Morgan problem #3, Chapter 1

Which of the following statements are true?

a. For all \( x \) there exists a \( y \) such that \( y > x^2 \). TRUE
   Proof: Given any value of \( x \), let \( y = x^2 + 1 \) so that \( y > x^2 \).

b. There exists a \( y \) such that for all \( x \), \( y > x^2 \). FALSE
   Proof: To show it is false, we must show that given any \( y \), there are values of \( x \) that make the inequality false, i.e. \( x^2 \geq y \). If \( y \) is any positive real number, let \( x = y + 1 \), then \( x^2 = y^2 + 2y + 1 > y \). (I have to fuss with the \( y + 1 \) to handle the case where \( y < 1 \).) On the other hand, if \( y \leq 0 \) then \( x^2 \geq y \) for any value of \( x \).

   Note that this is just another way of saying that \( \{ x^2 : x \in \mathbb{R} \} \) is unbounded.

c. There exists a \( y \) such that for all \( x \), \( y < x^2 \). TRUE
   Proof: We only have to give one value for \( y \) and then show that the inequality holds for all \( x \). Let \( y = -1 \) (or any negative number). Since \( x^2 \) is always positive, \( x^2 > y \).

d. For all \( a, b, c \), there exists an \( x \) such that \( ax^2 + bx + c = 0 \) FALSE
   Proof: We only need to show one instance of \( a, b \) and \( c \) and show that no values of \( x \) make the equality true. Let \( a = 1, b = 1 \) and \( c = 1 \). The equation \( x^2 + x + 1 = 0 \) has no real solutions because the discriminant (\( b^2 - 4ac \)) is negative.