of 0's has another representation that ends in a string of $9's$. And vice versa.

Exercise 2.34 Make a flowchart of theorems and connective arguments that take us from the Completeness Axiom to the representation of real numbers as decimals.

Chapter 3

Counting

3.1 Finite vs Infinite

We will be using the notion of a $1-1$ correspondence of two sets. This is just a bijection between the sets.

Definition Recall that we say that a set, S, is *finite* whenever there exists a integer, $N \geq 0$, and a $1-1$ correspondence between S and $\{n \in \mathbb{Z}^+ : n \leq N\}$. N is then the number of elements in the set. We say that a set is *infinite* if it is not finite.

editor: define cardinality

Definition We say that a set, S, is *countable* whenever there exists $1-1$ correspondence between S and a subset of $\mathbb{Z}^+$.

Exercise 3.1 Prove the following theorem.

Theorem 3.1 *Given two countable sets, U and V*

1. *Any subset of U is countable.*
2. *$U \cup V$ is countable*
3. *$U \cap V$ is countable*

Proof. □

True or False 12

Which of the following statements are true? Prove or give a counterexample.

a) If $A_0 \supset A_1 \supset A_2 \supset A_3 \supset A_4 \cdots$ are all infinite subsets of $\mathbb{R}$, then $\cap_{n=1}^{\infty} A_n$ is infinite.

b) If $A_0 \supset A_1 \supset A_2 \supset A_3 \supset A_4 \cdots$ are all finite, non-empty subsets of $\mathbb{R}$, then $\cap_{n=1}^{\infty} A_n$ is finite and non-empty.

3.1.1 The rational numbers

Theorem 3.2 *The rational numbers are countable.*

Proof. Consider this array that contains all the positive rational numbers. By counting along the diagonals, we can assign each rational number a positive integer to a rational number.

1 $\frac{1}{1}$	2 $\frac{2}{1}$	5 $\frac{3}{1}$	6 $\frac{4}{1}$	11 $\frac{5}{1}$	$\frac{6}{1}$	$\frac{7}{1}$	$\frac{8}{1}$	$\frac{9}{1}$	$\frac{10}{1}$	$\frac{11}{1}$	. . .
3 $\frac{1}{2}$	/ $\frac{2}{2}$	7 $\frac{3}{2}$	/ $\frac{4}{2}$	$\frac{5}{2}$	$\frac{6}{2}$	$\frac{7}{2}$	$\frac{8}{2}$	$\frac{9}{2}$	$\frac{10}{2}$	$\frac{11}{2}$	. . .
4 $\frac{1}{3}$	8 $\frac{2}{3}$	/ $\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$	$\frac{7}{3}$	$\frac{8}{3}$	$\frac{9}{3}$	$\frac{10}{3}$	$\frac{11}{3}$	. . .
9 $\frac{1}{4}$	/ $\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$	$\frac{7}{4}$	$\frac{8}{4}$	$\frac{9}{4}$	$\frac{10}{4}$	$\frac{11}{4}$	. . .
10 $\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	$\frac{6}{5}$	$\frac{7}{5}$	$\frac{8}{5}$	$\frac{9}{5}$	$\frac{10}{5}$	$\frac{11}{5}$	. . .
$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$\frac{7}{6}$	$\frac{8}{6}$	$\frac{9}{6}$	$\frac{10}{6}$	$\frac{11}{6}$	. . .
$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{6}{7}$	$\frac{7}{7}$	$\frac{8}{7}$	$\frac{9}{7}$	$\frac{10}{7}$	$\frac{11}{7}$	. . .
$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	$\frac{8}{8}$	$\frac{9}{8}$	$\frac{10}{8}$	$\frac{11}{8}$	. . .
.	.	.	.	.	.	.	.	.	.	.	. . .

□

Example 3.1 This procedure that gives a list of all positive rational numbers is an example of a sequence whose image is all positive rational numbers. This sequence clearly diverges as we can find a subsequence that is unbounded by picking out the sequence elements that are integers. We can also find subsequences that converge to any given positive number.

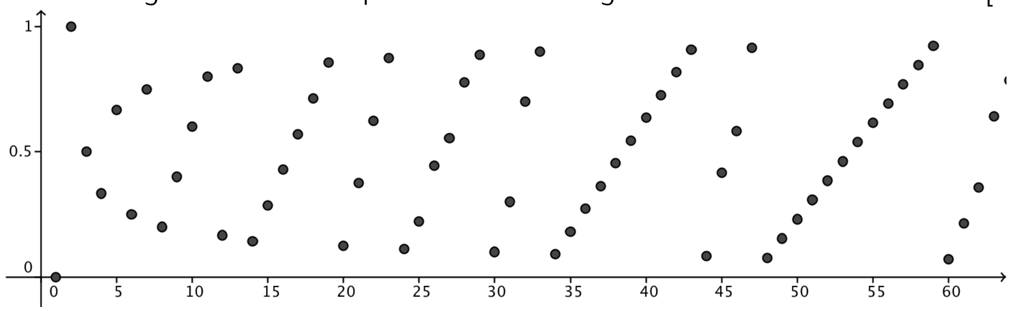
Exercise 3.2 Using the sequence of rational numbers given above, describe a procedure to find a subsequence that converges to a particular real number, r.

Example 3.2 There is a sequence whose image is the rational numbers between 0 and 1. List these numbers by first listing in order all fractions that have denominator 2, followed by those with denominator 3, and so on. In Figure 3.1 we show this sequence. In this case we eliminated duplicates as we listed the fractions.

Exercise 3.3 Using a similar argument show that $\mathbb{N}x\mathbb{N}$ is countable.

Exercise 3.4 Using a similar argument show that the countable union of countable sets is countable.

Figure 3.1: A sequence whose image is all rational numbers in [0,1]



3.1.2 How many decimals are there?

Theorem 3.3 *The real numbers are not countable.*

Proof. We will in fact show that the interval $(0,1)$ is not countable. The method of proof is called 'Cantor's diagonal argument' after the mathematician who first used it. Suppose we had had a correspondence between $\mathbb{Z}^+$ and the real numbers, using the decimal representation of the real numbers we could list them like this:

1	$0.a_{11}\, a_{12}\, a_{13}\, a_{14}\, a_{15}\, a_{16}\, a_{17}\, a_{18}\, a_{19} \cdots$
2	$0.a_{21}\, a_{22}\, a_{23}\, a_{24}\, a_{25}\, a_{26}\, a_{27}\, a_{28}\, a_{29} \cdots$
3	$0.a_{31}\, a_{32}\, a_{33}\, a_{34}\, a_{35}\, a_{36}\, a_{37}\, a_{38}\, a_{39} \cdots$
4	$0.a_{41}\, a_{42}\, a_{43}\, a_{44}\, a_{45}\, a_{46}\, a_{47}\, a_{48}\, a_{49} \cdots$
5	$0.a_{51}\, a_{52}\, a_{53}\, a_{54}\, a_{55}\, a_{56}\, a_{57}\, a_{58}\, a_{59} \cdots$
6	$0.a_{61}\, a_{62}\, a_{63}\, a_{64}\, a_{65}\, a_{66}\, a_{67}\, a_{68}\, a_{69} \cdots$
7	$0.a_{71}\, a_{72}\, a_{73}\, a_{74}\, a_{75}\, a_{76}\, a_{77}\, a_{78}\, a_{79} \cdots$
8	$0.a_{81}\, a_{82}\, a_{83}\, a_{84}\, a_{85}\, a_{86}\, a_{87}\, a_{88}\, a_{89} \cdots$
$\cdots$	$\cdots$

Make a number, $0.a_1\, a_2\, a_3\, a_4\, a_5\, a_6\, a_7\, a_8 \cdots$ that is not on the list by defining

$$a_k = a_{kk} + 1 \mod 10.$$

This number can not be on our list because it differs from the k^{th} element on the list in the k^{th} decimal place. $\square$

3.2 Cantor Sets

Draw a picture of the recursive procedure that defines the Cantor Set.

View the Cantor set as the intersection of a countable number of closed sets. Conclude that the Cantor Set is closed.

The Cantor Set is not countable.

The complement of the Cantor Set has accumulated length of 1.

The Cantor Set consists of all real numbers in $[0, 1]$ whose tertiary expansion can be written with no 2's

The dimension of the Cantor Set is log2 /log3

Exercises: construct other Cantor Sets.

Chapter 4

Functions

4.0 Limits

Definition Let $f : D \to \mathbb{R}$.

We write $\lim_{x\to p} f(x) = L$ and say the *limit as x approaches p of f(x) is L*, whenever,

$$\text{For all } \epsilon > 0, \text{ there exists } \delta > 0, \text{ so that for } x \in D,$$
$$0 < |x - p| < \delta \implies |f(x) - L| < \epsilon.$$

To emphasize the domain, we may say: the limit as x approaches p *in D*.

Note: The $0 <$ is included because it is not necessary for the limit to be equal to $f(p)$. It is not even necessary for f to be defined at p. This is important later on when we define derivatives as limits. The concern disappears when we use limits to define continuity and derivatives.

We don't go through the step of $L = 0$ as we did for limits of sequences because we are so much better at deconstructing limit statements now. However, it is true that

Theorem 4.1 $\lim_{x\to p} f(x) = L \iff \lim_{x\to p}(f(x) - L) = 0$

Definition We say that $f(x)$ *converges to L sequentially as* $x \to p$ whenever,

$$\text{for any sequence, } x_n \in D - \{p\}$$
$$x_n \to p \implies f(x_n) \to L$$

We do not consider values of p for the sequence because $f(p)$ may not be equal to L. Again, this won't be a concern with continuity.

Theorem 4.2 $\lim_{x\to p} f(x) = L$ *if and only if* $f(x)$ *converges to L sequentially as* $x \to p$

Proof. EFS □

Now, either definition of limit can be used in the proof of the following theorem.

Theorem 4.3 ALGEBRAIC PROPERTIES OF LIMITS 3 *Given two functions f and g defined on a domain, D, with $\lim_{x \to p} f(x) = A$ and $\lim_{x \to p} g(x) = B$, the following are true:*

1. $\lim_{x \to p}(f+g)(x) = A + B$
2. $\lim_{x \to p}(f \cdot g)(x) = A \cdot B$
3. *If $f(x) \neq 0$ for any $x \in D$ and if $A \neq 0$, then* $\lim_{x \to p}(\frac{1}{f})(x) = \frac{1}{A}$

Proof. A good way to prove these theorems is using sequential convergence and the corresponding theorems about sequences. □

Theorem 4.4 SQUEEZE THEOREM *Given two functions f and g defined on a domain, D, with $\lim_{x \to p} f(x) = \lim_{x \to p} g(x) = C$, and another function, h, defined on D, with $f(x) \leq h(x) \leq g(x)$ for all $x \in D$, then $\lim_{x \to p} h(x) = C$*

Proof. Given $\epsilon > 0$, find $\delta > 0$ such that

$$|x - p| < \delta \implies |f(x) - C| < \epsilon \text{ and } |f(x) - C| < \epsilon$$

The following inequalities demonstrate that $|h(x) - C| < \epsilon$, so $\lim_{x \to p} h(x) = C$

$$-\epsilon \leq -|f(x) - C| \leq f(x) - C \leq h(x) - C \leq g(x) - C \leq |g(x) - C| < \epsilon$$

□

Exercise 4.1 Because we are taking advantage of sequential limits, we did not need to prove the following theorem to prove Theorem 4.3 2. Prove this theorem:

Theorem 4.5 *If $\lim_{x \to p} f(x) = A$, then f is bounded in some open interval about p.*

4.1 Continuity

For the following, consider a function, $f : D \to \mathbb{R}$.

Definition CONTINUITY AT A POINT We say that the f is continuous at $p \in D$ if

$$\lim_{x \to p} f(x) = f(p).$$

Example 4.1 The function x^2 is continuous at the point $p = 3$.

Proof. directly from the definition: We need to show that $\lim_{x\to 3} x^2 = 9$.
Given $\epsilon > 0$, find a $\delta < \min\{1, \frac{\epsilon}{7}\}$. Then we can see that if $|x-3| < \delta$ we have

$$|x^2 - 9| = |x-3| \cdot |x+3| < \delta \cdot 7 < \frac{\epsilon}{7} \cdot 7 = \epsilon.$$

[Justifications: Because $\delta < \min\{1, \frac{\epsilon}{7}\}$, we know that δ is less than both 1 and $\frac{\epsilon}{7}$. if $|x-3| < 1$, then $|x+3| = |x-3+6| \leq |x-3| + |6| < 1+6 = 7$. This justifies the first '$<$'. The second '$<$' is justified because $\delta < \frac{\epsilon}{7}$]
So $|x-3| < \delta \implies |x^2 - 9| < \epsilon$, as we needed to show. □

Exercise 4.2 Directly from the definition of limit, show that the function $\sqrt{x}$ is continuous at the point $p = 4$.

Definition CONTINUITY ON A SUBSET OF THE DOMAIN We say that a function, f, is *continuous on $E \subset D$* if it is continuous at all points of E.

Exercise 4.3 The function x^2 is continuous on $\mathbb{R}$.

Exercise 4.4 The function $\sqrt{x}$ is continuous on $\mathbb{R}^{\geq}$.

Exercise 4.5 Give an example of a function is continuous on all but 3 points in $\mathbb{R}$.

4.1.1 Sequential continuity

We could have used a sequentially definition for continuous at a point. Here it is said in a theorem.

Theorem 4.6 *A function f is* continuous at a point $p \in D$ *if and only if*

for any sequence, $x_n \in D$
$$x_n \to p \implies f(x_n) \to f(p)$$

Proof. □

Another way to state the conclusion of this theorem is: *If f continuous on E and $s_n \in E$ converges to a point in E, then*

$$f(\lim_{n\to\infty} s_n) = \lim_{n\to\infty} f(s_n)$$

Exercise 4.6 Use the sequential definition of continuity to show that the function x^2 is continuous on $\mathbb{R}$.

Exercise 4.7 Give an example of a function, f, and a sequence of points, s_n, is the domain of f, such that $f(s_n)$ converges, but s_n does not.

Exercise 4.8 Prove the basic facts about continuous functions stated in the following theorem:

Theorem 4.7 ALGEBRAIC PROPERTIES OF CONTINUITY *Given two functions f and g defined and continuous on a domain, D, the following are true:*

1. *$f + g$ is continuous on D.*

2. *$f \cdot g$ is continuous on D.*

3. *If $f(x) \neq 0$ for any $x \in D$, $\frac{1}{f}$ is continuous on D.*

Proof. A good way to prove these theorems is to use the corresponding theorems about limits of functions. □

Exercise 4.9 Prove the following theorem.

Theorem 4.8 *The addition functions, $s_b : x \to x + b$, and multiplication functions, $t_m : x \to m \cdot x$, are continuous on $\mathbb{R}$.*

Theorem 4.9 *Polynomials are continuous.*

Proof. EFS First make a flowchart of the proof. What simpler facts should you prove first? What is the overall plan of attack. □

Theorem 4.10 *Rational functions are continuous wherever they are defined.*

Proof. EFS Use theorem 4.9 □

4.1.2 More Examples and Theorems

An aside to discussion inverse functions

The composition of two functions and the inverse of a function can be defined in the most abstract settings. We review those concepts here before considering the continuity of compositions and inverse functions.

Definition Given two functions, $f : D \to E$ and $g : E \to F$, the function, $g \circ f : D \to F$ is defined for each $x \in D$ by

$$g \circ f(x) = g(f(x)).$$

We write $g \circ f$ and say *composition of g with f.*

Theorem 4.11 *If $f : D \to E$ and $g : E \to F$ are continuous on their respective domains, then $g \circ f : D \to F$ is continuous on D.*

Proof. □

Definition We say that a function, $f : D \to E$ has an *inverse function*, if there exists a function, $f^{-1} : E \to D$ such that

$$\text{for all } d \in D, f^{-1} \circ f(d) = d.$$

Exercise 4.10 The proof of the following theorem is fundamental to understanding inverse functions.

Theorem 4.12 *If $f : D \to E$ has an inverse $f^{-1} : E \to D$, then f^{-1} has an inverse and it is f, that is*

$$\textit{for all } e \in E, f \circ f^{-1}(e) = e.$$

Proof. □

Exercise 4.11 There are not many examples available to us now without Υ. However, we can note that the addition function, $s_b : x \to x + b$ has an inverse. Show that

$$s_b^{-1} = s_{-b}.$$

Also, the multiplication function, $t_m : x \to m \cdot x$, has an inverse as long as $m \neq 0$. Show that

$$t_m^{-1} = t_{m^{-1}}.$$

Exercise 4.12 Write a formula for the composition function $t_m \circ s_b$ and another for $s_b \circ t_m$ What are the inverses of these composition functions?

Theorem 4.13 *Given two functions, $f : D \to E$ and $g : E \to F$, If both have inverses, so does $g \circ f : D \to F$. In fact,*

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

Proof. EFS □

A library of functions

It is good to have a well stocked library of functions to help think about the various properties and theorems.

1. polynomials and rational functions

2. piecewise continuous: step functions

3. piecewise differentiable: absolute value
4. characteristic function of the rationals
5. cantor function
6. inverse functions, like $\sqrt{x}$
7. Υ exponential, logarithms and trigonometric
8. power series

4.1.3 Uniform Continuity

Later we will need a stronger type of continuity:

Definition We say that a function f is *uniformly continuous on* $E \subset D$ whenever

$$\text{For all } \epsilon > 0, \text{ there exists a real number, } \delta > 0, \text{ such that}$$
$$\text{for all } x \text{ and } p \in E, \ |x - p| < \delta \implies |f(x) - f(p)| < \epsilon.$$

This defines a stronger condition than the just continuity. Given $\epsilon > 0$ one must find a δ that works for *all* $p \in E$. If one does not want to show *uniform* continuity, one can find δ that is dependent on the particular value of p.

Exercise 4.13 $f(x) = \sqrt{x}$ is uniformly continuous on $[1, \infty)$.

Exercise 4.14 $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 1]$.

Exercise 4.15 $f(x) = x^2$ is not uniformly continuous on $[1, \infty)$.

Exercise 4.16 $f(x) = x^2$ is uniformly continuous on $[0, 1]$.

Uniform continuity will be needed when we talk about integrals. The following theorem is very helpful.

Theorem 4.14 *A continuous function on a closed and bounded set is uniformly continuous on that set.*

Proof. Let $f{:}K \to \mathbb{R}$, where K is a closed and bounded subset of $\mathbb{R}$, be continuous at all points in K. Assume f is not uniformly continuous on K. Then there exists an $\epsilon > 0$ and sequences $x_n \in K$, $p_n \in K$ and $\delta_n \to 0$ with $|x_n - p_n| < \delta_n$ but $|f(x_n) - f(p_n)| \geq \epsilon$. x_n is bounded and so has a converging subsequence, x_{n_k}. Now p_{n_k} is bounded and so has a converging subsequence, $p_{n_{km}}$. For all m, we have

$$|x_{n_{km}} - p_{n_{km}}| < \delta_{n_{km}}.$$

So the limits are same:

$$\lim_{m\to\infty} x_{n_{km}} = \lim_{m\to\infty} p_{n_{km}}.$$

By sequential continuity, $\lim_{m\to\infty} |f(x_{n_{km}}) - f(y_{n_{km}})| \to 0$. So the difference cannot be bounded below by ϵ. □

Exercise 4.17 Is a linear combination of two uniformly continuous functions uniformly continuous? Explain.

Exercise 4.18 What are the other analogous facts for continuous functions and uniformly continuous functions from Theorem 4.3? Are they true or not?

4.2 Intermediate Value Theorem

Theorem 4.15 THE INTERMEDIATE VALUE THEOREM *A function that is continuous on a closed, bounded interval* $[a, b]$ *attains every value on the interval* $[f(a), f(b)]$.

Proof. If $f(a) = f(b)$, we are done. So assume, without lose of generality, that $f(a) < f(b)$ and let c be any number in $[f(a), f(b)]$. We will show that there exists two nested sequences of closed intervals, $[a_n, b_n]$ and $[f(a_n), f(b_n)]$ with $c \in [f(a_n), f(b_n)]$ such that the lengths of both go to zero.

We will then use the extensions of the nested interval theorem to show that the unique number in all the $[f(a_n), f(b_n)]$ is $f(c)$. We define the intervals inductively:

Base case: Let $a_0 = a$ and $b_0 = b$. c was chosen to be in $[f(a_n), f(b_n)]$. Assume that $[a_n, b_n]$ and $[f(a_n), f(b_n)]$ are defined as required, proceed inductively:

Let m be the midpoint of the interval $[a_n, b_n]$. There are three cases:

1. If $f(m) = c$, we are done.
2. If $f(m) < c < f(b_n)$, let $a_{n+1} = m$ and $b_{n+1} = b_n$.
3. If $f(a_n) < c < f(m)$, let $a_{n+1} = a_n$ and $b_{n+1} = m$.

In either case of the last two cases, $f(a_{n+1}) < c < f(b_{n+1})$. This new interval, $[a_{n+1}, b_{n+1}]$ is half the length of the previous because m in the midpoint. Because the intervals are cut in half at each step, the lengths converge to 0 and the extension of the Nested Interval Theorem: there is a point, p, in all of the intervals and $a_n \to p$ and $b_n \to p$. By sequential continuity $f(a_n) \to f(p)$ and $f(b_n) \to f(p)$. By the squeeze theorem, $f(p) = c$. □

Exercise 4.19 Show that there exists a real number c such that $c^4 - c^2 = 3$. Is c unique?

Theorem 4.16 *Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial of odd degree. There exists a real number, c, such that $p(c) = 0$*

Proof. EFS □

Exercise 4.20 Use the THE INTERMEDIATE VALUE THEOREM to show that there was a time in your life when your height in inches was equal to your weight in pounds.

Exercise 4.21 Prove: If $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$, $f(a) < g(a)$ and $g(b) < f(b)$, there exists a real number, $c \in (a, b)$, such that $f(c) = g(c)$.

Theorem 4.17 *If a function,$f : [a, b] \to \mathbb{R}$ is strictly increasing on the interval, $[a, b]$, then there exists an* inverse *function, $f^{-1} : [f(a), f(b)] \to [a, b]$. f^{-1} is increasing on the interval $[f(a), f(b)]$.*

Proof. Since f is increasing on the interval, $f(a)$ is the minimum value of f and $f(b)$ is the maximum value of f. We will define $f^{-1}(y)$ for each $y \in [f(a), f(b)]$, and then show that f^{-1} is indeed the inverse of f.

For $y \in f(a), f(b)]$, there is an unique $x \in [a, b]$ such that $f(x) = y$. The THE INTERMEDIATE VALUE THEOREM (Theorem 4.15) guarantees there is at least one such x. There cannot be two because the function is strictly increasing. So define $f^{-1}(y) = x$. Clearly, $f^{-1} \circ f(y) = y$. □

True or False 13

Which of the following statements are true? Explain.

a) A decreasing function defined on $[a, b]$ has an inverse defined on $[f(b), f(a)]$?

b) An increasing function defined on (a, b) has an inverse defined on $(f(a), f(b))$?

c) If a function has an inverse on an interval, it must be either increasing or decreasing on that interval.

d) The function $f : x \to x^2$ has an inverse.

4.3 Continuous images of sets

Subsets of $\mathbb{R}$ that are both closed and bounded are also called *compact*. There is a more general definition for compact in other topological spaces but, for this book, we stick to closed and bounded. Compact sets are important in topology for reasons given in the next few theorems.

Theorem 4.18 *The continuous image of a closed bounded set is closed and bounded.*

Proof. Setup: Let C be a closed bounded set and let $f : C \to \mathbb{R}$ be continuous. We will show that $f(C)$ is also closed and bounded.

First, $f(C)$ is bounded: Assume the image of C is not bounded. Then there exists a sequence of points in $f(C)$ that converge to $+\infty$ (or $-\infty$). Now, the x_n live back in C and so, by THE BOLZANO-WEIERSTRASS THEOREM(Theorem 2.25), have a convergent subsequence, say $x_{n_k} \to x$. Because C is closed $x \in C$. Now, f is continuous on C so $f(x_{n_k}) \to f(x)$. But a convergent sequence is bounded so $f(x_{n_k})$ cannot be a subsequence of a sequence that converges to $+\infty$ (or $-\infty$). This is a contradiction, so $f(C)$ is bounded.

$f(C)$ is closed: Let y be a limit point of C. So $y_n = f(x_n) \to y$. We will show that $y \in f(C)$ which will prove that $f(C)$ is closed. Because $x_n \in C$ and C is bounded, there is a subsequence $x_{n_k} \to x$ (BW again). Since C is closed, $x \in C$. Since f is continuous on C, $f(x_{n_k}) \to f(x)$. So $y = f(x)$ and $y \in f(C)$ □

Theorem 4.19 *The inverse image, under a continuous function, of an open set is open.*

True or False 14

Which of the following statements are true? Explain.

a) The continuous image of an open set is open.

b) The continuous image of an unbounded set is unbounded.

c) The inverse image, under a continuous function, of an closed set is closed.

d) The inverse image, under a continuous function, of a bounded set is bounded.

4.4 Optional: Connected Sets

Theorem 4.20 *The continuous image of a connect set is connected.*

An alternative proof of the intermediate value theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous. The image of $[a, b]$ is connected so it is an interval. It is bounded, so it is a finite interval. Finally it is closed, so it is a closed interval. So this interval must contain all of the interval $[f(a), f(b)]$. Hence every number in $[f(a), f(b)]$ is in the image of f. □

4.5 Existence of extrema

Theorem 4.21 *A continuous function defined on a closed, bounded set obtains a maximum (and a minimum value) on that set.*

Proof. □

Exercise 4.22 Find the maximum and minimum value of the function, $f(x) = x^2 - bx + c$ on the closed interval, $[0, 1]$. You will have to handle cases for different values of b and c.

4.6 Derivatives

4.6.1 Definitions

Definition DERIVATIVE AT A POINT We say that a function, $f : D \to \mathbb{R}$ is *differentiable* at a point $p \in D$, whenever the following limit exists.

$$\lim_{x \to p} \frac{f(x) - f(p)}{x - p}. \tag{4.1}$$

Notation We call the limit 4.1 the *derivative* of f at p, and write $f'(p)$.

$$f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p}. \tag{4.2}$$

Definition THE DERIVATIVE FUNCTION If the limit 4.1 exists for all p in a subset, E, of the domain of f, it defines a function which we write as f' and we say that f is *differentiable* on E

Theorem 4.22 *If $f : \mathbb{R} \to \mathbb{R}$ is constant on any open interval, then f' exists on the interval and $f'(x) = 0$ for all x in the interval.*

Exercise 4.23 Directly from the definition, find a formula for the derivative of the following functions at a point, p, in the domain of the function.

a) x^2

b) $\sqrt{x}$

c) x

4.6.2 Applications

editor: Ask students to bring in examples from their own discipline

practical significance The difference quotient represents an average rate – especially useful when the domain is a time variable. The derivative at a point p, $f'(p)$ represents an *instantaneous speed*.

geometric significance The derivative at a point p, $f'(p)$, represents the *slope* of the tangent line to the graph of f at the point $(p, f(p))$. [[insert picture]]

4.6.3 Basic Theorems

Theorem 4.23 *If $f : D \to \mathbb{R}$ is differentiable on D, then f is continuous on D.*

Theorem 4.24 *Given two real-valued functions f and g defined on a domain, $D \subset \mathbb{R}$ and $c \in \mathbb{R}$.*

1. *$c \cdot f + g$ is differentiable on D, and $(c \cdot f + g)'(x) = c \cdot f'(x) + g'(x)$*
2. *$f \cdot g$ is differentiable on D, and $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$*

Exercise 4.24 Restate the following theorem to include a formula for the derivative of a polynomial. Prove the theorem.

Theorem 4.25 *Polynomials are differentiable.*

Exercise 4.25 If $f(x) = \frac{1}{x}$, then f is differentiable on $(0, +\infty)$ and $f'(x) = -\frac{1}{x^2}$

Exercise 4.26 Carefully state a theorem that describes the derivative of a composition of two functions. Prove your theorem.

True or False 15

If $f'(x) > 0$ on an interval $(a, b) \subset D$, then $f(x)$ is increasing on (a, b).

4.6.4 Zero Derivative Theorem

The following series of theorems aims to show the converse of Theorem 4.22: If a function has a derivative that is zero on an open interval then the function is constant on that interval.

Theorem 4.26 *If a function, f, differentiable on (a, b) has a maximum value at $x_0 \in (a, b)$, then $f'(x_0) = 0$.*

Proof. □

Exercise 4.27 Give an example of a function that has a maximum value but the derivative at that point is not zero. Give one where the function is not differentiable and one where the interval is closed instead of open.

Exercise 4.28 Give an example of a differentiable function, f, and a point, p, such that $f'(p) = 0$, but f does not have a maximum or minimum value at p.

Theorem 4.27 ROLLE'S THEOREM *$f:[a, b] \to \mathbb{R}$, f continuous on $[a, b]$, and f differentiable on (a, b).*

$$f(a) = f(b) \implies \textit{There exists } x^* \in (a, b) \textit{ with } f'(x^*) = 0$$

Definition The *mean value of a function, f, over the interval* $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}$$

Note that this is the slope of the line connecting $(a, f(a))$ and $(b, f(b))$.

This is an example of an existence theorem. The proof relies ultimately on BW which was constructive but unwinding that construction may be hard. In calculus, we find typically find those points where a function has maximum by solving $f'(x) = 0$.

Exercise 4.29 Write an equation for the straight line determined by the two points, $(a, f(a))$ and $(b, f(b))$. Use the point-slope formula for the equation of a line.

Example 4.2 average speed

Theorem 4.28 THE MEAN VALUE THEOREM *$f:[a, b] \to \mathbb{R}$, f continuous on $[a, b]$, and f differentiable on (a, b).*

$$\textit{There exists } x^* \in (a, b) \textit{ with } f'(x^*) = \frac{f(b) - f(a)}{b - a}$$

Exercise 4.30 Where on the interval $[-1, 1]$ does the derivative of $f(x) = x^3$ assume mean value of the function? Sketch a graph.

Theorem 4.29 UNIQUENESS THEOREM *If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on the interval.*

Proof. Let z_1 and z_2 be any two points in the interval, (a, b). By THE MEAN VALUE THEOREM there is a point, $x^* \in (z_1, z_2)$, such that

$$f'(x^*) = \frac{f(z_2) - f(z_1)}{z_2 - z_1}.$$

But $f'(x^*) = 0$ by hypothesis. This then means that $f(z_1) = f(z_2)$. Since they were any two points in (a, b), f must be constant. $\square$

Chapter 5

Integration

5.0 Definition of Riemann Integral

5.0.1 Partitions of an interval

Definition A *partition of the interval* $[a, b]$ is a finite, increasing sequence of numbers, x_j, such that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

The points divide $[a, b]$ into n disjoint open subintervals, (x_i, x_{i+1}), such that the closed intervals cover $[a, b]$, i.e. $[a, b] = \bigcup_{i=0}^{n-1} [x_i, x_{i+1}]$

The *mesh* of a partition is the length of the largest sub-interval. If P is a partition of $[a, b]$ then we write $|P|$ for the mesh of P. The mesh is used as a way of ordering partitions. This kind of ordering is called a partial ordering because there is not trichotomy, i.e. two different partitions can have the same mesh. This ordering is useful because we want to talk about the concept of the limit as the mesh of the partitions goes to zero.

A *tagged partition of the interval* $[a, b]$ is a partition with the additional n points, $t_k^* \in [x_k, x_{k-1}]$. So we have

$$a = x_0 \leq t_1^* \leq x_1 \leq t_2^* \leq x_2 \leq t_3^* \cdots t_{n-1}^* \leq x_{n-1} \leq t_n^* \leq x_n = b.$$

There are lots of partitions of an interval. The x's may be placed with any spacing between them and the t^*'s can be any point with the interval. They are used to help us define the integral of a function. We need to consider, at least at the beginning, all possible partitions because we want to know that as long as we choose a reasonable way to compute the integral we get the same answer. Our definition of integral must make it so.

editor: stick in some pictures

5.0.2 Definition of Riemann Integral

Definition Given any function, $f\colon [a,b] \to \mathbb{R}$, and a partition of $[a,b]$, we define

$$P(f) = \sum_{k=1}^{n} f(t_k^*) \cdot (x_k - x_{k-1})$$

we call any such $P(f)$ a *Riemann sum of f over* $[a,b]$.

editor: insert picture

Definition We say the a function, $f\colon [a,b] \to \mathbb{R}$ is *Riemann-integrable* whenever there exists $I \in \mathbb{R}$ such that

$$\text{For all } \epsilon > 0 \text{, there exists } \delta > 0 \text{ so that, for any partition, } P$$
$$|P| < \delta \implies |P(f) - I| < \epsilon$$

We notate I by

$$\int_a^b f(x)dx$$

Named after Riemann xxxx was the first to give a precise definition of integral

Example 5.1 This is a pretty unmanageable definition but we can show that

$$\int_a^b 1 \cdot dx = b - a,$$

for let P be any tagged partition. Since the function value at each t_i^* is 1, the Riemann sum is just the sum of the lengths of all the intervals or $b - a$. And so all R-sums are the same.

With a definition that involves the existence of something over a seemingly unmanageable sets as the set of all tagged partitions of an interval, it is easier to find examples of what is not than to prove something is. Here are two examples of functions that are not integrable:

Example 5.2 The function

$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } 0 < x \leq 1 \\ 0, & \text{if } x = 0 \end{cases}$$

is not integrable on $[0,1]$.

Proof that f is not integrable: Consider a sequence of partition, P_n, with n equal-length intervals, given by points, $x_i = \frac{i}{n}$. Let $t_0^* = \frac{1}{n^2}$. For any choice of the rest of the t_k^*'s, the Riemann sum,

$$P_n(f) = \sum_{k=1}^{n} f(t_k^*) \cdot (x_k - x_{k-1}) > \frac{1}{\frac{1}{n^2}} \cdot \frac{1}{n} = n.$$

Although $mesh(P_n) \to 0$, $P_n \to \infty$. So f is not Riemann integrable. □

Similar considerations allow us to conclude the following theorem:

Theorem 5.1 *If f is integrable on $[a, b]$, then f is bounded on $[a, b]$.*

Proof. Hint: Assume f is not bounded and, for every $n > 0$, find a partition, P_n such that $|P_n(f)| < \frac{1}{n}$, but $P_n(f) > n$. □

Example 5.3 The function

$$g(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not integrable on $[0, 1]$.

Proof that g is not integrable: Let P_n be any partition with mesh less than $\frac{1}{n}$ and t_i^* all rational numbers. Let Q_n be the same except the tagged numbers are all irrational. Then $P_n(g) = 1$ and $Q_n(g) = 0$. We have given to sequences of Riemann sums that converge to different numbers, so g cannot be Riemann integrable. □

In each example, we only needed to find a sequence of partitions that converged to zero but that the corresponding Riemann sums did not converge. It would be so nice if we could simplify the existence of the integral by looking at a single sequence of partitions. That's what the next theorem allows:

Theorem 5.2 SEQUENTIAL DEFINITION OF RIEMANN INTEGRAL *A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if for every sequence of partitions, P_n, with $|P_n| \to 0 \implies P_n(f)$ converges.*

Proof. NOTE: If all partitions converge they necessarily converge to a common limit because we can intertwine the sequences. □

Example 5.4 Consider The function

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & \text{if } x \in (0, 1] \\ 0, & \text{if } x = 0. \end{cases}$$

Find two sequences of tagged partitions of $[0, 1]$, one diverging, one converging.

Now if we only knew that a function were integrable, we could pick out a convenient sequence of partitions and life would be easier. The next theorem gives us what we need for a large class of functions.

Theorem 5.3 *Continuous functions are Riemann integrable*

Proof. Let $f : [a, b] \to \mathbb{R}$ be continuous on [a,b]. We know by Theorem XXX, that f is uniformly continuous on $[a, b]$.
Let P_n be a sequence of tagged partitions such that mesh(P_n)$\to 0$. We will show that the corresponding Riemann sums is Cauchy and hence converges.
Given $\epsilon > 0$, use the uniform continuity of f to find δ such that $|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{b-a}$
Now, find a positive integer N such that

$$n > N \implies \text{mesh}(P_n) < \frac{\delta}{2}.$$

Consider any two partitions, P_n and P_m, with $n, m > N$. Label the tags from P_n with t^* and the tags from P_m with s^*. Consider a refinement of the two partitions, Q_{n+m} with $n + m$ elements, x_k, the combined points in the two given partitions. The difference between the Riemann sums will look like:

$$P_n(f) - P_m(f) = \sum_{k=1}^{n+m}(f(t_k^*) - f(s_k^*) \cdot (x_k - x_{k-1})$$

where the t_k^* is the t^* assigned to whatever P_n interval contains $[x_k, x_{k-1}]$ and similarly for s_k^*. We claim that $|t_k^* - s_k^*| <$ mesh(P_n)+mesh(P_m) $< \delta$. Proof of claim: Each interval in the refinement is contained in a unique interval from each of P_n and P_m. If both endpoints are from P_n, then t^* comes from that interval so $|t_k^* - s_k^*| <$ mesh(P_m). Similarly, $|t_k^* - s_k^*| <$ mesh(P_n) if both x_k and x_{k+1} are in P_m. If each endpoints is from a different partition, t^* and s^* come from two overlapping intervals – they may be as far apart as the length of one plus the length of the other. Since the lengths are less than respective mesh, we have $|t_k^* - s_k^*| <$ mesh(P_n)+mesh(P_m). Which means we also have:

$$|t_k^* - s_k^*| < \delta \implies |f(t_k^*) - f(s_k^*)| < \frac{\epsilon}{b-a}$$

Putting it all together:

$$\begin{aligned}
|P_n(f) - P_m(f)| &= |\sum_{k=1}^{n+m}(f(t_k^*) - f(s_k^*) \cdot (x_k - x_{k-1})| \\
&\le \sum_{k=1}^{n}|f(t_k^*) - f(s_k^*)| \cdot (x_k - x_{k-1}) \\
&\le \epsilon \cdot \sum_{k=1}^{n+m}(x_k - x_{k-1}) \\
&\le \epsilon(b - a)
\end{aligned}$$

□

Now we are free to use any of the techniques that are normally seen in a calculus courses: Consider partitions that are evenly spaced, tagged values that are endpoints, or maximum values or whatever.

Example 5.5 Compute $\int_a^b x^2 dx$

Exercise 5.1 Compute $\int_a^b x dx$ (Please do note that the answer is the area bounded by the curve, the x-axis, and the vertical lines $x = a$ and $x = b$.)

5.0.3 Theorems

Theorem 5.4 ALGEBRAIC PROPERTIES OF RIEMANN INTEGRALS *Given two functions, f and g, both integrable on the closed interval $[a, b]$. The following are true:*

1. $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
2. $\int_a^b c \cdot f(x)dx = c \cdot \int_a^b f(x)dx$
3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Theorem 5.5 ORDER PROPERTIES OF RIEMANN INTEGRALS *Given two functions, f and g, both integrable on the closed interval $[a, b]$. The following are true:*

1. *If $f(x) \leq g(x)$ for all $x \in [a, b]$, then*

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx)$$

2. *If M and m are respectively upper and lower bounds for f on $[a, b]$, then*

$$m \cdot (b - a) \leq \int_a^b f(x)dx \leq M \cdot (b - a)$$

3. $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$

We may know what integrable functions are but it can be cumbersome if not impossible to show the exact (or even approximate) value for a particular function, or class of functions. That is where the theorems of the next section make life easier.

There is nothing in the definition of the Riemann Integral that prohibits $b \leq a$.

Theorem 5.6 *For any function, f, integrable on $[a, b]$,*

1. $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. $\displaystyle\int_a^a f(x)dx = 0.$

Proof. □

5.1 Fundamental Theorem of Calculus

5.1.1 Integrals as Functions

Given a function, f, integrable on an interval $[A, B]$ and $a \in [A, B]$, we define a new function $F : [A, B] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(x)dx$$

Theorem 5.7 *For any function, f, integrable on $[A, B]$ and F,*

1. *On any interval where f is positive, F is increasing.*
2. $F(a) = 0$

Proof. □

5.1.2 Statement of the Theorem

Theorem 5.8 FIRST FUNDAMENTAL THEOREM OF CALCULUS, AN EXISTENCE THEOREM *Let f be integrable on $[a, b]$ and continuous on (a, b). For $x \in [a, b]$ define*

$$F(x) = \int_a^x f(t)dt.$$

Then the derivative of F exists for all $x \in (a, b)$ and

$$F'(x) = f(x).$$

editor: The conclusion holds for any point,x, where f is continuous. f need not be continuous on all of the open interval.

Proof. □

Theorem 5.9 SECOND FUNDAMENTAL THEOREM OF CALCULUS, A UNIQUENESS THEOREM *Let f be continuous on an open interval, I, let F be an antiderivative for f on I, that is $F'(x) = f(x)$ for all $x \in I$. Then, for $a \in I$ and each $x \in I$,*

$$F(x) = F(a) + \int_a^x f(t)dt.$$

editor: Well, unique up to a constant.

5.2 Computing integrals

5.3 Application: Logarithm and Exponential Functions

Definition We define a function, $\log : \mathbb{R}^+ \to \mathbb{R}$, using Riemann integral of $\frac{1}{x}$. That is,

$$\log(x) = \int_1^x \frac{1}{t} dt$$

NOTE: Because $\frac{1}{t}$ is not integrable on any interval, $(0, a)$, the log is not defined for negative values.

Common properties Υ of logarithms can be derived from properties of integrals.

Exercise 5.2 Prove the follow theorem:

Theorem 5.10 *1.* $\log(1) = 0$

2. For all $x, y \in \mathbb{R}^+$, $\log(xy) = \log(x) + \log(y)$

5.4 Flowchart