Theorem: Bolzano-Weierstrass

Every bounded sequence in \( \mathbb{R} \) has a convergent subsequence.

The proof presented here uses only the mathematics developmented by Apostol on pages 17-28 of the handout. We make particular use of

**AXIOM 10** Every nonempty set \( S \) of real numbers which is bounded above has a supremum (least upper bound). Every nonempty set \( S \) of real numbers which is bounded below has an infimum (greatest lower bound).

and

**THEOREM I.32** Let \( h \) be a given positive number and \( S \) be a set of real numbers.

If \( S \) has a supremum, then for some \( x \) in \( S \) we have \( x > \sup S - h \), and

If \( S \) has an infimum, then for some \( x \) in \( S \) we have \( x < \inf S + h \)

The idea of the proof of the Bolzano-Weierstrass Theorem is to let \( S \) be the set of all of the terms in the sequence. Since this set is bounded it must have a least upper bound, \( s \). We can then find a converging subsequence by choosing, for each \( n \), an element from \( S \) that satisfies **THEOREM I.32** with \( h = \frac{1}{n} \). We must be careful with each choice to pick a term that is further out the sequence than the last term we picked so that we truly get a subsequence. The trouble is that \( s \) may not be an accumulation point. It may be an element in \( S \) and the only element in \( S \) that satisfies the inequality. Even this would be OK if there were an infinite number of the terms in the sequence were equal to \( s \) – we could make a subsequence of nothing but \( s \)'s which would of course converge to \( s \). We are then left with the possibility that there are a finite number of \( s \)'s among the terms of the sequence. In the proof, we must cope with this possibility repeatedly.

**Proof of Theorem**

Let \( a_n \) be a bounded sequence and let \( S = \{a_n : n > 0\} \). Since the sequence is bounded, so is \( S \) and so \( S \) has a least upper bound, \( s_0 \), by **AXIOM 10**.
If \( s_0 \not\in S \), find a subsequence that converges to \( s_0 \) using I.32 for \( h = \frac{1}{n} \) for all positive integers \( n \).

If there are an infinite number of \( a_n \)'s that are equal to \( s_0 \), then the constant sequence, \( s_0 \), is a converging subsequence and we are done.

Otherwise, there are a finite number of \( a_n \)'s in \( S \) that are equal to \( s_0 \). Pick \( N_1 \) large enough so that \( s_0 \) is not in the set \( S_1 = \{ a_n : n > N_1 \} \). \( S_1 \) is bounded because it is a subset of the bounded set, \( S \). It has a least upper bound, \( s_1 \) and \( s_1 < s_0 \). As above, we can either find a subsequence that converges to \( s_1 \) or there are a finite number of the \( s_1 \)'s in \( S_1 \). In the first case, we are done, otherwise we continue by defining \( S_2 = \{ a_n : n > N_2 \} \) and it’s supremum, \( s_2 \), with \( s_2 < s_1 < s_0 \).

We continue inductively. At any stage, \( m \), we either find a converging subsequence, or we let \( S_m = \{ a_n : n > N_m \} \) and it’s supremum, \( s_m \), with \( s_m < s_{m-1} < \cdots < s_1 < s_0 \).

If we never find a supremum that is an accumulation point or a supremum that occurs an infinite number of times, we the process continues indefinitely. We are left with a decreasing subsequence, \( s_n \), of our original sequence. Being a subsequence of \( a_n \) it is bounded below and therefore has a greatest lower bound, \( t \). Because \( s_n \) is decreasing, \( t \) cannot be an element of the set \( T = \{ s_n : n > 1 \} \). (Otherwise it would be greater than the next term in the sequence and hence not a lower bound). So we proceed, via the second part of I.32 to find a subsequence that converges to \( t \). And then we are truly done.