You may use one page of handwritten notes. Staple that page to the exam when you have finished.

NAME:____________________________________

EXAMPLES (5 pts each)
Provide an example with a brief explanation or prove an example does not exist.

1. A boundary point that is not an accumulation point.
An isolated point of a set $S$ is a boundary point of $S$ but not an accumulation point.
PROOF: The only point in that is in $S$ and in a ball about an isolated point contains is the point itself so the point cannot be an accumulation point.

2. A function, $\mathbb{R} \rightarrow \mathbb{R}$, that is not continuous at every point.
EXAMPLES include
$f(x) = [x]$ discontinuous at all integers

$$f(x) = \begin{cases} 
1, & \text{if } x \text{ is rational} \\
0, & \text{if } s \text{ is irrational}
\end{cases} \text{ discontinuous at every point}$$

$$f(x) = \begin{cases} 
\frac{1}{x}, & \text{if } x \neq 0 \\
0, & \text{if } x = 0
\end{cases} \text{ discontinuous at } 0. \text{ You must define } f(0)$$

3. A union of closed sets that is not closed.
Examples must be an infinite union of closed sets: a finite union of closed sets is closed.

$$\bigcup_{n=1}^{\infty} \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] = (0, 1)$$

Let $s_n$ be an ordering of the rational numbers, then $\bigcup_{n=1}^{\infty} \{s_n\} = \mathbb{Q}$

4. A set in which every point is boundary point.
EXAMPLES include: $\mathbb{Z}$, any finite set of points.

5. A bounded sequence that does not have a convergent subsequence.
By Bolzano-Weierstrass, every bounded sequence has a convergent subsequence.

6. A sequence that converges to the real number 0.9.
EXAMPLES include: $s_n = 0.9$, a constant sequence, $s_n = 0.9 + \frac{1}{n}$, $s_n = \frac{9n}{10n + 1}$
TRUE OR FALSE (5 POINTS EACH)
Circle TRUE or FALSE.
Provide a counterexample or a brief explanation.
Provide a rigorous proof of one of the true statements.

1. TRUE OR FALSE Every bounded sequence converges.
   Examples included $s_n = 1 + (-1)^n$  $s_n = \sin(n)$

2. TRUE OR FALSE The empty set and $\mathbb{R}$ are the only two subsets of $\mathbb{R}$ that are both open and closed.

3. TRUE OR FALSE If the product of two functions is continuous, then both of the functions are continuous.
   Examples included
   
   $$f(x) = \begin{cases} 
   1, & \text{if } x \neq 0 \\
   \frac{1}{x}, & \text{if } x = 0 
   \end{cases}$$
   
   and
   
   $$g(x) = x$$

4. TRUE OR FALSE A closed set contains all of its accumulation points.
   By definition, a closed set contains all of it’s boundary points. If $p$ is an accumulation point of a closed set $S$, then every ball about $p$ contains points is $S\setminus\{p\}$ If $p$ is not in $S$, then $p$ is a boundary point – but $S$ contains all it’s boundary points. A contradiction so $p$ is in $S$. Hence, $S$ contains all of it’s boundary points.

5. For each one, circle T, TRUE OR F, FALSE.
   
   $\begin{align*}
   &T \quad F \quad \partial \mathbb{Z} = \mathbb{Z} \quad \text{T \quad F \quad} \partial \mathbb{Q} = \mathbb{Q}, \partial \mathbb{Q} = \mathbb{R} \\
   &T \quad F \quad \partial \mathbb{R} = \mathbb{R}, \partial \mathbb{R} = \emptyset \quad \text{T \quad F \quad} \partial [0,1] = [0,1], \partial [0,1] = \{0,1\}
   \end{align*}$

6. TRUE OR FALSE An accumulation point is either an interior point or a boundary point.
   Let $p$ be an accumulation point of a set $S$. Suppose it is not an interior point, then every open ball open $p$ contains points not in $S$ . Because $p$ is an accumulation point the open ball contains points in $S$. Hence $p$ is a boundary point.
PROVE ONE OF THE FOLLOWING. (20 POINTS)
Prove rigorously and directly form the definitions used in Morgan. Correct statements of the definitions should be included in the proofs. Circle the statement that you are proving.

1. A finite set has no accumulations points.

2. If \( f \) is continuous from \( \mathbb{R} \to \mathbb{R} \), use the \( \epsilon \)-\( \delta \) definition of continuous to show that if \( a_n \to a \), then \( f(a_n) \to f(a) \).

3. Using the \( \epsilon \)-definition of convergent sequences, prove:
   
   if \( a_n \to 0 \) and \( b_n \to b \), then \( a_n \cdot b_n \to 0 \)

Proof of 1: Let \( r \) be less than the minimum distance between all of the points. Since the set is finite, such a minimum exists. Then any ball of radius \( r \) about any point contains no other points in the set. Hence it is not an accumulation point.

Proof of 2:
Given \( \epsilon > 0 \),
Find \( \delta > 0 \) such that whenever \( |a - y| < \delta \), then \( |f(a) - f(y)| < \epsilon \). (Continuity of \( f \) at \( a \)).

Now find \( N \), such whenever \( n > N \), \( |a - a_n| < \delta \) (limit of \( a_n \))
Put it all together, whenever \( n > N \), \( |a - a_n| < \delta \) so then \( |f(a) - f(a_n)| < \epsilon \).
Hence \( f(a_n) \to f(a) \).

Proof of 3: Since \( b_n \) converges, it is bounded and we can find non-zero number \( B \) such that \( |b_n| < B \), for all \( n \).

Given \( \epsilon > 0 \), find \( N \) such that whenever \( n > N \), \( |a_n| < \frac{\epsilon}{B} \) by the existence of limit \( a \).

So now whenever \( n > N \), we have

\[
|a_n b_n - 0| \leq |a_n| |b_n|
\]

\[
< \frac{\epsilon}{B} \cdot B = \epsilon
\]

which shows that \( \lim_{n \to 0} a_n b_n = ab \)