3.1 HOW DO WE MEASURE DISTANCE TRAVELED?

If the velocity of a moving object is a constant, we can find the distance it travels using the formula

$$\text{Distance} = \text{Velocity} \times \text{Time}.$$

In this section we see how to estimate the distance when the velocity is not a constant.

A Thought Experiment: How Far Did the Car Go?

Velocity Data Every Two Seconds

Suppose a car is moving with increasing velocity. Suppose we measure the car's velocity every two seconds, and obtain the data in Table 3.1:

TABLE 3.1 *Velocity of car every two seconds*

Time (sec)	0	2	4	6	8	10
Velocity (ft/sec)	20	30	38	44	48	50

How far has the car traveled? Since we don't know how fast the car is moving at every moment, we can't calculate the distance exactly, but we can make an estimate. The velocity is increasing, so the car is going at least 20 ft/sec for the first two seconds. Since Distance = Velocity × Time, the car goes at least $(20)(2) = 40$ feet during the first two seconds. Likewise, it goes at least $(30)(2) = 60$ feet during the next two seconds, and so on. During the ten-second period it goes at least

$$(20)(2) + (30)(2) + (38)(2) + (44)(2) + (48)(2) = 360 \text{ feet.}$$

Thus, 360 feet is an underestimate of the total distance traveled during the ten seconds.

To get an overestimate, we can reason this way: During the first two seconds, the car's velocity is at most 30 ft/sec, so it moved at most $(30)(2) = 60$ feet. In the next two seconds it moved at most $(38)(2) = 76$ feet, and so on. Therefore, over the ten-second period it moved at most

$$(30)(2) + (38)(2) + (44)(2) + (48)(2) + (50)(2) = 420 \text{ feet.}$$

Therefore,

$$360 \text{ feet} \leq \text{Total distance traveled} \leq 420 \text{ feet.}$$

There is a difference of 60 feet between the upper and lower estimates.

Velocity Data Every One Second

What if we want a more accurate estimate? Then we should make more frequent velocity measurements, say every second. The data is in Table 3.2.

As before, we get a lower estimate for each second by using the velocity at the beginning of that second. During the first second the velocity is at least 20 ft/sec, so the car travels at least $(20)(1) = 20$ feet. During the next second the car moves at least 26 feet, and so on. So now we can say

$$\begin{aligned}\text{New lower estimate} &= (20)(1) + (26)(1) + (30)(1) + (35)(1) + (38)(1)\\ &\quad + (42)(1) + (44)(1) + (46)(1) + (48)(1) + (49)(1)\\ &= 378 \text{ feet.}\end{aligned}$$

TABLE 3.2 *Velocity of car every second*

Time (sec)	0	1	2	3	4	5	6	7	8	9	10
Velocity (ft/sec)	20	26	30	35	38	42	44	46	48	49	50

Notice that this is greater than the old lower estimate of 360 feet.

We get a new upper estimate by considering the velocity at the end of each second. During the first second the velocity is at most 26 ft/sec, and so the car moves at most $(26)(1) = 26$ feet; in the next second it moves at most 30 feet, and so on.

$$\begin{aligned}\text{New upper estimate} &= (26)(1) + (30)(1) + (35)(1) + (38)(1) + (42)(1) \\ &\quad + (44)(1) + (46)(1) + (48)(1) + (49)(1) + (50)(1) \\ &= 408 \text{ feet.}\end{aligned}$$

This is less than the old upper estimate of 420 feet. Now we know that

$$378 \text{ feet} \leq \text{Total distance traveled} \leq 408 \text{ feet.}$$

Notice that the difference between the new upper and lower estimates is now 30 feet, half of what it was before. By halving the interval of measurement, we have halved the difference between the upper and lower estimates.

Visualizing Distance on the Velocity Graph: Two Second Data

We can represent both upper and lower estimates on a graph of the velocity. On such a graph we can also see how changing the time interval between velocity measurements changes the accuracy of our estimates.

The velocity can be graphed by plotting the two second data in Table 3.1 and drawing a smooth curve through the data points. (See Figure 3.1.) The area of the first dark rectangle is $(20)(2) = 40$, the lower estimate of the distance moved during the first two seconds. The area of the second dark rectangle is $(30)(2) = 60$, the lower estimate for the distance moved in the next two seconds. The total area of the dark rectangles represents the lower estimate for the total distance moved during the ten seconds.

If the dark and light rectangles are considered together, the first area is $(30)(2) = 60$, the upper estimate for the distance moved in the first two seconds. The second area is $(38)(2) = 76$, the upper estimate for the next two seconds. Continuing this calculation suggests that the upper estimate for the total distance is represented by the sum of the areas of the dark and light rectangles. Therefore, the area of the light rectangles alone represents the difference between the two estimates.

To calculate the difference between the two estimates, look at Figure 3.1 and imagine the light rectangles all pushed to the right and stacked on top of each other. This gives a rectangle of width 2 and height 30. Notice that the height, 30, is just the difference between the initial and final values of the velocity: $30 = 50 - 20$. The width, 2, is the time interval between velocity measurements.

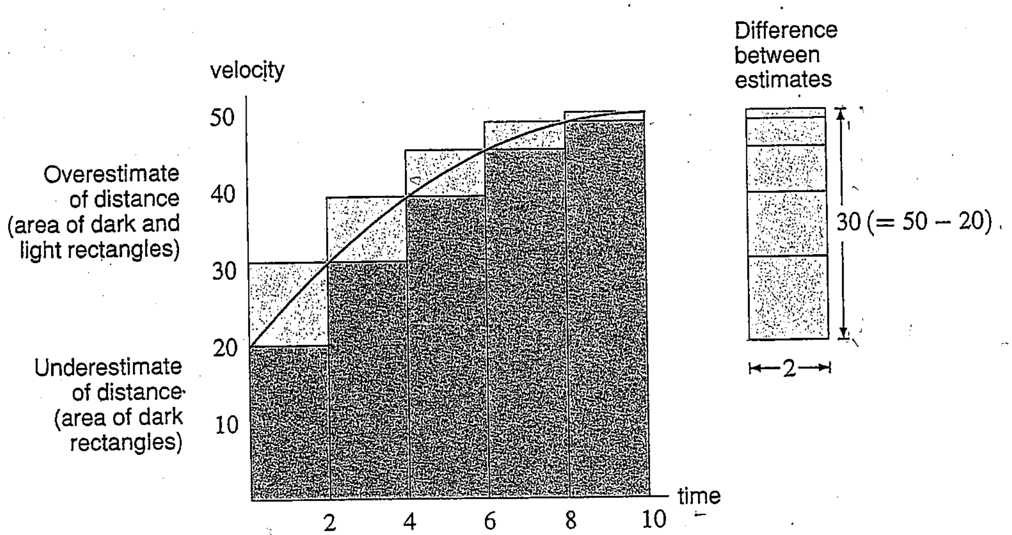

Figure 3.1: Velocity measured every 2 seconds

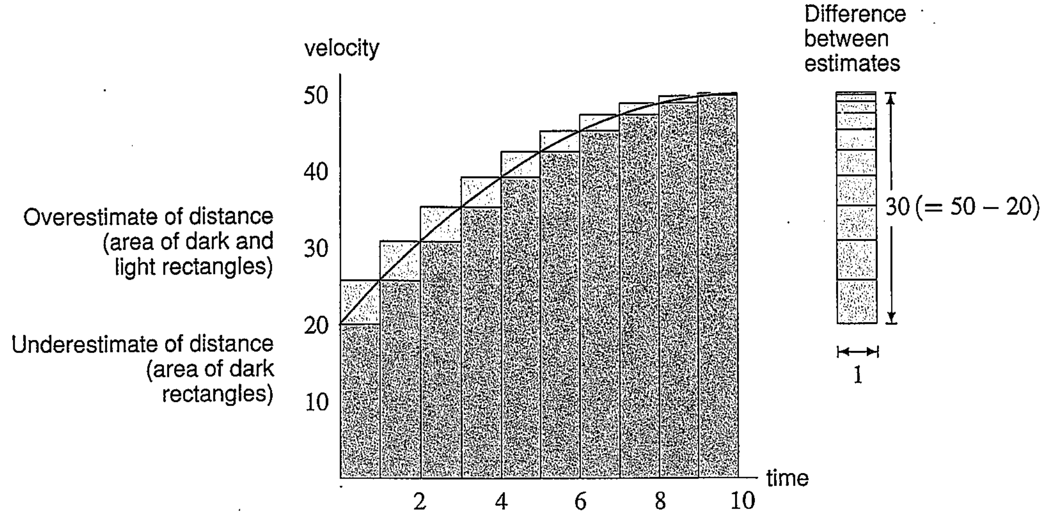

Figure 3.2: Velocity measured every second

Visualizing Distance on the Velocity Graph: One Second Data

The data for the velocities measured every second is graphed in Figure 3.2. The area of the dark rectangles again represents the lower estimate, and the dark and light rectangles together represent the upper estimate. As before, the difference between the two estimates is represented by the area of the light rectangles. This difference can be calculated by stacking the light rectangles vertically, giving a rectangle of the same height as before but of half the width. Its area is therefore half what it was before. Again, the height of this stack is $50 - 20 = 30$, but its width is the time interval between measurements, which is 1.

Example 1 What would be the difference between the upper and lower estimates if the velocity were given every tenth of a second? Every hundredth of a second? Every thousandth of a second?

Solution Every tenth of a second: Difference between estimates $= (50 - 20)(1/10) = 3$ feet.
Every hundredth of a second: Difference between estimates $= (50 - 20)(1/100) = 0.3$ feet.
Every thousandth of a second: Difference between estimates $= (50 - 20)(1/1000) = 0.03$ feet.

Example 2 How frequently must the velocity be recorded in order to estimate the total distance traveled to within 0.1 feet?

Solution The difference between the velocity at the beginning and end of the observation period is $50 - 20 = 30$. If the time between the measurements is h, then the difference between the upper and lower estimates is $(30)h$. We want

$$(30)h < 0.1,$$

or

$$h < \frac{0.1}{30} \approx 0.0033.$$

So if the measurements are made less than 0.0033 seconds apart, the distance estimate will be accurate to within 0.1 feet.

Making the Estimates for Distance Precise

We now obtain an exact expression for the total distance traveled. We express the exact total distance traveled as a limit of upper or lower estimates.

We want to know the distance traveled by a moving object over the time interval $a \le t \le b$. Let the velocity at time t be given by the function $v = f(t)$. We take measurements of $f(t)$ at equally spaced times $t_0, t_1, t_2, \ldots, t_n$, with time $t_0 = a$ and time $t_n = b$. The time interval between any two consecutive measurements is

$$\Delta t = \frac{b-a}{n},$$

where Δt means the change, or increment, in t.

During the first time interval, the velocity can be approximated by $f(t_0)$, so the distance traveled is approximately

$$f(t_0)\Delta t.$$

During the second time interval, the velocity is about $f(t_1)$, so the distance traveled is about

$$f(t_1)\Delta t.$$

Continuing in this way and adding up all the estimates, we get an estimate for the total distance. In the last interval, the velocity is approximately $f(t_{n-1})$, so the last term is $f(t_{n-1})\Delta t$:

$$\begin{array}{c}\text{Total distance traveled}\\ \text{between } t=a \text{ and } t=b\end{array} \approx f(t_0)\Delta t + f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_{n-1})\Delta t.$$

This is called a *left-hand sum* because we used the value of velocity from the left end of each time interval. It is represented by the sum of the areas of the rectangles in Figure 3.3.

We can also calculate a *right-hand sum* by using the value of the velocity at the right end of each time interval. In that case the estimate for the first interval is $f(t_1)\Delta t$, for the second interval it is $f(t_2)\Delta t$, and so on. The estimate for the last interval is now $f(t_n)\Delta t$, so

$$\begin{array}{c}\text{Total distance traveled}\\ \text{between } t=a \text{ and } t=b\end{array} \approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t + \cdots + f(t_n)\Delta t.$$

The right-hand sum is represented by the area of the rectangles in Figure 3.4.

If f is an increasing function, the left-hand sum is an underestimate of the total distance traveled. The reason is that a left-hand sum uses the velocity at the start of each interval to compute the distance

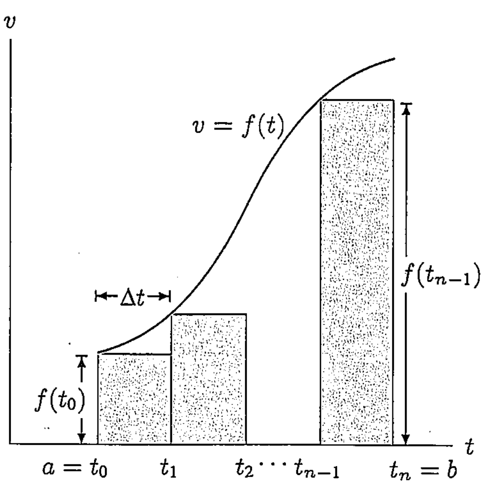

Figure 3.3: Left-hand sums

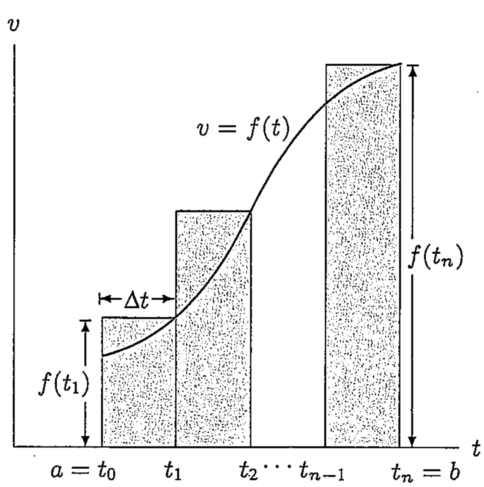

Figure 3.4: Right-hand sums

traveled, whereas in fact the velocity continues to increase after that measurement. Similarly, if f is increasing, the right-hand sum is an overestimate. If f is decreasing, as in Figure 3.5, then the roles of the two sums are reversed.

For either increasing or decreasing functions, the exact value of the distance traveled lies somewhere between the two estimates. Thus the accuracy of our estimate depends on how close these two sums are. For a function which is increasing throughout or decreasing throughout the interval $[a, b]$:

$$\left|\begin{array}{c}\text{Difference between}\\ \text{upper and lower estimates}\end{array}\right| = \left|\begin{array}{c}\text{Difference between}\\ f(a) \text{ and } f(b)\end{array}\right| \times \Delta t = |f(b) - f(a)| \cdot \Delta t.$$

(Absolute values are used to make the difference nonnegative.) In Figure 3.5, the area of the light rectangles is the difference between estimates. By making the time interval, Δt, between measurements small enough, we can make this difference between lower and upper estimates as small as we like.

In the car example, as n increased, the overestimates of the distance traveled decreased and the underestimates increased, trapping the exact distance between them. This suggests that to find exactly the total distance traveled between $t = a$ and $t = b$, we take the limit of the sums, as n goes to infinity.

$$\begin{aligned}\begin{array}{c}\text{Total distance traveled}\\ \text{between } t = a \text{ and } t = b\end{array} &= \lim_{n\to\infty} (\text{Left-hand sum})\\ &= \lim_{n\to\infty} \left[f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-1})\Delta t\right]\\ &= \text{Area under curve } f(t) \text{ from } t = a \text{ to } t = b\end{aligned}$$

and

$$\begin{aligned}\begin{array}{c}\text{Total distance traveled}\\ \text{between } t = a \text{ and } t = b\end{array} &= \lim_{n\to\infty} (\text{Right-hand sum})\\ &= \lim_{n\to\infty} \left[f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_n)\Delta t\right]\\ &= \text{Area under curve } f(t) \text{ from } t = a \text{ to } t = b.\end{aligned}$$

Provided f is continuous, the limits of the left and the right sums are both equal to the total distance traveled. This method of calculating the distance by taking the limit of a sum works even if the velocity is not increasing throughout, or decreasing throughout, the interval.

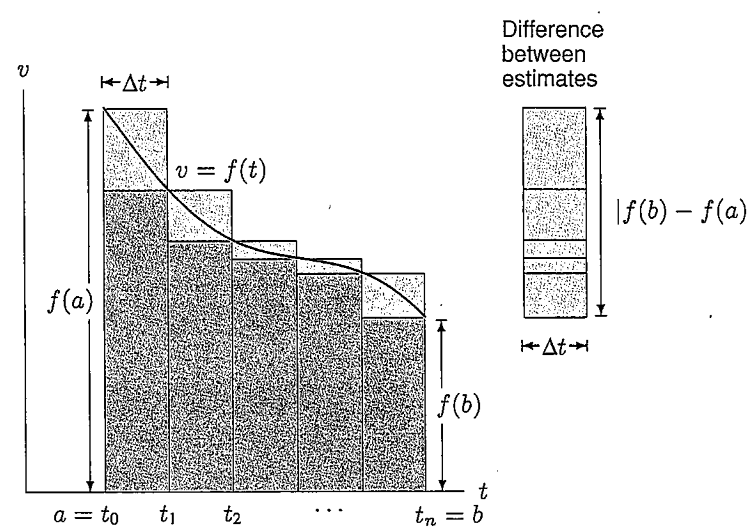

Figure 3.5: Left and right sums if f is decreasing

Problems for Section 3.1

1. A car comes to a stop six seconds after the driver applies the brakes. While the brakes are on, the following velocities are recorded:

Time since brakes applied (sec)	0	2	4	6
Velocity (ft/sec)	88	45	16	0

(a) Give lower and upper estimates for the distance the car traveled after the brakes were applied.
(b) On a sketch of velocity against time, show the lower and upper estimates of part (a).

2. A student is speeding down Route 11 in his fancy red Porche when his radar system warns him of an obstacle 400 feet ahead. He immediately applies the brakes, starts to slow down, and spots a skunk in the road directly ahead of him.

Suppose that the "black box" in the Porche records the car's speed every two seconds, producing the following table. Assume that the speed decreases throughout the 10 seconds it takes to stop, although not necessarily at a uniform rate.

Time since brakes applied (sec)	0	2	4	6	8	10
Speed (ft/sec)	100	80	50	25	10	0

(a) Using the information in this table, what is your best estimate of the total distance that the student's car traveled before coming to rest?
(b) Which statement below can you justify from the information given in the story and data table? (Choose one and justify it.)
(i) The car stopped before getting to the skunk.
(ii) The "black box" data is inconclusive. The skunk may or may not have been hit.
(iii) The unfortunate skunk was hit by the car.

3. Roger decides to run a marathon. Roger's friend Jeff rides behind him on a bicycle and clocks his pace every 15 minutes. Roger starts out strong, but after an hour and a half he is so exhausted that he has to stop. The data Jeff collected are summarized below:

Time spent running (min)	0	15	30	45	60	75	90
Speed (mph)	12	11	10	10	8	7	0

(a) Assuming that Roger's speed is never increasing, give upper and lower estimates for the distance Roger ran during the first half hour.
(b) Give upper and lower estimates for the distance Roger ran in total during the entire hour and a half.
(c) How often would Jeff have needed to measure Roger's pace in order to find lower and upper estimates within 0.1 mile of the actual distance that he ran?

4. Coal gas is produced at a gasworks. Pollutants in the gas are removed by scrubbers, which become less and less efficient as time goes on. The following measurements, made at the start of each month, show the rate at which pollutants are escaping in the gas:

Time (months)	0	1	2	3	4	5	6
Rate pollutants are escaping (tons/month)	5	7	8	10	13	16	20

(a) Make an overestimate and an underestimate of the total quantity of pollutants that escaped during the first month.

(b) Make an overestimate and an underestimate of the total quantity of pollutants that escaped during the six months.

(c) How often would measurements have to be made in order to find overestimates and underestimates which differ by less than 1 ton from the exact quantity of pollutants that escaped during the first six months?

5. Suppose time, t, is given in seconds and your velocity, v, in meters/second, is given by

$$v(t) = 1 + t^2 \quad \text{for} \quad 0 \leq t \leq 6.$$

Use $\Delta t = 2$ to estimate the distance traveled during this time. Find the left- and right-hand sums, and then average the two.

6. For $0 \leq t \leq 1$, a bug is crawling at a velocity, v, determined by the formula

$$v = \frac{1}{1+t},$$

where t is in hours and v is in meters/hour. Use $\Delta t = 0.2$ to estimate the distance that the bug crawls during this hour. Find an overestimate and an underestimate. Then average the two to get a new estimate.

7. In Figure 3.6, use the grid to estimate the area of the region bounded by the curve, the horizontal axis and the vertical lines $x = 3$ and $x = -3$. Get an upper and a lower estimate that are within 4 square units of one another. Explain your answer.

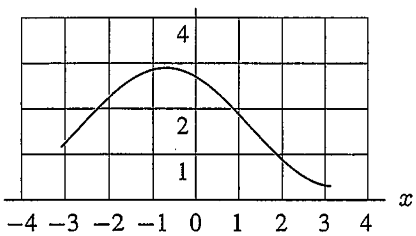

Figure 3.6

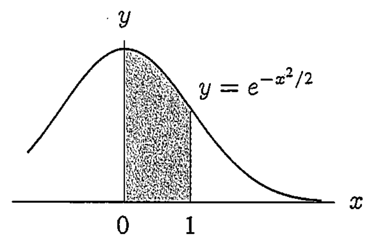

Figure 3.7

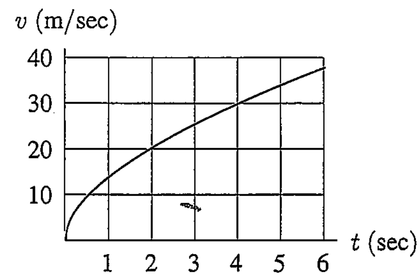

Figure 3.8

8. (a) In Figure 3.7 estimate the shaded area with an error of at most 0.1.

(b) How can you approximate this shaded area to any desired degree of accuracy?

9. Figure 3.8 shows the graph of the velocity, v, of an object (in m/sec). Estimate the total distance the object traveled between $t = 0$ and $t = 6$.

10. You jump out of an airplane. Before your parachute opens you fall faster and faster, but your acceleration decreases as you fall because of air resistance. The table below gives your acceleration, a (in m/sec^2), after t seconds.

t	0	1	2	3	4	5
a	9.81	8.03	6.53	5.38	4.41	3.61

(a) Give upper and lower estimates of your speed at $t = 5$.

(b) Get a new estimate by taking the average of your upper and lower estimates. What does the concavity of the graph of acceleration tell you about your new estimate?

11. When an aircraft attempts to climb as rapidly as possible, its climb rate decreases with altitude. (This occurs because the air is less dense at higher altitudes.) Table 3.3 shows performance data for a certain single-engine aircraft.

TABLE 3.3

Altitude (1000 ft)	0	1	2	3	4	5	6	7	8	9	10
Climb rate (ft/min)	925	875	830	780	730	685	635	585	535	490	440

(a) Calculate upper and lower estimates for the time required for this aircraft to climb from sea level to 10,000 ft.

(b) If climb rate data were available in increments of 500 ft, what would be the difference between a lower and upper estimate of climb time based on 20 subdivisions?

3.2 THE DEFINITE INTEGRAL

In Section 3.1 we saw how distance traveled can be approximated by sums and expressed exactly as the limit of a sum. In this section we show how these sums can be defined for any function f, whether or not it represents a velocity. Suppose we have a function $f(t)$ which is continuous for $a \leq t \leq b$. We divide the interval from a to b into n equal subdivisions, and we call the width of an individual subdivision Δt, so

$$\Delta t = \frac{b-a}{n}.$$

We let $t_0, t_1, t_2, \ldots, t_n$ be endpoints of the subdivisions, as in Figures 3.9 and 3.10. As before, we construct two sums:

$$\text{Left-hand sum} = f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-1})\Delta t$$

and

$$\text{Right-hand sum} = f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_n)\Delta t.$$

These sums represent the areas in Figures 3.9 and 3.10, provided $f(t) \geq 0$. In Figure 3.9, the first rectangle has width Δt and height $f(t_0)$, since the top of its left edge just touches the curve, and hence it has area $f(t_0)\Delta t$. The second rectangle has width Δt and height $f(t_1)$, and hence has area $f(t_1)\Delta t$, and so on. The sum of all these areas is the left-hand sum. The right-hand sum, shown in Figure 3.10, is constructed in the same way, except that each rectangle touches the curve on its right edge instead of its left.

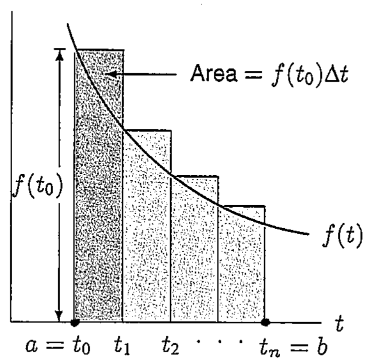

Figure 3.9: Left-hand sum

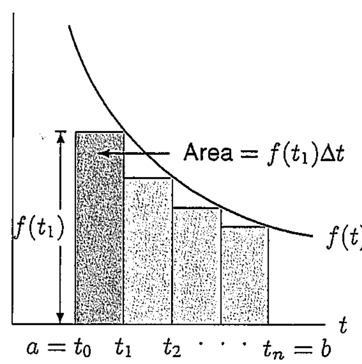

Figure 3.10: Right-hand sum

Writing Left and Right Sums Using Sigma Notation

Both the left-hand and right-hand sums can be written more compactly using *sigma*, or summation, notation. The symbol $\sum$ is a capital sigma, or Greek letter "S." We write

$$\text{Right-hand sum} = \sum_{i=1}^{n} f(t_i)\Delta t = f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_n)\Delta t.$$

The $\sum$ tells us to add terms of the form $f(t_i)\Delta t$. The "$i = 1$" at the base of the sigma sign tells us to start at $i = 1$, and the "n" at the top tells us to stop at $i = n$.

In the left-hand sum we start at $i = 0$ and stop at $i = n - 1$, so we write

$$\text{Left-hand sum} = \sum_{i=0}^{n-1} f(t_i)\Delta t = f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-1})\Delta t.$$

Taking the Limit to Obtain the Definite Integral

In the previous section we took the limit of these sums as n went to infinity, and we do the same here. If f is continuous for $a \leq t \leq b$, the limits of the left- and right-hand sums exist and are equal. The *definite integral* is the limit of these sums. A formal definition of the definite integral is given in the Focus on Theory section on page 181.

Suppose f is continuous for $a \le t \le b$. The **definite integral** of f from a to b, written

$$\int_a^b f(t)\,dt,$$

is the limit of the left-hand or right-hand sums with n subdivisions of $[a, b]$ as n gets arbitrarily large. In other words,

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} (\text{Left-hand sum}) = \lim_{n\to\infty}\left(\sum_{i=0}^{n-1} f(t_i)\Delta t\right)$$

and

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} (\text{Right-hand sum}) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(t_i)\Delta t\right).$$

Each of these sums is called a *Riemann sum*, f is called the *integrand*, and a and b are called the *limits of integration*.

The "$\int$" notation comes from an old-fashioned "S," which stands for "sum" in the same way that $\sum$ does. The "dt" in the integral comes from the factor Δt. Notice that the limits on the $\sum$ symbol are 0 and $n-1$ for the left-hand sum, and 1 and n for the right-hand sum, whereas the limits on the $\int$ sign are a and b.

Computing a Definite Integral

In practice, we often approximate definite integrals numerically using a calculator or computer. They use programs which compute sums for larger and larger values of n, and eventually give a value for the integral. Different calculators and computers may give slightly different estimates, owing to round-off error and the fact that they may use different approximation methods. Some (but not all) definite integrals can be computed exactly. However, any definite integral can be approximated numerically.

In the next example, we see how numerical approximation works. For each value of n, we calculate an over- and an under-estimate for the integral. As we increase the value of n the over- and under-estimates get closer together, trapping the value of the integral between them. By increasing the value of n sufficiently, we can calculate the integral to any desired accuracy.

Example 1 Calculate the left-hand and right-hand sums with $n = 2$ and $n = 10$ for $\int_1^2 \frac{1}{t}\,dt$. How do the values of these sums compare with the exact value of the integral?

Solution Here $a = 1$ and $b = 2$, so for $n = 2$, $\Delta t = (2-1)/2 = 0.5$. Therefore, $t_0 = 1$, $t_1 = 1.5$ and $t_2 = 2$. (See Figure 3.11.) We have

$$\begin{aligned}
\text{Left-hand sum} &= f(1)\Delta t + f(1.5)\Delta t \\
&= 1(0.5) + \frac{1}{1.5}(0.5) \\
&\approx 0.8333, \\
\text{Right-hand sum} &= f(1.5)\Delta t + f(2)\Delta t \\
&= \frac{1}{1.5}(0.5) + \frac{1}{2}(0.5) \\
&\approx 0.5833.
\end{aligned}$$

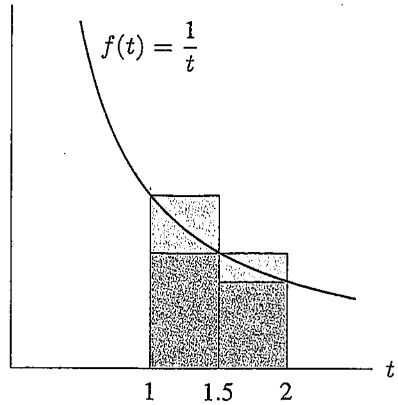

Figure 3.11: Approximating $\int_1^2 \frac{1}{t}\,dt$ with $n = 2$

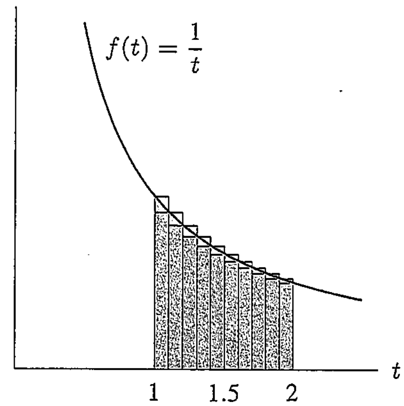

Figure 3.12: Approximating $\int_1^2 \frac{1}{t}\,dt$ with $n = 10$

From Figure 3.11 we see that the left-hand sum is bigger than the area under the curve and the right-hand sum is smaller. So the area under the curve $f(t) = 1/t$ from $t = 1$ to $t = 2$ is between 0.5833 and 0.8333 :

$$0.5833 < \int_1^2 \frac{1}{t}\,dt < 0.8333.$$

When $n = 10$, we have $\Delta t = (2-1)/10 = 0.1$ (see Figure 3.12), so

$$\begin{aligned}
\text{Left-hand sum} &= f(1)\Delta t + f(1.1)\Delta t + \cdots + f(1.9)\Delta t \\
&= \left(1 + \frac{1}{1.1} + \cdots + \frac{1}{1.9}\right) 0.1 \\
&\approx 0.7188, \\
\text{Right-hand sum} &= f(1.1)\Delta t + f(1.2)\Delta t + \cdots + f(2)\Delta t \\
&= \left(\frac{1}{1.1} + \frac{1}{1.2} + \cdots + \frac{1}{2}\right) 0.1 \\
&\approx 0.6688.
\end{aligned}$$

From Figure 3.12 you can see that the left-hand sum is larger than the area under the curve, and the right-hand sum smaller, so

$$0.6688 < \int_1^2 \frac{1}{t}\,dt < 0.7188.$$

Notice that the left- and right-hand sums trap the exact value of the integral between them. As the subdivisions become finer, the left- and right-hand sums get closer together.

Example 2 Use left and right sums with $n = 250$ for $\int_1^2 \frac{1}{t}\,dt$ to estimate the value of the integral.

Solution Using a program on a calculator or computer, we see that

$$0.6921 < \int_1^2 \frac{1}{t}\,dt < 0.6941.$$

So we can say that

$$\int_1^2 \frac{1}{t}\,dt \approx 0.69.$$

to two decimal places. The exact value is known to be $\int_1^2 \frac{1}{t}\,dt = \ln 2 = 0.693147\ldots.$

The Definite Integral as an Area

If $f(x)$ is positive we can interpret each term $f(x_0)\Delta x, f(x_1)\Delta x, \ldots$ in a left- or right-hand Riemann sum as the area of a rectangle. See Figure 3.13. As the width Δx of the rectangles approaches zero, the rectangles fit the curve of the graph more exactly, and the sum of their areas gets closer and closer to the area under the curve shaded in Figure 3.14. This suggests that:

When $f(x)$ is positive and $a < b$:

$$\text{Area under graph of } f \text{ between } a \text{ and } b = \int_a^b f(x)\,dx.$$

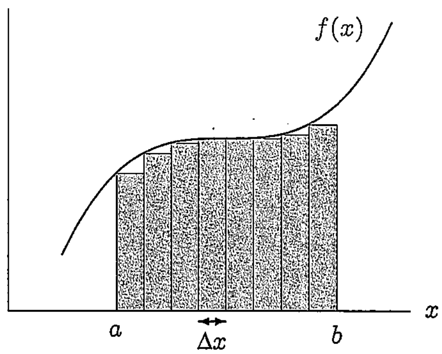

Figure 3.13: Area of rectangles approximating the area under the curve

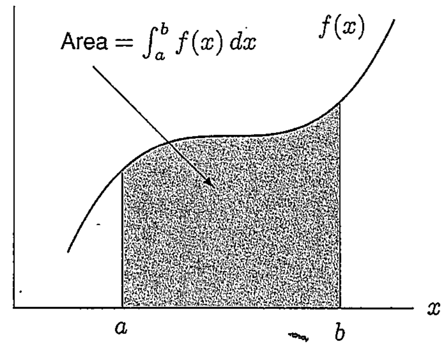

Figure 3.14: The definite integral $\int_a^b f(x)\,dx$

Example 3 Consider the integral $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$.

(a) Interpret the integral as an area, and find its exact value.

(b) Estimate the integral using a calculator or computer. Compare your answer to the exact value.

Solution (a) The integral is the area under the graph of $y = \sqrt{1-x^2}$ between -1 and 1. Rewriting this equation as $x^2 + y^2 = 1$, we see that the graph is a semicircle of radius 1 and area $\pi/2$ (see Figure 3.15).

(b) A calculator estimates the integral as $1.5707966\ldots$. For comparison, $\pi/2 = 1.5707963\ldots$.

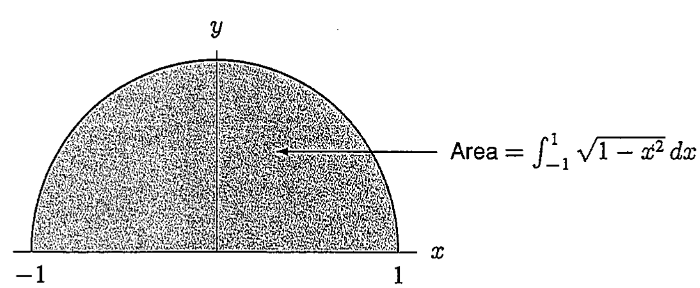

Figure 3.15: Area interpretation of $\int_{-1}^{1} \sqrt{1-x^2}\,dx$

When $f(x)$ is not positive

We have assumed in drawing Figure 3.14 that the graph of $f(x)$ lies above the x-axis. If the graph lies below the x-axis, then each value of $f(x)$ is negative, so each $f(x)\Delta x$ is negative, and the area gets counted negatively. In that case, the definite integral is the negative of the area.

Example 4 How does the definite integral $\int_{-1}^{1}(x^2-1)\,dx$ relate to the area between the parabola $y = x^2 - 1$ and the x-axis?

Solution Using a calculator, we find $\int_{-1}^{1}(x^2-1)\,dx \approx -1.33$. The parabola lies below the axis between $x = -1$ and $x = 1$. (See Figure 3.16.) So the area between the parabola and the x-axis is approximately 1.33.

> When $f(x)$ is positive for some x values and negative for others, and $a < b$:
> $\int_a^b f(x)\,dx$ is the sum of the areas above the x-axis, counted positively, and the areas below the x-axis, counted negatively.

Example 5 Interpret the definite integral $\int_0^{\sqrt{2\pi}}\sin(x^2)\,dx$ in terms of areas.

Solution The integral is the area above the x-axis, A_1, minus the area below the x-axis, A_2. See Figure 3.17. Approximating the integral with a calculator gives

$$\int_0^{\sqrt{2\pi}}\sin(x^2)\,dx \approx 0.43.$$

The graph of $y = \sin(x^2)$ crosses the x-axis where $x^2 = \pi$, that is, at $x = \sqrt{\pi}$. The next crossing is at $x = \sqrt{2\pi}$. Breaking the integral into two parts and calculating each one separately gives

$$\int_0^{\sqrt{\pi}}\sin(x^2)\,dx \approx 0.89 \quad \text{and} \quad \int_{\sqrt{\pi}}^{\sqrt{2\pi}}\sin(x^2)\,dx \approx -0.46.$$

So $A_1 \approx 0.89$ and $A_2 \approx 0.46$. Then, as we would expect,

$$\int_0^{\sqrt{2\pi}}\sin(x^2)\,dx = A_1 - A_2 \approx 0.89 - 0.46 = 0.43.$$

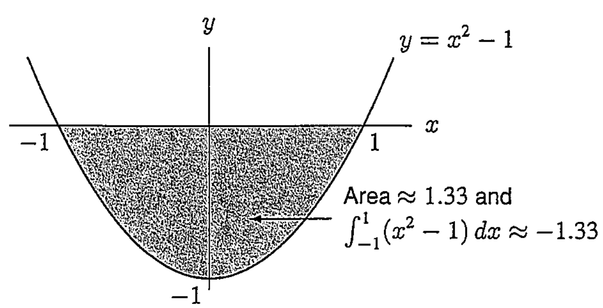

Figure 3.16: Integral $\int_{-1}^{1}(x^2-1)\,dx$ is negative of shaded area

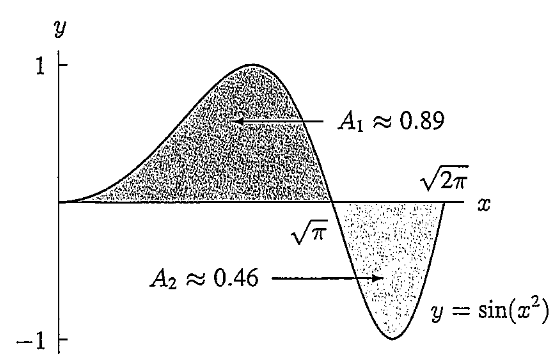

Figure 3.17: Integral $\int_0^{\sqrt{2\pi}}\sin(x^2)\,dx = A_1 - A_2$

Problems for Section 3.2

1. On a copy of Figure 3.18, draw rectangles representing each of the following Riemann sums for the function f on the interval $0 \le t \le 8$. Calculate the value of each sum.
   (a) Left-hand sum with $\Delta t = 4$
   (b) Right-hand sum with $\Delta t = 4$
   (c) Left-hand sum with $\Delta t = 2$
   (d) Right-hand sum with $\Delta t = 2$

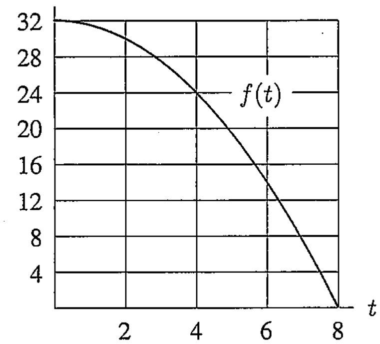

Figure 3.18

2. Write out the terms of the right-hand sum with $n = 5$ that could be used to approximate $\int_3^7 \frac{1}{1+x}\,dx$. Do not evaluate the terms or the sum.

For Problems 3–8, construct a table of left- and right-hand sums with 2, 10, 50, and 250 subdivisions. Observe the limit to which your sums are tending as the number of subdivisions gets larger, and hence estimate the value of the definite integral.

3. $\int_0^1 x^3\,dx$
4. $\int_0^{\pi/2} \cos x\,dx$
5. $\int_2^3 \sin(t^2)\,dt$
6. $\int_0^1 e^{t^2}\,dt$
7. $\int_{0.2}^3 \sin\left(1/x\right)\,dx$
8. $\int_1^2 x^x\,dx$

9. Estimate $\int_1^2 x^2\,dx$ using left- and right-hand sums with four subdivisions. How far from the true value of the integral could your estimate be?

10. (a) Use a calculator or computer to find $\int_0^6 (x^2+1)\,dx$. Represent this value as the area under a curve.
    (b) Estimate $\int_0^6 (x^2+1)\,dx$ using a left-hand sum with $n = 3$. Represent this sum graphically on a sketch of $f(x) = x^2 + 1$. Is this sum an overestimate or underestimate of the true value found in part (a)?
    (c) Estimate $\int_0^6 (x^2+1)\,dx$ using a right-hand sum with $n = 3$. Represent this sum on your sketch. Is this sum an overestimate or underestimate?

11. A table of values for $f(t)$ is given. Estimate $\int_0^{100} f(t)\,dt$.

t	0	20	40	60	80	100
$f(t)$	1.2	2.8	4.0	4.7	5.1	5.2

Estimate the area of the regions in Problems 12–17.

12. Under the curve $y = \cos t$ for $0 \le t \le \pi/2$.
13. Under the curve $y = 7 - x^2$ and above the x-axis.
14. Under the curve $y = \cos\sqrt{x}$ for $0 \le x \le 2$.
15. Under the curve $y = e^x$ and above the line $y = 1$ for $0 \le x \le 2$.
16. Between $y = x^2$ and $y = x^3$ for $0 \le x \le 1$.
17. Between $y = x^{1/2}$ and $y = x^{1/3}$ for $0 \le x \le 1$.
18. In Problems 16 and 17 we calculated the areas between $y = x^2$ and $y = x^3$ and between $y = x^{1/2}$ and $y = x^{1/3}$ on $0 \le x \le 1$. Explain why you would expect these two areas to be equal.
19. (a) Sketch a graph of $f(x) = x(x+2)(x-1)$.
    (b) Find the total area between the graph and the x-axis between $x = -2$ and $x = 1$.
    (c) Find $\int_{-2}^{1} f(x)\,dx$ and interpret it in terms of areas.
20. Compute the definite integral $\int_0^4 \cos\sqrt{x}\,dx$ and interpret the result in terms of areas.
21. Without computing the integral, decide if

$$\int_0^{2\pi} e^{-x}\sin x\,dx$$

is positive or negative, and explain your decision. [Hint: Sketch $e^{-x}\sin x$.]

22. Suppose that we use $n = 500$ subintervals to approximate $\int_{-1}^{1} (2x^3 + 4)\,dx$. Without computing the Riemann sums, find the difference between the right- and left-hand Riemann sums.
23. The graph of a function $f(t)$ is given in Figure 3.19. Which of the following four numbers could be an estimate of $\int_0^1 f(t)dt$ accurate to two decimal places? Explain how you chose your answer.

    (a) -98.35 (b) 71.84 (c) 100.12 (d) 93.47

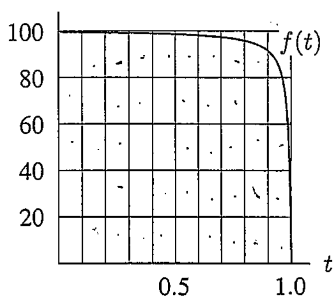

Figure 3.19

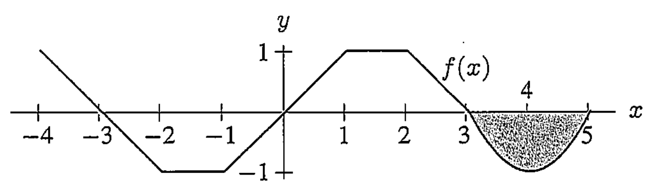

Figure 3.20

24. The graph of $y = f(x)$ is given in Figure 3.20.
    (a) What is $\int_{-3}^{0} f(x)\,dx$?
    (b) If the area of the shaded region is A, estimate $\int_{-3}^{4} f(x)\,dx$.

25. Consider the definite integral $\int_0^1 x^4\,dx$.

    (a) Write an expression for a right-hand Riemann sum approximation for this integral using n subdivisions. Express each x_i, $i = 1,2,\ldots,n$, in terms of i.
    (b) Use a computer algebra system to obtain a formula for the sum you wrote in part (a) in terms of n.
    (c) Take the limit of this expression for the sum as $n \to \infty$, thereby finding the exact value of this integral.

26. Repeat Problem 25, using the definite integral $\int_0^1 x^5\,dx$.

27. Three terms of a left-hand sum used to approximate a definite integral $\int_a^b f(x)\,dx$ are as follows.

$$\left(2+0\cdot\frac{4}{3}\right)^2\cdot\frac{4}{3}+\left(2+1\cdot\frac{4}{3}\right)^2\cdot\frac{4}{3}+\left(2+2\cdot\frac{4}{3}\right)^2\cdot\frac{4}{3}.$$

Find possible values for a and b and a possible formula for $f(x)$.

28. Consider the integral $\int_1^2 \frac{1}{t}\,dt$. In Example 1, by dividing the interval $1 \le t \le 2$ into 10 equal parts, we showed that

$$0.1\left[\frac{1}{1.1}+\frac{1}{1.2}+\ldots+\frac{1}{2.0}\right] \le \int_1^2 \frac{1}{t}\,dt \le 0.1\left[\frac{1}{1}+\frac{1}{1.1}+\ldots+\frac{1}{1.9}\right].$$

    (a) Now divide the interval $1 \le t \le 2$ into n equal parts to show that

$$\sum_{r=1}^{n}\frac{1}{n+r} < \int_1^2 \frac{1}{t}\,dt < \sum_{r=0}^{n-1}\frac{1}{n+r}.$$

    (b) Show that the difference between the upper and lower sums in part (a) is $1/2n$.
    (c) The exact value of $\int_1^2 (1/t)\,dt$ is $\ln(2)$. How large should n be to approximate $\ln(2)$ with an error of at most $5\cdot 10^{-6}$, using one of the sums in part (a)?

3.3 INTERPRETATIONS OF THE DEFINITE INTEGRAL

The Notation and Units for the Definite Integral

Just as the Leibniz notation dy/dx for the derivative reminds us that the derivative is the limit of a ratio of differences, the notation for the definite integral helps us recall the meaning of the integral. The symbol

$$\int_a^b f(x)\,dx$$

reminds us that an integral is a limit of sums (the integral sign is an old-fashioned S) of terms of the form "$f(x)$ times a small difference of x." Officially, dx is not a separate entity, but a part of the whole integral symbol. Just as one thinks of d/dx as a single symbol meaning "the derivative with respect to x of...," one can think of $\int_a^b \ldots dx$ as a single symbol meaning "the integral of ... with respect to x."

However, many scientists and mathematicians informally think of dx as an "infinitesimally" small bit of x which in this context is multiplied by a function value $f(x)$. This viewpoint is often the key to interpreting the meaning of a definite integral. For example, if $f(t)$ is the velocity of a moving particle at time t, then $f(t)dt$ may by thought of informally as velocity $\times$ time, giving the distance traveled by the particle during a small bit of time dt. The integral $\int_a^b f(t)\,dt$ may then be thought of as the sum of all these small distances, giving us the net change in position of the particle between $t = a$ and $t = b$.