

25. Consider the definite integral $\int_0^1 x^4 dx$.
- Write an expression for a right-hand Riemann sum approximation for this integral using n subdivisions. Express each x_i , $i = 1, 2, \dots, n$, in terms of i .
 - Use a computer algebra system to obtain a formula for the sum you wrote in part (a) in terms of n .
 - Take the limit of this expression for the sum as $n \rightarrow \infty$, thereby finding the exact value of this integral.
26. Repeat Problem 25, using the definite integral $\int_0^1 x^5 dx$.
27. Three terms of a left-hand sum used to approximate a definite integral $\int_a^b f(x) dx$ are as follows.

$$\left(2 + 0 \cdot \frac{4}{3}\right)^2 \cdot \frac{4}{3} + \left(2 + 1 \cdot \frac{4}{3}\right)^2 \cdot \frac{4}{3} + \left(2 + 2 \cdot \frac{4}{3}\right)^2 \cdot \frac{4}{3}.$$

Find possible values for a and b and a possible formula for $f(x)$.

28. Consider the integral $\int_1^2 \frac{1}{t} dt$. In Example 1, by dividing the interval $1 \leq t \leq 2$ into 10 equal parts, we showed that

$$0.1 \left[\frac{1}{1.1} + \frac{1}{1.2} + \dots + \frac{1}{2.0} \right] \leq \int_1^2 \frac{1}{t} dt \leq 0.1 \left[\frac{1}{1} + \frac{1}{1.1} + \dots + \frac{1}{1.9} \right].$$

- (a) Now divide the interval $1 \leq t \leq 2$ into n equal parts to show that

$$\sum_{r=1}^n \frac{1}{n+r} < \int_1^2 \frac{1}{t} dt < \sum_{r=0}^{n-1} \frac{1}{n+r}.$$

- (b) Show that the difference between the upper and lower sums in part (a) is $1/2n$.
- (c) The exact value of $\int_1^2 (1/t) dt$ is $\ln(2)$. How large should n be to approximate $\ln(2)$ with an error of at most $5 \cdot 10^{-6}$, using one of the sums in part (a)?

3.3 INTERPRETATIONS OF THE DEFINITE INTEGRAL

The Notation and Units for the Definite Integral

Just as the Leibniz notation dy/dx for the derivative reminds us that the derivative is the limit of a ratio of differences, the notation for the definite integral helps us recall the meaning of the integral. The symbol

$$\int_a^b f(x) dx$$

reminds us that an integral is a limit of sums (the integral sign is an old-fashioned S) of terms of the form “ $f(x)$ times a small difference of x .” Officially, dx is not a separate entity, but a part of the whole integral symbol. Just as one thinks of d/dx as a single symbol meaning “the derivative with respect to x of . . .,” one can think of $\int_a^b \dots dx$ as a single symbol meaning “the integral of . . . with respect to x .”

However, many scientists and mathematicians informally think of dx as an “infinitesimally” small bit of x which in this context is multiplied by a function value $f(x)$. This viewpoint is often the key to interpreting the meaning of a definite integral. For example, if $f(t)$ is the velocity of a moving particle at time t , then $f(t)dt$ may be thought of informally as velocity \times time, giving the distance traveled by the particle during a small bit of time dt . The integral $\int_a^b f(t) dt$ may then be thought of as the sum of all these small distances, giving us the net change in position of the particle between $t = a$ and $t = b$.

The notation for the integral also helps us determine what units should be used for the value of the integral. Since the terms being added up are products of the form “ $f(x)$ times a difference in x ,” the unit of measurement for $\int_a^b f(x) dx$ is the product of the units for $f(x)$ and the units for x . For example, if $f(t)$ is velocity measured in meters/second and t is time measured in seconds, then

$$\int_a^b f(t) dt$$

has units of (meters/sec) \times (sec) = meters. This is what we expect, since the value of this integral represents change in position.

As another example, graph $y = f(x)$ with the same units of measurement of length along the x - and y -axes, say cm. Then $f(x)$ and x are measured in the same units, so

$$\int_a^b f(x) dx$$

is measured in square units of $\text{cm} \times \text{cm} = \text{cm}^2$. Again, this is what we would expect since in this context the integral represents an area.

The Definite Integral as an Average

We know how to find the average of n numbers: Add them and divide by n . But how do we find the average value of a continuously varying function? Let us consider an example. Suppose $f(t)$ is the temperature at time t , measured in hours since midnight, and that we want to calculate the average temperature over a 24-hour period. One way to start would be to average the temperatures at n equally spaced times, t_1, t_2, \dots, t_n , during the day.

$$\text{Average temperature} \approx \frac{f(t_1) + f(t_2) + \dots + f(t_n)}{n}$$

The larger we make n , the better the approximation. We can rewrite this expression as a Riemann sum over the interval $0 \leq t \leq 24$ if we use the fact that $\Delta t = 24/n$, so $n = 24/\Delta t$:

$$\begin{aligned} \text{Average temperature} &\approx \frac{f(t_1) + f(t_2) + \dots + f(t_n)}{24/\Delta t} \\ &= \frac{f(t_1)\Delta t + f(t_2)\Delta t + \dots + f(t_n)\Delta t}{24} \\ &= \frac{1}{24} \sum_{i=1}^n f(t_i)\Delta t. \end{aligned}$$

As $n \rightarrow \infty$, the Riemann sum tends towards an integral, and $1/24$ of the sum also approximates the average temperature better. It makes sense, then, to write

$$\begin{aligned} \text{Average temperature} &= \lim_{n \rightarrow \infty} \frac{1}{24} \sum_{i=1}^n f(t_i)\Delta t \\ &= \frac{1}{24} \int_0^{24} f(t) dt. \end{aligned}$$

We have found a way of expressing the average temperature over an interval in terms of an integral. Generalizing for any function f , if $a < b$, we define

$$\begin{array}{l} \text{Average value of } f \\ \text{from } a \text{ to } b \end{array} = \frac{1}{b-a} \int_a^b f(x) dx$$

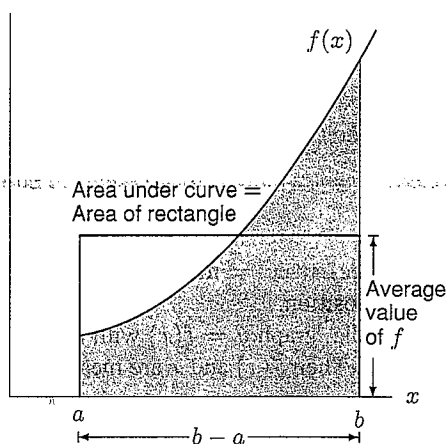


Figure 3.21: Area and average value

How to Visualize the Average on a Graph

The definition of average value tells us that

$$(\text{Average value of } f) \cdot (b - a) = \int_a^b f(x) dx.$$

Let's interpret the integral as the area under the graph of f . Then the average value of f is the height of a rectangle whose base is $(b - a)$ and whose area is the same as the area under the graph of f . (See Figure 3.21.)

Example 1 Suppose that $C(t)$ represents the daily cost of heating your house, measured in dollars per day, where t is time measured in days and $t = 0$ corresponds to January 1, 1997. Interpret $\int_0^{90} C(t) dt$ and $\frac{1}{90-0} \int_0^{90} C(t) dt$.

Solution The units for the integral $\int_0^{90} C(t) dt$ are (dollars/day) \times (days) = dollars. The integral represents the total cost in dollars to heat your house for the first 90 days of 1997, namely the months of January, February, and March. The second expression is measured in (1/days)(dollars) or dollars per day, the same units as $C(t)$. It represents the average cost per day to heat your house during the first 90 days of 1997.

Example 2 On page 14, we saw that the population of Mexico could be modeled by the function

$$P = f(t) = 67.38(1.026)^t,$$

where P is in millions of people and t is in years since 1980. Use this function to predict the average population of Mexico between the years 2000 and 2020.

Solution We want the average value of $f(t)$ between $t = 20$ and $t = 40$. This is given by

$$\text{Average population} = \frac{1}{40-20} \int_{20}^{40} f(t) dt \approx \frac{1}{20}(2942.66) = 147.1.$$

We used a calculator to evaluate the integral. We see that the average population of Mexico between 2000 and 2020 is predicted to be about 147 million people.

Applications of the Definite Integral

The total distance traveled by a moving object in a given time interval may be represented by a definite integral of the velocity. The following examples show how representing a quantity as a definite integral, and thereby as an area, can be helpful even if we don't evaluate the integral.

Example 3 Two cars start from rest at a traffic light and accelerate for several minutes. Figure 3.22 shows their velocities as a function of time. (a) Which car is ahead after one minute? (b) Which car is ahead after two minutes?

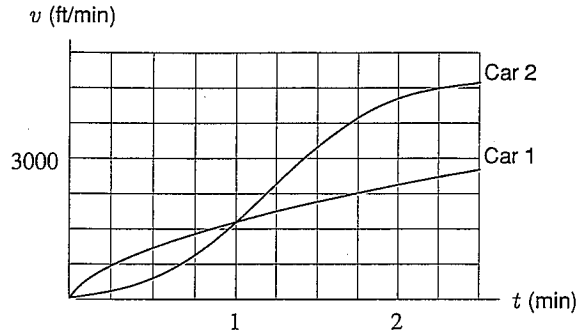


Figure 3.22: Velocities of two cars. Which is ahead when?

- Solution**
- (a) For the first minute car 1 goes faster than car 2, and therefore car 1 must be ahead at the end of one minute.
- (b) At the end of two minutes the situation is less clear, since car 1 was going faster for the first minute and car 2 for the second. However, if $v = f(t)$ is the velocity of a car after t minutes, then we know that

$$\text{Distance traveled in two minutes} = \int_0^2 f(t) dt,$$

since the integral of velocity is distance traveled. This definite integral may also be interpreted as the area under the graph of f between 0 and 2. Since the area representing the distance traveled by car 2 is clearly larger than the area for car 1 (see Figure 3.22), we know that car 2 has traveled farther than car 1.

Example 4 A car starts at noon and travels with the velocity shown in Figure 3.23. A truck starts at 1 pm from the same place and travels at a constant velocity of 50 mph.

- (a) How far away is the car when the truck starts?
- (b) During the period when the car is ahead of the truck, when is the distance between them greatest, and what is that greatest distance?
- (c) When does the truck overtake the car, and how far have both traveled then?

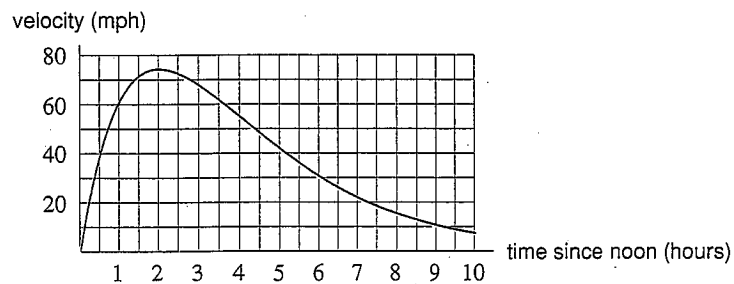


Figure 3.23: Velocity of car for Example 4

Solution To find distances from the velocity graph, we use the fact that if t is the time measured from noon, and v is the velocity, then

$$\begin{array}{l} \text{Distance traveled} \\ \text{by car up to time } T \end{array} = \int_0^T v \, dt = \begin{array}{l} \text{Area under velocity} \\ \text{graph between 0 and } T. \end{array}$$

The truck's motion can be represented on the same graph by the horizontal line $v = 50$, starting at $t = 1$. The distance traveled by the truck is then the rectangular area under this line, and the distance between the two vehicles is the difference between these areas. Note each small rectangle on the graph corresponds to moving at 10 mph for a half hour (i.e., to a distance of 5 miles).

- (a) The distance traveled by the car when the truck starts is represented by the shaded area in Figure 3.24, which totals about seven rectangles or about 35 miles.

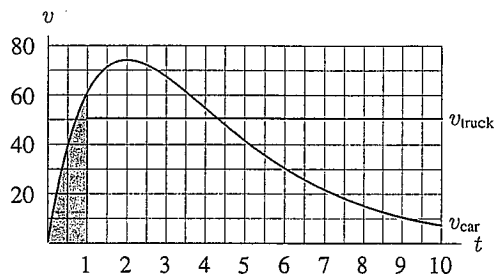


Figure 3.24: Shaded area represents distance traveled by car from noon to 1pm

- (b) The car starts ahead of the truck, and the distance between them increases as long as the velocity of the car is greater than the velocity of the truck. Later, when the truck's velocity exceeds the car's, the truck starts to gain on the car. In other words, the distance between the car and the truck will increase as long as $v_{car} > v_{truck}$, and it will decrease when $v_{car} < v_{truck}$. Therefore, the maximum distance occurs when $v_{car} = v_{truck}$, that is, when $t \approx 4.3$ hours (at about 4:20 pm). (See Figure 3.25.) The distance traveled by the car is the area under the v_{car} graph between $t = 0$ and $t = 4.3$; the distance traveled by the truck is the area under the v_{truck} line between $t = 1$ (when it started) and $t = 4.3$. So the distance between the car and truck is represented by the shaded area in Figure 3.25, which is approximately

$$35 \text{ miles} + 50 \text{ miles} = 85 \text{ miles.}$$

- (c) The truck overtakes the car when both have traveled the same distance. This occurs when the area under the curve (v_{car}) up to that time equals the area under the line (v_{truck}) up to that time. Since the areas under v_{car} and v_{truck} overlap (see Figure 3.26), they are equal when the lightly shaded area equals the heavily shaded area (which we know is about 85 miles). This happens when $t \approx 8.3$ hours, or about 8:20 pm.

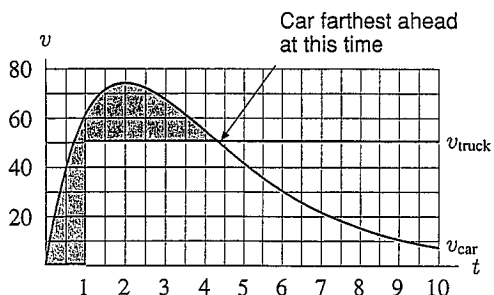


Figure 3.25: Shaded area = distance by which car is ahead at about 4:20 pm

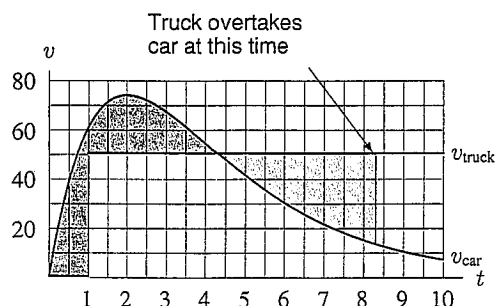


Figure 3.26: Truck overtakes car when dark and light shaded areas are equal

Problems for Section 3.3

- For the two cars in Example 3, page 163, estimate:
 - The distances moved by car 1 and car 2 during the first minute.
 - The time at which the two cars have gone the same distance.
- Consider the car and the truck in Example 4, page 163.
 - How fast is the distance between the car and the truck increasing or decreasing at 3 pm?
 - What is the practical significance (in terms of the distance between the car and the truck) of the fact that the car's velocity is maximized at about 2 pm?
- Consider the car and the truck in Example 4, page 163, but suppose the truck starts at noon. (Everything else remains the same.)
 - Sketch a new graph showing the velocities of both car and truck against time.
 - How many times do the two graphs intersect? What does each intersection mean in terms of the distance between the two?
- If $f(t)$ is measured in meters/second² and t is measured in seconds, what are the units of $\int_a^b f(t) dt$?
- If $f(t)$ is measured in dollars per year and t is measured in years, what are the units of $\int_a^b f(t) dt$?
- If $f(x)$ is measured in pounds and x is measured in feet, what are the units of $\int_a^b f(x) dx$?
- Oil is leaking out of a ruptured tanker at a rate of $r = f(t)$ gallons per minute, where t is in minutes. Write a definite integral expressing the total quantity of oil which leaks out of the tanker in the first hour.

In Problems 8–10, find the average value of the function over the given interval.

8. $g(t) = 1 + t$ over $[0, 2]$ 9. $f(x) = 4x + 7$ over $[1, 3]$ 10. $g(t) = e^t$ over $[0, 10]$

11. If the graph of f is in Figure 3.27:

- (a) What is $\int_1^6 f(x) dx$? (b) What is the average value of f on $[1, 6]$?

12. A two-day environmental clean up operation started at 9 am on the first day. The number of workers fluctuated as shown in Figure 3.28. If the workers were paid \$10 per hour, how much was the total personnel cost of the operation?

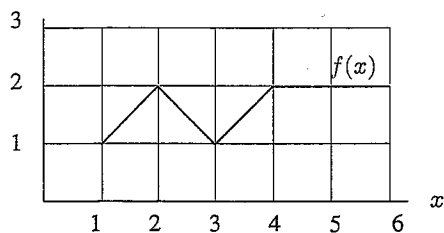


Figure 3.27

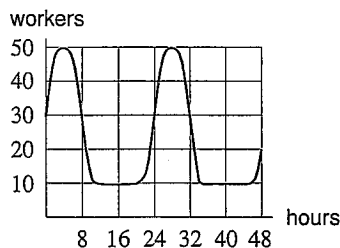


Figure 3.28

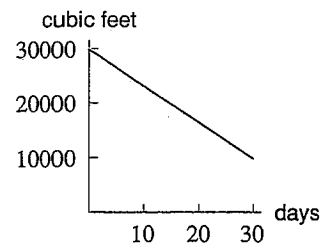


Figure 3.29

13. Suppose in Problem 12 that the workers were paid \$10 per hour for work during the time period 9 am to 5 pm and were paid \$15 per hour for work during the rest of the day. What would the total personnel costs of the operation have been under these conditions?
14. A warehouse charges its customers \$5 per day for every 10 cubic feet of space used for storage. Figure 3.29 records the storage used by one company over a month. How much will the company have to pay?

15. If $f(x) = 2$, show that the average value of $f(x)$ over the interval $[a, b]$ is 2.
16. (a) Without computing any integrals, explain why the average value of $f(x) = \sin x$ on $[0, \pi]$ must be between 0.5 and 1.
 (b) Compute this average. Give 2 decimal places in your answer.
17. (a) What is the average value of $f(x) = \sqrt{1-x^2}$ over the interval $0 \leq x \leq 1$?
 (b) How can you tell whether this average value is more or less than 0.5 without doing any calculations?
18. How do the units for the average value relate to the units for $f(x)$ and the units for x ?
19. A bar of metal is cooling from 1000°C to room temperature, 20°C . The temperature, H , of the bar t minutes after it starts cooling is given, in $^\circ\text{C}$, by

$$H = 20 + 980e^{-0.1t}.$$

- (a) Find the temperature of the bar at the end of one hour.
 (b) Find the average value of the temperature over the first hour.
 (c) Is your answer to part (b) greater or smaller than the average of the temperatures at the beginning and the end of the hour? Explain this in terms of the concavity of the graph of H .
20. The value, V , of a Tiffany lamp, worth \$225 in 1965, increases at 15% per year. Its value in dollars t years after 1965 is given by

$$V = 225(1.15)^t.$$

Find the average value of the lamp over the period 1965–2000.

21. The number of hours, H , of daylight in Madrid as a function of date is approximated by the formula

$$H = 12 + 2.4 \sin[0.0172(t - 80)],$$

where t is the number of days since the start of the year. Find the average number of hours of daylight in Madrid:

- (a) in January (b) in June (c) over a whole year
 (d) Comment on the relative magnitudes of your answers to parts (a), (b), and (c). Why are they reasonable?
22. A bicyclist is pedaling along a straight road with velocity, v , given in Figure 3.30. Suppose the cyclist starts 5 miles from a lake, and that positive velocities take her away from the lake and negative velocities towards the lake. When is the cyclist farthest from the lake, and how far away is she then?
23. A force F parallel to the x -axis is given by the graph in Figure 3.31. Estimate the work, W , done by the force, where $W = \int_0^{16} F(x) dx$.

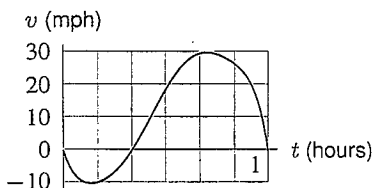


Figure 3.30

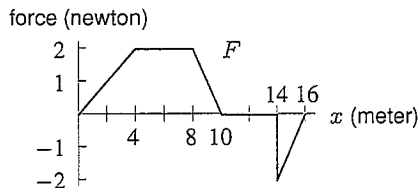


Figure 3.31

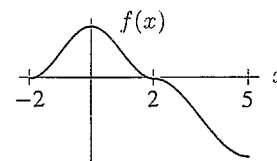


Figure 3.32

24. For the even function f in Figure 3.32, write an expression involving one or more definite integrals that denotes:
 (a) The average value of f for $0 \leq x \leq 5$. (b) The average value of $|f|$ for $0 \leq x \leq 5$.
25. For the even function f in Figure 3.32, consider the average value of f over the following intervals:
 (I) $0 \leq x \leq 1$ (II) $0 \leq x \leq 2$ (III) $0 \leq x \leq 5$ (IV) $-2 \leq x \leq 2$
 (a) For which interval is the average value of f least?
 (b) For which interval is the average value of f greatest?
 (c) For which pair of intervals are the average values equal?

3.4 THEOREMS ABOUT DEFINITE INTEGRALS

The Definite Integral of a Rate Gives Total Change

We have seen how the definite integral of a velocity function can be interpreted as total distance traveled. If $v(t)$ is velocity and $s(t)$ is position, then $v(t) = s'(t)$ and we know that

$$\text{Total change in position} = s(b) - s(a) = \int_a^b s'(t) dt.$$

In this section we generalize this result to explain why the integral of the rate of change of any quantity gives the total change in that quantity.

Suppose $F'(t)$ is the rate of change of some quantity $F(t)$ with respect to time, and that we are interested in the total change in $F(t)$ between $t = a$ and $t = b$. We divide the interval $a \leq t \leq b$ into n subintervals, each of length Δt . For each small interval, we estimate the change in $F(t)$, written ΔF , and then add all these up. In each subinterval we assume the rate of change of $F(t)$ is approximately constant, so that we can say

$$\Delta F \approx \text{Rate of change of } F \times \text{Time elapsed.}$$

For the first subinterval, from t_0 to t_1 , the rate of change of $F(t)$ is approximately $F'(t_0)$, so

$$\Delta F \approx F'(t_0) \Delta t.$$

Similarly, for the second interval

$$\Delta F \approx F'(t_1) \Delta t.$$

Summing over all the subintervals, we get

$$\text{Total change in } F = \sum_{i=0}^{n-1} \Delta F \approx \sum_{i=0}^{n-1} F'(t_i) \Delta t.$$

We have approximated the change in $F(t)$ as a left-hand sum.

However, the total change in $F(t)$ between the times $t = a$ and $t = b$ is simply $F(b) - F(a)$. Taking the limit as n goes to infinity suggests the following result:¹

$$F(b) - F(a) = \begin{array}{l} \text{Total change in } F(t) \\ \text{between } t = a \text{ and } t = b \end{array} = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} F'(t_i) \Delta t = \int_a^b F'(t) dt.$$

This result is one of the most important in calculus because it makes the connection between the derivative and the definite integral. It is called the Fundamental Theorem of Calculus and is often stated as follows:

The Fundamental Theorem of Calculus

If f is continuous on the interval $[a, b]$ and $f(t) = F'(t)$, then

$$\int_a^b f(t) dt = F(b) - F(a).$$

In words:

The definite integral of a rate of change gives total change.

¹We could equally well have used a right-hand sum, since the definite integral is their common limit.

What Is Involved in Proving the Fundamental Theorem?

The argument we have given makes the Fundamental Theorem plausible. However, it is not a mathematical proof. Each of the approximations

$$\Delta F \approx F'(t_i) \Delta t, \quad i = 1, \dots, n,$$

involves a small error. To prove that the total change in F is well-approximated by the Riemann sum $\sum_{i=0}^{n-1} F'(t_i) \Delta t$, we must show that the sum of all the errors is small (in fact, that it can be made as small as we like by choosing n large enough). We will not give the details of this argument. However, the key idea brings out an important fact about local linearization: the error in the local linearization $\Delta F \approx F'(t_i) \Delta t$ is not merely small, it is small relative to the size of Δt . This follows from the definition of the derivative: if we choose Δt small enough, we can ensure that the error in the approximation

$$F'(t_i) \approx \frac{\Delta F}{\Delta t}$$

is as small as we like. In other words,

$$F'(t_i) = \frac{\Delta F}{\Delta t} + \epsilon,$$

where ϵ can be made as small as we like. Multiplying through by Δt , we get an approximation

$$F'(t_i) \Delta t = \Delta F + \epsilon \Delta t.$$

Thus, in the approximation $F'(t_i) \Delta t \approx \Delta F$, the error is $\epsilon \Delta t$, which is small even relative to Δt . When we add these small errors, we still get a small total error. For a proof of the Fundamental Theorem from another point of view, see Problem 14 on page 291.

Using the Fundamental Theorem

The Fundamental Theorem provides a precise way of computing certain definite integrals.

Example 1 Compute $\int_1^3 2x \, dx$ by two different methods.

Solution Using left- and right-hand sums, we can approximate this integral as accurately as we want. With $n = 100$, for example, the left-sum is 7.96 and the right sum is 8.04. Using $n = 500$ we learn

$$7.992 < \int_1^3 2x \, dx < 8.008.$$

The Fundamental Theorem, on the other hand, allows us to compute the integral exactly. We take $f(x) = 2x$. By Example 4, on page 108, we know that if $F(x) = x^2$, then $F'(x) = 2x$. So we use $f(x) = 2x$ and $F(x) = x^2$ and obtain

$$\int_1^3 2x \, dx = F(3) - F(1) = 3^2 - 1^2 = 8.$$

The Fundamental Theorem can also be used when the rate, $F'(t)$, is known and we want to find the total change $F(b) - F(a)$. If we also know $F(a)$, the theorem enables us to reconstruct the function F from knowledge about its derivative $F' = f$.

Example 2 Let $F(t)$ represent a bacteria population which is 5 million at time $t = 0$. Suppose that after t hours, the population is growing at an instantaneous rate of 2^t million bacteria per hour. Estimate the total increase in the bacteria population during the first hour, and the population at $t = 1$.

Solution Since the rate at which the population is growing is $F'(t) = 2^t$, we have

$$\text{Change in population} = F(1) - F(0) = \int_0^1 2^t \, dt.$$

Using a calculator gives

$$\text{Change in population} = \int_0^1 2^t dt \approx 1.44 \text{ million bacteria.}$$

Since $F(0) = 5$, the population at $t = 1$ is given by

$$\text{Population} = F(1) = F(0) + \int_0^1 2^t dt \approx 5 + 1.44 = 6.44 \text{ million.}$$

Properties of the Definite Integral

For the definite integral $\int_a^b f(x) dx$, we have so far only considered the case $a < b$. We now allow $a \geq b$. We still set $x_0 = a$, $x_n = b$, and $\Delta x = (b - a)/n$. As before, we have $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$.

Theorem: Properties of Limits of Integration

If a , b , and c are any numbers and f is a continuous function, then

1. $\int_b^a f(x) dx = - \int_a^b f(x) dx$.
2. $\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$.

In words:

1. The integral from b to a is the negative of the integral from a to b .
2. The integral from a to c plus the integral from c to b is the integral from a to b .

By interpreting the integrals as areas, we can justify these results for $f \geq 0$. In fact, they are true for all functions for which the integrals make sense.

Why is $\int_b^a f(x) dx = - \int_a^b f(x) dx$?

By definition, both integrals are approximated by sums of the form $\sum f(x_i) \Delta x$. The x_i s are the same in each case: the only difference in the sums for $\int_b^a f(x) dx$ and $\int_a^b f(x) dx$ is that in the first $\Delta x = (a - b)/n = -(b - a)/n$ and in the second $\Delta x = (b - a)/n$. Since everything else about the sums is the same, we must have $\int_b^a f(x) dx = - \int_a^b f(x) dx$.

Why is $\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$?

Suppose $a < c < b$. Figure 3.33 suggests that $\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$ since the area under f from a to c plus the area under f from c to b together make up the whole area under f from a to b .

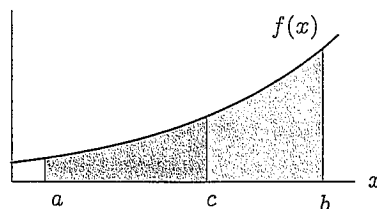


Figure 3.33: Additivity of the definite integral ($a < c < b$)

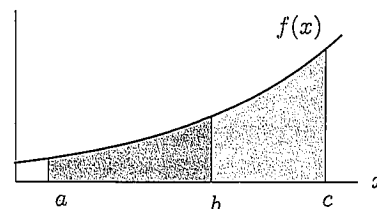


Figure 3.34: Additivity of the definite integral ($a < b < c$)

Actually, this property holds for all numbers a , b , and c , not just those satisfying $a < c < b$. (See Figure 3.34.) For example, the area under f from 3 to 6 is equal to the area from 3 to 8 *minus* the area from 6 to 8, so

$$\int_3^6 f(x) dx = \int_3^8 f(x) dx - \int_6^8 f(x) dx = \int_3^8 f(x) dx + \int_8^6 f(x) dx.$$

Example 3 Suppose you know that $\int_0^{1.25} \cos(x^2) dx = 0.98$ and $\int_0^1 \cos(x^2) dx = 0.90$. (See Figure 3.35.) What are the values of the following integrals?

(a) $\int_1^{1.25} \cos(x^2) dx$ (b) $\int_{-1}^1 \cos(x^2) dx$ (c) $\int_{1.25}^{-1} \cos(x^2) dx$

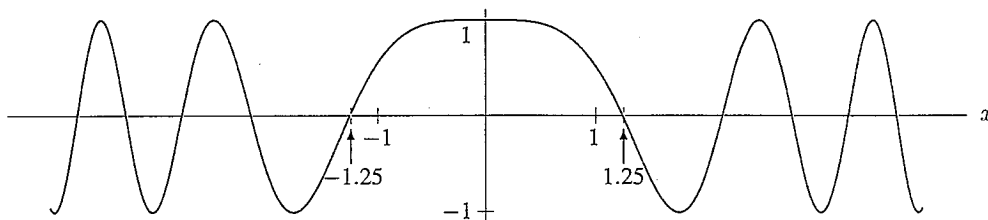


Figure 3.35: Graph of $f(x) = \cos(x^2)$

- Solution** (a) Since $\int_0^{1.25} \cos(x^2) dx = \int_0^1 \cos(x^2) dx + \int_1^{1.25} \cos(x^2) dx$ by the additivity property, we get $0.98 = 0.90 + \int_1^{1.25} \cos(x^2) dx$, so $\int_1^{1.25} \cos(x^2) dx = 0.08$.
- (b) $\int_{-1}^1 \cos(x^2) dx = \int_{-1}^0 \cos(x^2) dx + \int_0^1 \cos(x^2) dx$.
By the symmetry of $\cos(x^2)$ about the y -axis, $\int_{-1}^0 \cos(x^2) dx = \int_0^1 \cos(x^2) dx = 0.90$.
So $\int_{-1}^1 \cos(x^2) dx = 0.90 + 0.90 = 1.80$.
- (c) $\int_{1.25}^{-1} \cos(x^2) dx = -\int_{-1}^{1.25} \cos(x^2) dx = -(\int_{-1}^0 \cos(x^2) dx + \int_0^{1.25} \cos(x^2) dx) = -(0.90 + 0.98) = -1.88$.

Theorem: Properties of Sums and Constant Multiples of the Integrand

Let f and g be continuous functions and let c be a constant.

- $\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx.$
- $\int_a^b cf(x) dx = c \int_a^b f(x) dx.$

In words:

- The integral of the sum (or difference) of two functions is the sum (or difference) of their integrals.
- The integral of a constant times a function is that constant times the integral of the function.

Why Do these Properties Hold?

Both can be visualized by thinking of the definition of the definite integral as the limit of a sum of areas of rectangles.

For property 1, let's suppose that f and g are positive on the interval $[a, b]$ so that the area under $f(x) + g(x)$ is approximated by the sum of the areas of rectangles like the one shaded in Figure 3.36. The area of this rectangle is

$$[f(x_i) + g(x_i)]\Delta x = f(x_i)\Delta x + g(x_i)\Delta x.$$

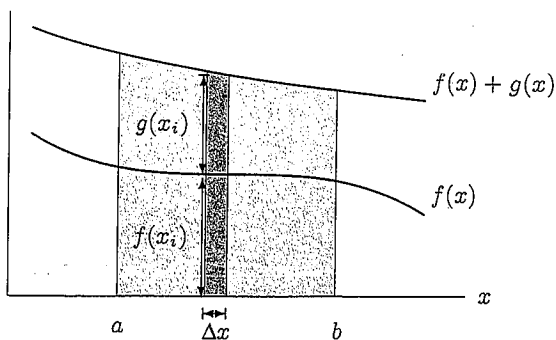


Figure 3.36: Area = $\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$

Since $f(x_i)\Delta x$ is the area of a rectangle under the graph of f , and $g(x_i)\Delta x$ is the area of a rectangle under the graph of g , the area under $f(x) + g(x)$ is the sum of the areas under $f(x)$ and $g(x)$.

For property 2, notice that multiplying a function by c stretches or flattens the graph in the vertical direction by a factor of c . Thus, it stretches or flattens the height of each approximating rectangle by c , and hence multiplies the area by c .

Example 4 Evaluate the definite integral $\int_0^2 (1 + 3x) dx$ exactly.

Solution We can break this integral up as follows:

$$\int_0^2 (1 + 3x) dx = \int_0^2 1 dx + \int_0^2 3x dx = \int_0^2 1 dx + 3 \int_0^2 x dx.$$

This expresses our original integral in terms of two simpler integrals. From the area interpretation of the integral, we see that

$$\int_0^2 1 dx = 2,$$

since it represents the area under the horizontal line $y = 1$ between $x = 0$ and $x = 2$ (see Figure 3.37). Similarly,

$$\int_0^2 x dx = \frac{1}{2} \cdot 2 \cdot 2 = 2$$

because it is the area of the triangle in Figure 3.38. Therefore,

$$\int_0^2 (1 + 3x) dx = \int_0^2 1 dx + 3 \int_0^2 x dx = 2 + 3(2) = 8.$$

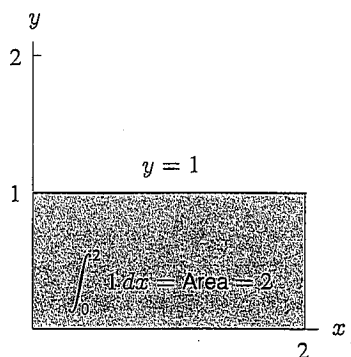


Figure 3.37: Area representing $\int_0^2 1 dx$

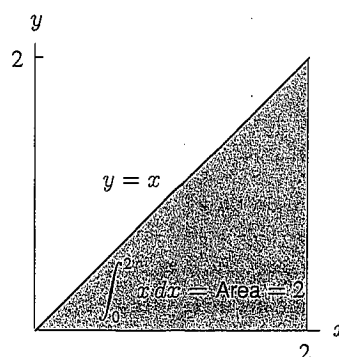


Figure 3.38: Area representing $\int_0^2 x dx$

Comparing Integrals

Suppose we have constants m and M such that $m \leq f(x) \leq M$ for $a \leq x \leq b$. We say f is *bounded above* by M and *bounded below* by m . Then the graph of f lies between the horizontal lines $y = m$ and $y = M$. So the definite integral lies between $m(b - a)$ and $M(b - a)$. See Figure 3.39.

Suppose $f(x) \leq g(x)$ for $a \leq x \leq b$, as in Figure 3.40. By a similar argument, the definite integral of f is less than or equal to the definite integral of g . This leads us to the following results:

Theorem: Results About Comparison of Definite Integrals

Let f and g be continuous functions.

1. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b - a) \leq \int_a^b f(x) dx \leq M(b - a)$.
2. If $f(x) \leq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x) dx \leq \int_a^b g(x) dx$.

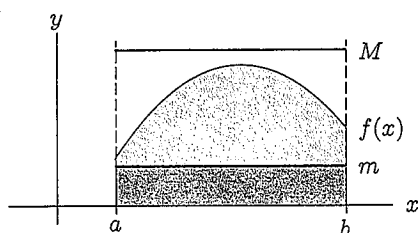


Figure 3.39: The area under the graph of f lies between the areas of the rectangles

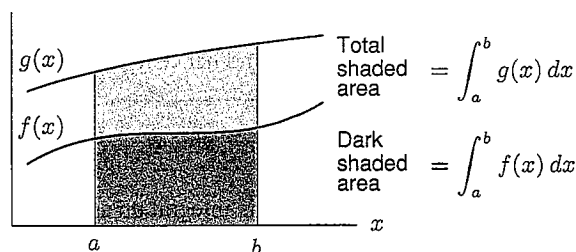


Figure 3.40: If $f(x) \leq g(x)$ then $\int_a^b f(x) dx \leq \int_a^b g(x) dx$

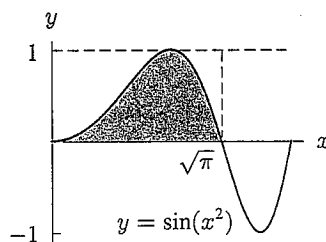


Figure 3.41: Graph showing that $\int_0^{\sqrt{\pi}} \sin(x^2) dx < \sqrt{\pi}$

Example 5 Explain why $\int_0^{\sqrt{\pi}} \sin(x^2) dx \leq \sqrt{\pi}$.

Solution Since $\sin(x^2) \leq 1$ for all x (see Figure 3.41), part 2 of the theorem gives

$$\int_0^{\sqrt{\pi}} \sin(x^2) dx \leq \int_0^{\sqrt{\pi}} 1 dx = \sqrt{\pi}.$$

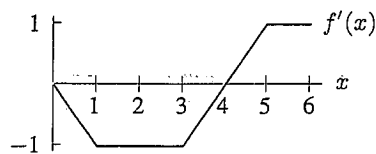
Example 6 Show that $2 \leq \int_0^2 \sqrt{1+x^3} dx \leq 6$.

Solution Notice that $f(x) = \sqrt{1+x^3}$ is increasing for $0 \leq x \leq 2$, since x^3 gets bigger as x increases. This means that $f(0) \leq f(x) \leq f(2)$. For this function, $f(0) = 1$ and $f(2) = 3$. Thus we can imagine the area under $f(x)$ as lying between the area under the line $y = 1$ and the area under the line $y = 3$ on the interval $0 \leq x \leq 2$. That is,

$$1(2 - 0) \leq \int_0^2 \sqrt{1+x^3} dx \leq 3(2 - 0).$$

Problems for Section 3.4

1. The graph of a derivative $f'(x)$ is shown in Figure 3.42. Fill in the table of values for $f(x)$ given that $f(0) = 2$.



x	0	1	2	3	4	5	6
$f(x)$	2						

Figure 3.42: Graph of f' , not f

2. Water is leaking out of a tank at a rate of $R(t)$ gallons/hour, where t is measured in hours.
- Write a definite integral that expresses the total amount of water that leaks out in the first two hours.
 - Figure 3.43 is a graph of $R(t)$. On a sketch, shade in the region whose area represents the total amount of water that leaks out in the first two hours.
 - Give an upper and lower estimate of the total amount of water that leaks out in the first two hours.
3. Figure 3.44 shows the graph of f . If $F' = f$ and $F(0) = 0$, find $F(b)$ for $b = 1, 2, 3, 4, 5, 6$.

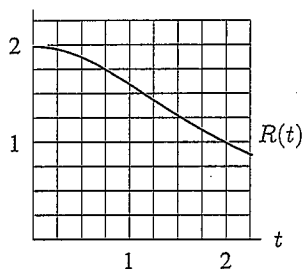


Figure 3.43

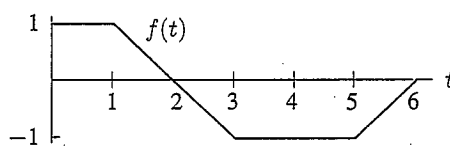
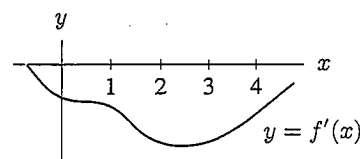


Figure 3.44

Figure 3.45: Graph of f' , not f

Problems 4–5 concern the graph of f' in Figure 3.45.

- Which is greater, $f(0)$ or $f(1)$?
- List the following in increasing order: $\frac{f(4) - f(2)}{2}$, $f(3) - f(2)$, $f(4) - f(3)$.
- A news broadcast in early 1993 said the average American's annual income is changing at a rate of $r(t) = 40(1.002)^t$ dollars per month, where t is in months from January 1, 1993. How much did the average American's income change during 1993?
- A cup of coffee at 90°C is put into a 20°C room when $t = 0$. If the coffee's temperature is changing at a rate given in $^\circ\text{C}$ per minute by

$$r(t) = -7e^{-0.1t}, \quad t \text{ in minutes,}$$

estimate, to one decimal place, the coffee's temperature when $t = 10$.

- The rate at which the world's oil is being consumed is continuously increasing. Suppose the rate (in billions of barrels per year) is given by the function $r = f(t)$, where t is measured in years and $t = 0$ is the start of 1990.
 - Write a definite integral which represents the total quantity of oil used between the start of 1990 and the start of 1995.
 - Suppose $r = 32e^{0.05t}$. Using a left-hand sum with five subdivisions, find an approximate value for the total quantity of oil used between the start of 1990 and the start of 1995.
 - Interpret each of the five terms in the sum from part (b) in terms of oil consumption.

9. The graph of a function $y = f(x)$ is given in Figure 3.46. Suppose $f(x)$ is the rate (in thousands of algae per hour) at which a population of algae is growing, where x is in hours.
- Estimate the average value of the rate of growth of this population over the interval $x = -1$ to $x = 3$. Explain how you arrived at your answer.
 - Estimate the total change in the algae population over the interval $x = -3$ to $x = 3$.

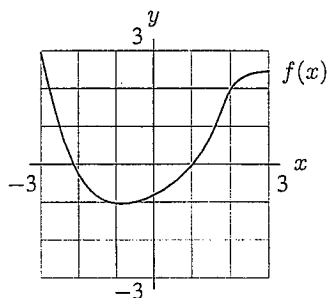


Figure 3.46

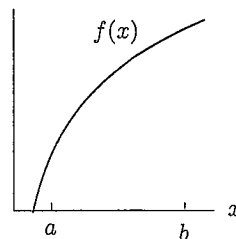


Figure 3.47

For Problems 10–13, mark the following quantities on a copy of the graph of f in Figure 3.47.

- A length representing $f(b) - f(a)$.
- A slope representing $\frac{f(b) - f(a)}{b - a}$.
- An area representing $F(b) - F(a)$, where $F' = f$.
- A length roughly approximating

$$\frac{F(b) - F(a)}{b - a}, \text{ where } F' = f.$$

Suppose $\int_a^b f(x) dx = 8$, $\int_a^b (f(x))^2 dx = 12$, $\int_a^b g(t) dt = 2$, and $\int_a^b (g(t))^2 dt = 3$. Find the integrals in Problems 14–19.

- $\int_a^b (f(x) + g(x)) dx$
- $\int_a^b ((f(x))^2 - (g(x))^2) dx$
- $\int_a^b (f(x))^2 dx - (\int_a^b f(x) dx)^2$
- $\int_a^b cf(z) dz$
- $\int_a^b (c_1g(x) + (c_2f(x))^2) dx$
- $\int_{a+5}^{b+5} f(x - 5) dx$
- The function for the *standard normal distribution*, which is often used in statistics, has the formula

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

and the graph in Figure 3.48. Statistics books usually contain tables such as the one below, showing only the area under the curve from 0 to b , for different values of b .

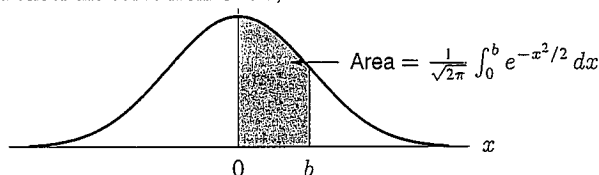


Figure 3.48

b	$\frac{1}{\sqrt{2\pi}} \int_0^b e^{-x^2/2} dx$
1	0.3413
2	0.4772
3	0.4987
4	0.5000

Use the information given in the table and the symmetry of the standard normal curve about the y -axis to find:

- $\frac{1}{\sqrt{2\pi}} \int_1^3 e^{-x^2/2} dx.$
- $\frac{1}{\sqrt{2\pi}} \int_{-2}^3 e^{-x^2/2} dx.$

21. (a) Is $\int_{-1}^1 e^{x^2} dx$ positive, negative, or zero? Explain.
 (b) Explain why $0 < \int_0^1 e^{x^2} dx < 3$.
22. Without calculating the integral, explain why the following statements are false.
 (a) $\int_{-2}^{-1} e^{x^2} dx = -3$ (b) $\int_{-1}^1 \left| \frac{\cos(x+2)}{1+\tan^2 x} \right| dx = 0$
23. Without doing any computations, find the values of
 (a) $\int_{-2}^2 \sin x dx$, (b) $\int_{-\pi}^{\pi} x^{113} dx$.
24. Use the property $\int_b^a f(x) dx = -\int_a^b f(x) dx$ to show that $\int_a^a f(x) dx = 0$.
25. The average value of $y = v(x)$ equals 4 for $1 \leq x \leq 6$, and equals 5 for $6 \leq x \leq 8$. What is the average value of $v(x)$ for $1 \leq x \leq 8$?

CHAPTER SUMMARY

- **Definite integral as limit of right-hand or left-hand sums**
- **Interpretations of the definite integral**
 Total change from rate of change, change in position given velocity, area, $(b - a) \cdot$ Average value
- **Properties of the definite integral**
 Properties involving integrand, properties involving limits, comparison between integrals.
- **Fundamental Theorem of Calculus**
- **Working with the definite integral**
 Estimate definite integral from graph, table of values, or formula.

REVIEW PROBLEMS FOR CHAPTER THREE

1. A car going 80 ft/sec (about 55 mph) brakes to a stop in 8 seconds. Its velocity is recorded every 2 seconds and is given in the following table.
- (a) Give your best estimate of the distance traveled by the car during the 8 seconds.
 (b) To estimate the distance traveled accurate to within 20 feet, how often should you record the velocity?

t (seconds)	0	2	4	6	8
$v(t)$ (ft/sec)	80	52	28	10	0

2. Fill in Table 3.4 with the appropriate left- and right-hand sums with n subdivisions for the integral $\int_{-1}^2 5e^{-x^2} dx$. Use the table to estimate the value of the definite integral.

TABLE 3.4

n	Left	Right
5		
25		
500		

3. A graph of $y = f(x)$ is given in Figure 3.49. Estimate $\int_0^{20} f(x) dx$.

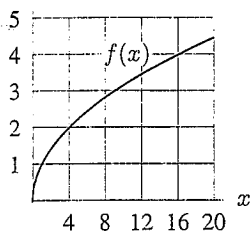


Figure 3.49

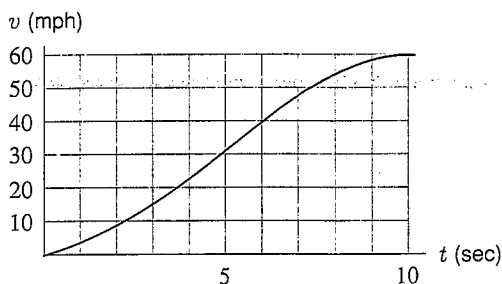


Figure 3.50

4. A car accelerates smoothly from 0 to 60 mph in 10 seconds. Suppose the car's velocity as a function of time is given in Figure 3.50. Estimate how far the car travels during the 10-second period.
5. Your velocity is given by

$$v(t) = \sin(t^2) \quad \text{for } 0 \leq t \leq 1.1.$$

Represent the distance traveled during this time by an integral, and use a calculator or computer to find the distance traveled.

Find the area of the regions in Problems 6–9.

6. Between $y = x^2 - 9$ and the x -axis. 7. Under one arch of the curve $y = \sin x$.
8. Between the parabola $y = 4 - x^2$ and the x -axis. 9. Between the line $y = 1$ and one arch of the curve $y = \sin \theta$.
10. If $\int_2^5 (2f(x) + 3) dx = 17$, find $\int_2^5 f(x) dx$.
11. The graph of a continuous function f is given in Figure 3.51. Rank the following integrals in ascending numerical order. Explain your reasons.

(i) $\int_0^2 f(x) dx$ (ii) $\int_0^1 f(x) dx$ (iii) $\int_0^2 (f(x))^{1/2} dx$ (iv) $\int_0^2 (f(x))^2 dx$.

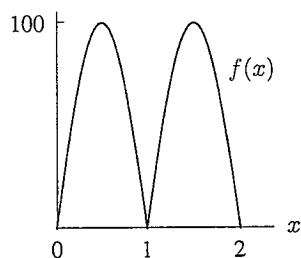


Figure 3.51

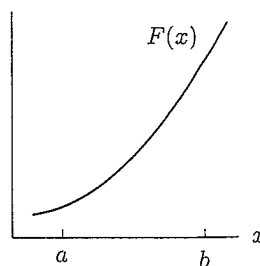


Figure 3.52

For Problems 12–14, assuming $F' = f$, mark the following quantities on a copy of Figure 3.52.

12. A slope representing $f(a)$. 13. A length representing $\int_a^b f(x) dx$.
14. A slope representing $\frac{1}{b-a} \int_a^b f(x) dx$.

15. (a) Sketch a graph of $f(x) = \sin(x^2)$ and mark on it the points $x = \sqrt{\pi}, \sqrt{2\pi}, \sqrt{3\pi}, \sqrt{4\pi}$.
 (b) Use your graph to decide which of the four numbers

$$\int_0^{\sqrt{n\pi}} \sin(x^2) dx \quad n = 1, 2, 3, 4$$

is largest. Which is smallest? How many of the numbers are positive?

16. The Environmental Protection Agency was recently asked to investigate a spill of radioactive iodine. Measurements showed the ambient radiation levels at the site to be four times the maximum acceptable limit, so the EPA ordered an evacuation of the surrounding neighborhood.

It is known that the level of radiation from an iodine source decreases according to the formula

$$R(t) = R_0 e^{-0.004t}$$

where R is the radiation level (in millirems/hour) at time t , R_0 is the initial radiation level (at $t = 0$), and t is the time measured in hours.

- (a) How long will it take for the site to reach an acceptable level of radiation?
 (b) How much total radiation (in millirems) will have been emitted by that time, assuming the maximum acceptable limit is 0.6 millirems/hour?
17. Assume the population, P , of Mexico (in millions), is given by

$$P = 67.38(1.026)^t,$$

where t is the number of years since 1980.

- (a) What was the average population of Mexico between 1980 and 1990?
 (b) What is the average of the population of Mexico in 1980 and the population in 1990?
 (c) Explain, in terms of the concavity of the graph of P (see Figure 1.18 on page 15), why your answer to part (b) is larger or smaller than your answer to part (a).
18. The graphs in Figure 3.53 represent the velocity, v , of a particle moving along the x -axis for time $0 \leq t \leq 5$. The vertical scales of all graphs are the same. Identify the graph showing which particle

- (a) has a constant acceleration.
 (b) ends up farthest to the left of where it started.
 (c) ends up the farthest from its starting point.
 (d) experiences the greatest initial acceleration.
 (e) has the greatest average velocity.
 (f) has the greatest average acceleration.

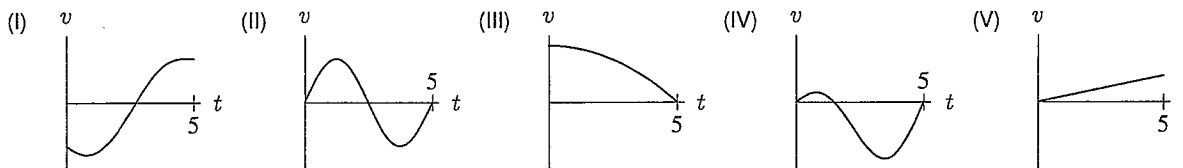


Figure 3.53

19. A new sales agent finds that as she gains experience, she increases the number of large appliances she sells each month. In the first month she sells only seven, but each month she sells two more than the month before, so that the number she sells in month t is $2t + 5$.
- (a) Find the average number of large appliances she sells per month over the first year arithmetically, by calculating the number of appliances sold each month and then taking the average over 12 months.
 (b) Now find the average by integration as though the sales function applied for all values of t (instead of just for integers).
 (c) How well do the two results compare?
 (d) If the answer you found in part (a) is viewed as the true answer, and the integral answer as an approximation, why would anyone want to use the integral answer instead of the true answer?
 (e) Draw a picture showing both answers as the area of some region. Mark on your picture a region representing the error in the integral answer.

20. Water is run into a large tank through a hose at a constant rate. After 5 minutes a hole is opened in the bottom of the tank, and water starts to flow out. Initially the flow rate through the hole is twice as great as the rate through the hose, but as the water level in the tank goes down, the flow rate through the hole decreases; after another 10 minutes the water level in the tank appears to be constant. Plot graphs of the flow rates through the hose and through the hole against time on the same pair of axes. Show how the volume of water in the tank at any time can be interpreted as an area (or the difference between two areas) on the graph. In particular, interpret the steady-state volume of water in the tank.²
21. The Glen Canyon Dam at the top of the Grand Canyon prevents natural flooding. In 1996, scientists decided an artificial flood was necessary to restore the environmental balance. Water was released through the dam at a controlled rate³ shown in Figure 3.54. The figure also shows the rate of flow of the last natural flood in 1957.
- At what rate was water passing through the dam in 1996 before the artificial flood?
 - At what rate was water passing down the river in the pre-flood season in 1957?
 - Estimate the maximum rates of discharge for the 1996 and 1957 floods.
 - Approximately how long did the 1996 flood last? How long did the 1957 flood last?
 - Estimate how much additional water passed down the river in 1996 as a result of the artificial flood.
 - Estimate how much additional water passed down the river in 1957 as a result of the flood.

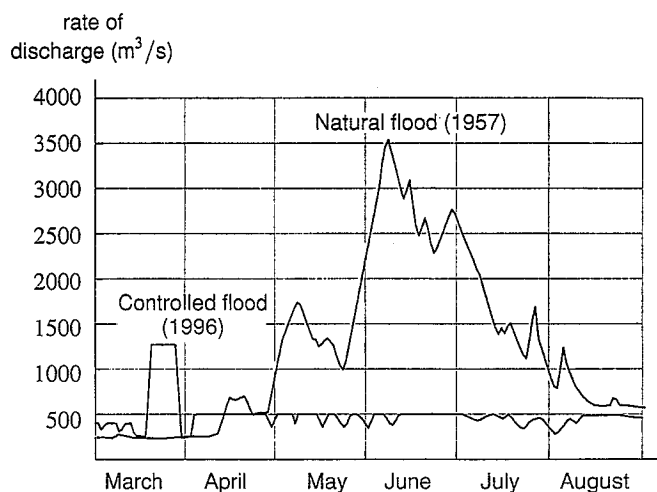


Figure 3.54

22. The Montgolfier brothers (Joseph and Etienne) were eighteenth-century pioneers in the field of hot-air ballooning. Had they had the appropriate instruments, they might have left us a record of one of their early experiments, like that shown in Figure 3.55. The graph shows their vertical velocity, v , with upward as positive.
- Over what intervals was the acceleration positive? Negative?
 - What was the greatest altitude achieved, and at what time?
 - At what time was the upward acceleration greatest?
 - At what time was the deceleration greatest?
 - What might have happened during this flight to explain the answer to part (d)?
 - This particular flight ended on top of a hill. How do you know that it did, and what was the height of the hill above the starting point?

²From *Calculus: The Analysis of Functions*, by Peter D. Taylor (Toronto: Wall & Emerson, Inc., 1992)

³Adapted from M. Collier, R. Webb, E. Andrews "Experimental Flooding in Grand Canyon" in *Scientific American* (January 1997).

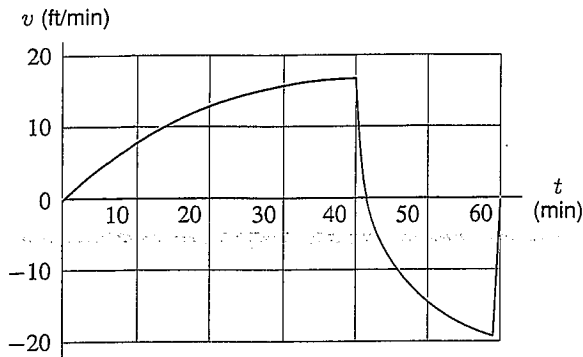


Figure 3.55

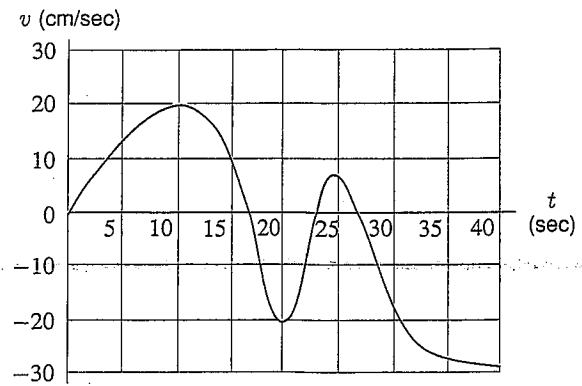


Figure 3.56

23. A mouse moves back and forth in a tunnel, attracted to bits of cheddar cheese alternately introduced and removed from the ends (right and left) of the tunnel. The graph of the mouse's velocity, v , is given in Figure 3.56, with positive velocity corresponding to motion toward the right end. Assuming that the mouse starts ($t = 0$) at the center of the tunnel, use the graph to estimate the time(s) at which
- The mouse changes direction.
 - The mouse is moving most rapidly to the right; to the left.
 - The mouse is farthest to the right of center; farthest to the left.
 - The mouse's speed (i.e., the magnitude of its velocity) is decreasing.
 - The mouse is at the center of the tunnel.
24. For the even function f graphed in Figure 3.57:
- Suppose you know $\int_0^2 f(x) dx$. What is $\int_{-2}^2 f(x) dx$?
 - Suppose you know $\int_0^5 f(x) dx$ and $\int_2^5 f(x) dx$. What is $\int_0^2 f(x) dx$?
 - Suppose you know $\int_{-2}^5 f(x) dx$ and $\int_{-2}^2 f(x) dx$. What is $\int_0^5 f(x) dx$?
25. For the even function f graphed in Figure 3.57:
- Suppose you know $\int_{-2}^2 f(x) dx$ and $\int_0^5 f(x) dx$. What is $\int_2^5 f(x) dx$?
 - Suppose you know $\int_{-2}^5 f(x) dx$ and $\int_{-2}^0 f(x) dx$. What is $\int_2^5 f(x) dx$?
 - Suppose you know $\int_2^5 f(x) dx$ and $\int_{-2}^2 f(x) dx$. What is $\int_0^2 f(x) dx$?
26. The graph of some function f is given in Figure 3.58. List, from *least* to *greatest*,
- $f'(1)$.
 - The average value of $f(x)$, $0 \leq x \leq a$.
 - The average value of the rate of change of $f(x)$, for $0 \leq x \leq a$.
 - $\int_0^a f(x) dx$.

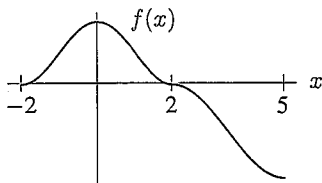


Figure 3.57

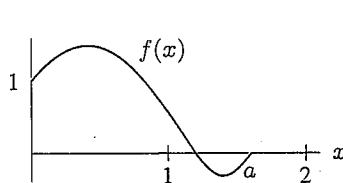


Figure 3.58

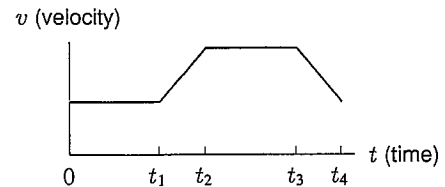


Figure 3.59

27. A force, F , acts on a particle of mass m , moving along a line. The velocity v of the particle is shown in Figure 3.59. Using $F = ma$, where a is the acceleration of the particle, decide if the work done by the force on the particle in each of the intervals $[0, t_1]$, $[t_1, t_2]$, $[t_2, t_3]$, $[t_3, t_4]$, is positive, negative, or zero. (Work done = Force \cdot Distance, when force is constant.)