

# CHAPTER TWO

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## KEY CONCEPT: THE DERIVATIVE

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We begin this chapter by investigating the problem of speed: How can we measure the speed of a moving object at a given instant in time? Or, more fundamentally, what do we mean by the term *speed*? We'll come up with a definition of speed that has wide-ranging implications — not just for the speed problem, but for measuring the rate of change of any quantity. Our journey will lead us to the key concept of *derivative*, which forms the basis for our study of calculus.

The derivative can be interpreted geometrically as the slope of a curve, and physically as a rate of change. Because derivatives can be used to represent everything from fluctuations in interest rates to the rates at which fish populations vary and gas molecules move, they have applications throughout the sciences.

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## 2.1 HOW DO WE MEASURE SPEED?

The speed of an object at an instant in time is surprisingly difficult to define precisely. Consider the statement “At the instant it crossed the finish line, the horse was traveling at 42 mph.” How can such a claim be substantiated? A photograph taken at that instant will show the horse motionless—it is no help at all. There is some paradox in trying to study the horse’s motion at a particular instant in time, since by focusing on a single instant we stop the motion!

Problems of motion were of central concern to Zeno and other philosophers as early as the fifth century B.C. The modern approach, made famous by Newton’s calculus, is to stop looking for a simple notion of speed at an instant, and instead to look at speed over small intervals containing the instant. This method sidesteps the philosophical problems mentioned earlier but introduces new ones of its own.

We illustrate the ideas discussed above by an idealized example, called a thought experiment. It is idealized in the sense that we assume that we can make measurements of distance and time as accurately as we wish.

### A Thought Experiment: Average and Instantaneous Velocity

We look at the speed of a small object (say, a grapefruit) that is thrown straight upward into the air at  $t = 0$  seconds. The grapefruit leaves the thrower’s hand at high speed, slows down until it reaches its maximum height, and then gradually speeds up in the downward direction and finally, “Splat!” (See Figure 2.1.)

But suppose that we would like to be more precise in our determination of the speed, say, at  $t = 1$  second. We assume that we can measure the height of the grapefruit above the ground at any time  $t$ ; we think of the height,  $y$ , as a function of time. (See Table 2.1.) “Splat!” comes sometime between 6 and 7 seconds. The numbers show the behavior noted above: During the first second the grapefruit travels  $90 - 6 = 84$  feet, and during the second second it travels only  $142 - 90 = 52$  feet. Hence the grapefruit traveled faster over the first interval,  $0 \leq t \leq 1$ , than the second interval,  $1 \leq t \leq 2$ .

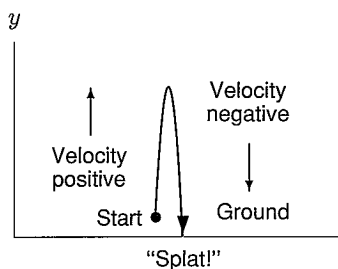


TABLE 2.1 Height of the grapefruit above the ground

$t$ (sec)	0	1	2	3	4	5	6
$y$ (feet)	6	90	142	162	150	106	30

Figure 2.1: The grapefruit’s path is straight up and down

### Velocity versus Speed

From now on, we will make a distinction between velocity and speed. Suppose an object moves along a line. We pick one direction to be positive and say that the *velocity* is positive if it is in the same direction, and negative if it is in the opposite direction. For the grapefruit, upward is positive and downward is negative. (See Figure 2.1.) *Speed* is the magnitude of the velocity and so is always positive or zero.

If  $s(t)$  is the position of an object at time  $t$ , then the **average velocity** of the object over the interval  $a \leq t \leq b$  is

$$\text{Average velocity} = \frac{\text{Change in position}}{\text{Change in time}} = \frac{s(b) - s(a)}{b - a}$$

In words, the **average velocity** of an object over an interval is the net change in position during the interval divided by the change in time.

**Example 1** Compute the average velocity of the grapefruit over the interval  $4 \leq t \leq 5$ . What is the significance of the sign of your answer?

**Solution** During this interval, the grapefruit moves  $(106 - 150) = -44$  feet. Therefore the average velocity is  $-44$  ft/sec. The negative sign means the height is decreasing and the grapefruit is moving downward.

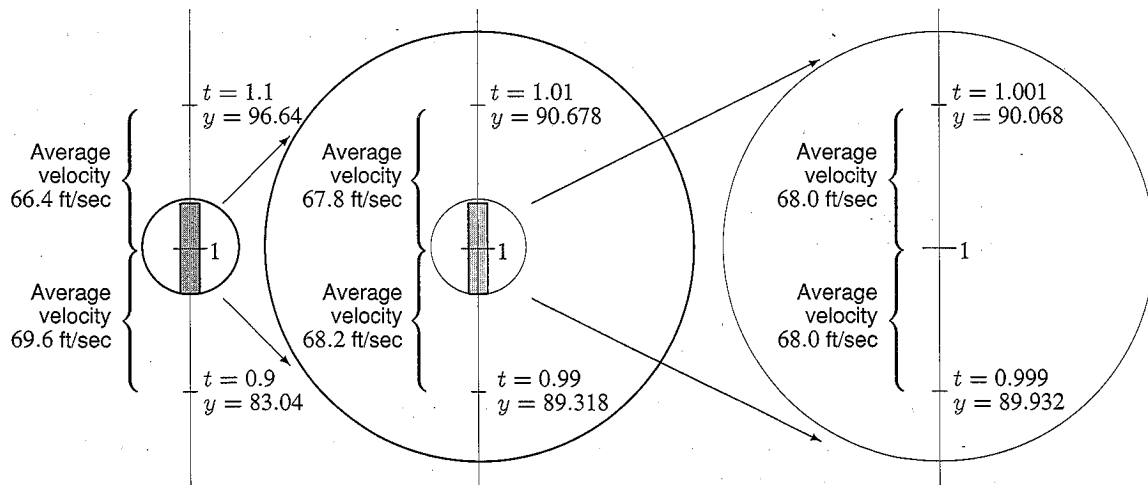
**Example 2** Compute the average velocity of the grapefruit over the interval  $1 \leq t \leq 3$ .

**Solution** Average velocity =  $(162 - 90)/(3 - 1) = 72/2 = 36$  ft/sec.

The average velocity is a useful concept since it gives a rough idea of the behavior of the grapefruit: if two grapefruits are hurled into the air, and one has an average velocity of 10 ft/sec over the interval  $0 \leq t \leq 1$  while the second has an average velocity of 100 ft/sec over the same interval, clearly the second one is moving faster.

But average velocity over an interval doesn't solve the problem of measuring the velocity of the grapefruit at *exactly*  $t = 1$  second. To get closer to an answer to that question, we have to look at what happens near  $t = 1$  in more detail. The data<sup>1</sup> in Figure 2.2 shows the average velocity over small intervals on either side of  $t = 1$ .

Notice that the average velocity before  $t = 1$  is slightly more than the average velocity after  $t = 1$ . We expect to define the velocity at  $t = 1$  to be something between these two average velocities. As the size of the interval shrinks, the values of the velocity before  $t = 1$  and the velocity after  $t = 1$  get closer together. In the smallest interval in Figure 2.2, both velocities are 68.0 ft/sec (to one decimal place), so we will define the velocity at  $t = 1$  to be 68.0 ft/sec (to one decimal place).



**Figure 2.2:** Average velocities over intervals on either side of  $t = 1$ : Showing successively smaller intervals

<sup>1</sup>The data is in fact calculated from the formula  $y = 6 + 100t - 16t^2$ .

Of course, if we calculate to more decimal places, the average velocities before and after  $t = 1$  would no longer agree. To calculate the velocity at  $t = 1$  to more decimal places of accuracy, we would take smaller and smaller intervals on either side of  $t = 1$  until the average velocities agree to the number of decimal places we wanted. In this way, we could estimate the velocity at  $t = 1$  to any accuracy.

## Defining Instantaneous Velocity Using Limit Notation

When we take smaller and smaller intervals, it turns out that the average velocities are always just above or just below 68 ft/sec. It seems natural, then, to define velocity at the instant  $t = 1$  to be 68 ft/sec. This is called the *instantaneous velocity* at this point. Its definition depends on our being convinced that smaller and smaller intervals will provide average speeds that come arbitrarily close to 68. This process is called *taking the limit*.

Notice how we have replaced the original difficulty of computing velocity at a point by a search for an argument to convince ourselves that the average velocities do approach a number as the time intervals shrink in size. In a sense, we have traded one hard question for another, since we don't yet have any idea how to be certain what number the average velocities are approaching. In the thought experiment, the number seems to be exactly 68, but what if it were 68.000001? How can we be sure that we have taken small enough intervals? Showing that the limit is exactly 68 requires more precise knowledge of the limiting process.

We now define instantaneous velocity at an arbitrary point  $t = a$ . We use the same method as for  $t = 1$ : we look at smaller and smaller intervals of size  $h$  around  $t = a$ . Then, over the interval  $a \leq t \leq a + h$ ,

$$\text{Average velocity} = \frac{s(a+h) - s(a)}{h}.$$

The same formula holds when  $h < 0$ . The instantaneous velocity is the number that the average velocities approach as the intervals decrease in size, that is, as  $h$  becomes smaller. So we define

$$\text{Instantaneous velocity} = \text{Limit, as } h \text{ approaches 0, of } \frac{s(a+h) - s(a)}{h}.$$

This is written more compactly using limit notation, as follows:

Let  $s(t)$  give the position at time  $t$ . Then the **instantaneous velocity** at  $t = a$  is

$$\text{Instantaneous velocity at } t = a = \lim_{h \rightarrow 0} \frac{s(a+h) - s(a)}{h}.$$

In words, the **instantaneous velocity** of an object at time  $t = a$  is given by the limit of the average velocity over an interval, as the interval shrinks around  $a$ .

This expression forms the foundation of the rest of calculus. Be sure that you are not confused by the notation and recognize it for what it is: the number that the average velocities approach as the intervals shrink. To estimate that number, the limit, we look at intervals of smaller and smaller, but never zero, length.

## Visualizing Velocity: Slope of Curve

Now we will see how to visualize velocity using a graph of height. Let's go back to the grapefruit. Suppose that Figure 2.3 shows the height of the grapefruit plotted against time. (Note that this is not a picture of the grapefruit's path, which is straight up and down.)

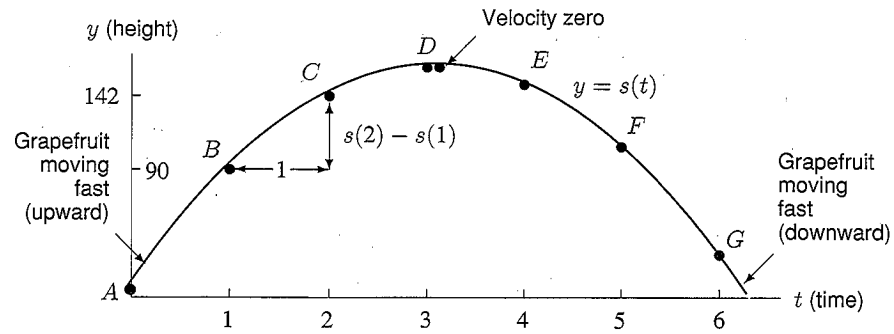


Figure 2.3: The height,  $y$ , of the grapefruit at time  $t$

How can we visualize the average velocity on this graph? Suppose  $y = s(t)$ . Let's consider the interval  $1 \leq t \leq 2$  and the expression

$$\text{Average velocity} = \frac{\text{Change in position}}{\text{Change in time}} = \frac{s(2) - s(1)}{2 - 1} = \frac{142 - 90}{1} = 52 \text{ ft/sec.}$$

Now  $s(2) - s(1)$  is the change in position over the interval, and it is marked vertically in Figure 2.3. The 1 in the denominator is the time elapsed and is marked horizontally in Figure 2.3. Therefore,

$$\text{Average velocity} = \frac{\text{Change in position}}{\text{Change in time}} = \text{Slope of line joining } B \text{ and } C.$$

(See Figure 2.3.) A similar argument shows the following:

The **average velocity** over any time interval  $a \leq t \leq b$  is the slope of the line joining the points on the graph of  $s(t)$  corresponding to  $t = a$  and  $t = b$ .

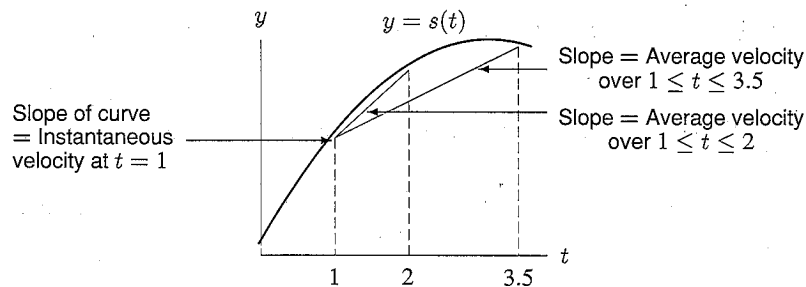


Figure 2.4: Average velocities over small intervals

The next question is how to visualize the instantaneous velocity. Let's think about how we found the instantaneous velocity. We took average velocities over smaller and smaller intervals beginning at the point  $t = 1$ . Two such velocities are represented by the slopes of the lines in Figure 2.4. As the length of the interval shrinks, the slope of the line gets closer to the slope of the curve at  $t = 1$ .

The cornerstone of the idea is the fact that, on a very small scale, most functions look almost like straight lines. Imagine taking the graph of a function near a point and "zooming in" to get a close-up view. (See Figure 2.5.) The more we zoom in, the more the curve will appear to be a straight line. In other words, if we repeatedly zoom in on a section of the curve centered at a point of interest, the section of curve will eventually look like a straight line. We call the slope of this line the *slope of the curve* at the point. Therefore, the slope of the magnified line is the instantaneous velocity. Thus, we can say:

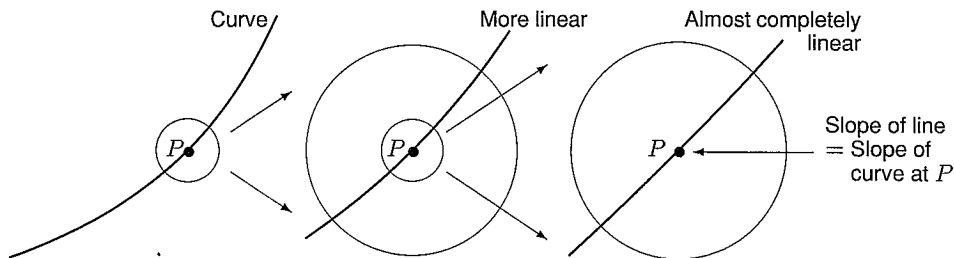


Figure 2.5: Estimating the slope of the curve at the point by “zooming in”

The **instantaneous velocity** is the slope of the curve at a point.

Look back at the graph of the grapefruit’s height as a function of time in Figure 2.3. If we think of the velocity at any point as the slope of the curve there, we can see how the grapefruit’s velocity varies during its journey. At points *A* and *B* the curve has a large positive slope, indicating that the grapefruit is traveling up rapidly. Point *D* is almost at the top: the grapefruit is slowing down. At the peak, the slope of the curve is zero: the fruit has slowed to zero velocity for an instant in preparation for its return to earth. At point *E* the curve has a small negative slope, indicating a slow velocity of descent. Finally, the slope of the curve at point *G* is large and negative, indicating a large downward velocity that is responsible for the “Splat.”

### The Idea of a Limit

In the process of defining the instantaneous velocity, we observed average velocities as time intervals shrank around a point and introduced the notation for a limit. Now we look a bit more at the idea of a limit, which is considered in more detail in the Focus on Theory section on page 127. We talk about the limit of the function at the point *c*.

We write  $\lim_{x \rightarrow c} f(x)$  to represent the number *L* approached by *f(x)* as *x* approaches *c*.

**Example 3** Investigate  $\lim_{x \rightarrow 2} x^2$ .

**Solution** Notice that we can make  $x^2$  as close to 4 as we like by taking *x* sufficiently close to 2. (Look at the values of  $1.9^2$ ,  $1.99^2$ ,  $1.999^2$ , and  $2.1^2$ ,  $2.01^2$ ,  $2.001^2$ ; they seem to be approaching 4 in Table 2.2.) We write

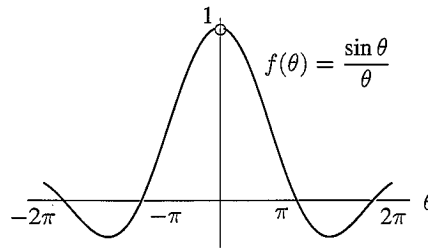
$$\lim_{x \rightarrow 2} x^2 = 4,$$

which is read “the limit, as *x* approaches 2, of  $x^2$  is 4.” Notice that the limit doesn’t ask what happens at  $x = 2$ , so it is not sufficient to substitute 2 to find the answer. The limit describes behavior of a function *near* a point, not *at* the point.

**TABLE 2.2** Values of  $x^2$

<i>x</i>	1.9	1.99	1.999	2.001	2.01	2.1
$x^2$	3.61	3.96	3.996	4.004	4.04	4.41

**Example 4** Use a graph to estimate  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}$ . (Be sure to use radians.)



**Figure 2.6:** Find the limit as  $\theta \rightarrow 0$

**Solution** Figure 2.6 shows that as  $\theta$  approaches 0 from either side, the value of  $\frac{\sin \theta}{\theta}$  appears to approach 1, suggesting that  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$ . Zooming in on the graph near  $\theta = 0$  provides further support for this conclusion. Notice that  $\frac{\sin \theta}{\theta}$  is undefined at  $\theta = 0$ .

**Example 5** Estimate  $\lim_{h \rightarrow 0} \frac{(3 + h)^2 - 9}{h}$  numerically.

**Solution** The limit is the value approached by this expression as  $h$  approaches 0. The values in Table 2.3 seem to be converging to 6 as  $h \rightarrow 0$ . So it is a reasonable guess that

$$\lim_{h \rightarrow 0} \frac{(3 + h)^2 - 9}{h} = 6.$$

However, we cannot be sure that the limit is *exactly* 6 by looking at the table. To calculate the limit exactly requires algebra.

**TABLE 2.3** Values of  $((3 + h)^2 - 9) / h$

$h$	-0.1	-0.01	-0.001	0.001	0.01	0.1
$((3 + h)^2 - 9) / h$	5.9	5.99	5.999	6.001	6.01	6.1

**Example 6** Use algebra to find  $\lim_{h \rightarrow 0} \frac{(3 + h)^2 - 9}{h}$ .

**Solution** Expanding the numerator gives

$$\frac{(3 + h)^2 - 9}{h} = \frac{9 + 6h + h^2 - 9}{h} = \frac{6h + h^2}{h}$$

Since taking the limit as  $h \rightarrow 0$  means looking at values of  $h$  near, but not equal, to 0, we can cancel  $h$ , giving

$$\lim_{h \rightarrow 0} \frac{(3 + h)^2 - 9}{h} = \lim_{h \rightarrow 0} (6 + h).$$

As  $h$  approaches 0, the values of  $(6 + h)$  approach 6, so

$$\lim_{h \rightarrow 0} \frac{(3 + h)^2 - 9}{h} = \lim_{h \rightarrow 0} (6 + h) = 6.$$

**Problems for Section 2.1**

1. A car is driven at a constant speed. Sketch a graph of the distance the car has traveled as a function of time.
2. A car is driven at an increasing speed. Sketch a graph of the distance the car has traveled as a function of time.
3. A car starts at a high speed, and its speed then decreases slowly. Sketch a graph of the distance the car has traveled as a function of time.

4. For the function shown in Figure 2.7, at what labeled points is the slope of the graph positive? Negative? At which labeled point does the graph have the greatest (i.e., most positive) slope? The least slope (i.e., negative and with the largest magnitude)?

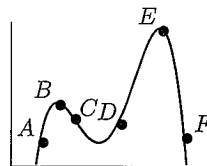


Figure 2.7

5. Match the points labeled on the curve in Figure 2.8 with the given slopes.

Slope	Point
-3	
-1	
0	
1/2	
1	
2	

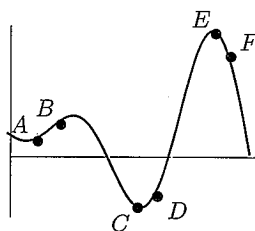


Figure 2.8

6. For the graph  $y = f(x)$  shown in Figure 2.9, arrange the following numbers in ascending (i.e., smallest to largest) order:

- The slope of the graph at A.
- The slope of the graph at B.
- The slope of the graph at C.
- The slope of the line AB.
- The number 0.
- The number 1.

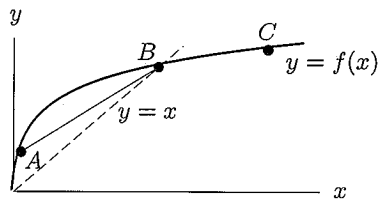


Figure 2.9

7. The graph of  $f(t)$  in Figure 2.10 gives the position of a particle at time  $t$ . List the following quantities in order, smallest to largest.

- A, the average velocity between  $t = 1$  and  $t = 3$ ,
- B, the average velocity between  $t = 5$  and  $t = 6$ ,
- C, the instantaneous velocity at  $t = 1$ ,
- D, the instantaneous velocity at  $t = 3$ ,
- E, the instantaneous velocity at  $t = 5$ ,
- F, the instantaneous velocity at  $t = 6$ .

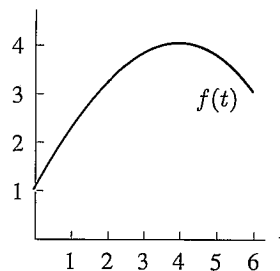


Figure 2.10



8. Suppose a particle is moving at varying velocity along a straight line and that  $s = f(t)$  represents the distance of the particle from a point as a function of time,  $t$ . Sketch a possible graph for  $f$  if the average velocity of the particle between  $t = 2$  and  $t = 6$  is the same as the instantaneous velocity at  $t = 5$ .

Estimate the limits in Problems 9–12 by substituting smaller and smaller values of  $h$ . Give your answers to one decimal place.

$$9. \lim_{h \rightarrow 0} \frac{(3+h)^3 - 27}{h} \quad 10. \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} \quad 11. \lim_{h \rightarrow 0} \frac{7^h - 1}{h} \quad 12. \lim_{h \rightarrow 0} \frac{e^{1+h} - e}{h}$$

For each of the functions in Problems 13–22, do the following four things:

- Make a table of values of  $f(x)$  for  $x = 0.1, 0.01, 0.001, 0.0001, -0.1, -0.01, -0.001, \text{ and } -0.0001$ .
- Make a conjecture about the value of  $\lim_{x \rightarrow 0} f(x)$ .
- Graph the function to see if it is consistent with your answers to parts (a) and (b).
- Find an interval for  $x$  near 0 such that the difference between your conjectured limit and the value of the function is less than 0.01. (In other words, find a window of height 0.02 such that the graph exits the sides of the window and not the top or bottom of the window.)

$$13. f(x) = 3x + 1$$

$$14. f(x) = x^2 - 1$$

$$15. f(x) = \sin 2x$$

$$16. f(x) = \sin 3x$$

$$17. f(x) = \frac{\sin 2x}{x}$$

$$18. f(x) = \frac{\sin 3x}{x}$$

$$19. f(x) = \frac{e^x - 1}{x}$$

$$20. f(x) = \frac{e^{2x} - 1}{x}$$

$$21. f(x) = \frac{\cos 2x - 1 + 2x^2}{x^3}$$

$$22. f(x) = \frac{\cos 3x - 1 + 4.5x^2}{x^3}$$

## 2.2 THE DERIVATIVE AT A POINT

### Average Rate of Change

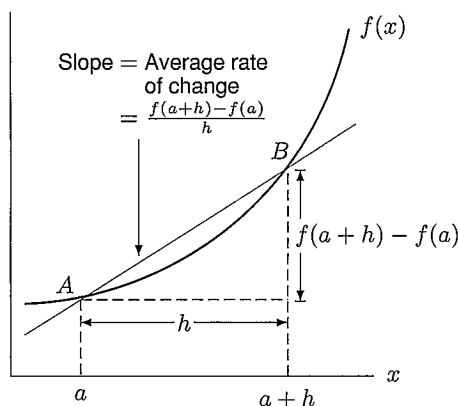
Now we will apply the analysis of Section 2.1 to any function  $y = f(x)$ , not just to height as a function of time. In the case of height, we looked at the change in height divided by the change in time, which tells us

$$\text{Average rate of change of height with respect to time} = \frac{s(a+h) - s(a)}{h}$$

This ratio is called the *difference quotient*. For any function  $f$ , we say:

$$\begin{array}{l} \text{Average rate of change} \\ \text{of } f \text{ over the interval} \\ \text{from } a \text{ to } a+h \end{array} = \frac{f(a+h) - f(a)}{h}$$

The numerator,  $f(a+h) - f(a)$ , measures the change in the value of  $f$  over the interval from  $a$  to  $a+h$ . Therefore, the difference quotient is the change in  $f$  divided by the change in  $x$ . See Figure 2.11.



**Figure 2.11:** Visualizing the average rate of change of  $f$

Although the interval is no longer necessarily a time interval, we still talk about the *average rate of change* of  $f$  over the interval. If we want to emphasize the independent variable, we talk about the average rate of change of  $f$  with respect to  $x$ .

### Average Rate of Change versus Absolute Change

The average rate of change of a function over an interval is not the same as the absolute change. Absolute change is just the difference in the values of  $f$  at the ends of the interval, that is,

$$f(a+h) - f(a).$$

The average rate of change, however, is the absolute change divided by the size of the interval,

$$\frac{f(a+h) - f(a)}{h}.$$

The average rate of change tells how quickly (or slowly) the function changes from one end of the interval to the other, relative to the size of the interval. It is often more useful to know the rate of change than the absolute change. For example, if someone offers you a \$150 return on a \$100 investment, you will want to know how long it is going to take to make that money. Just knowing the absolute change in your money, \$50, is not enough, but knowing the rate of change (i.e., \$50 divided by the time it takes to make it) will help you decide whether or not to make the investment.

### Blowing Up a Balloon

Consider the function which gives the radius of a sphere in terms of its volume. For example, think of blowing air into a balloon. You've probably noticed that a balloon seems to blow up faster at the start and then slows down as you blow more air into it. What you're seeing is variation in the rate of change of the radius with respect to volume.

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**Example 1** The volume,  $V$ , of a sphere is given by  $V = 4\pi r^3/3$ . Solving for  $r$  in terms of  $V$  gives

$$r = f(V) = \left(\frac{3V}{4\pi}\right)^{1/3}.$$

Calculate the average rate of change of  $r$  with respect to  $V$  over the intervals  $0.5 \leq V \leq 1$  and  $1 \leq V \leq 1.5$ .

**Solution** Using the formula for the average rate of change gives

$$\begin{array}{l} \text{Average rate of change} \\ \text{of radius for } 0.5 \leq V \leq 1 \end{array} = \frac{f(1) - f(0.5)}{0.5} = 2 \left( \left( \frac{3}{4\pi} \right)^{1/3} - \left( \frac{1.5}{4\pi} \right)^{1/3} \right) \approx 0.26$$

$$\begin{array}{l} \text{Average rate of change} \\ \text{of radius for } 1 \leq V \leq 1.5 \end{array} = \frac{f(1.5) - f(1)}{0.5} = 2 \left( \left( \frac{4.5}{4\pi} \right)^{1/3} - \left( \frac{3}{4\pi} \right)^{1/3} \right) \approx 0.18.$$

So we see that the rate decreases as the volume increases.

## Instantaneous Rate of Change: The Derivative

We can also define the *instantaneous rate of change* of a function at a point in the same way that we defined instantaneous velocity: we look at the average rate of change over smaller and smaller intervals. This instantaneous rate of change is so important that it is given its own name, the *derivative of  $f$  at  $a$* , denoted by  $f'(a)$ . We define it as follows:

The **derivative of  $f$  at  $a$** , written  $f'(a)$ , is defined as

$$\begin{array}{l} \text{Rate of change} \\ \text{of } f \text{ at } a \end{array} = f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

If the limit exists, then  $f$  is said to be **differentiable at  $a$** .

To emphasize that  $f'(a)$  is the rate of change of  $f(x)$  as the variable  $x$  changes, we call  $f'(a)$  the derivative of  $f$  with respect to  $x$  at  $x = a$ . When the function  $y = s(t)$  represents the position of an object, the derivative  $s'(t)$  is the velocity.

**Example 2** By choosing small values for  $h$ , estimate the instantaneous rate of change of the radius of a sphere with respect to change in volume at  $V = 1$ .

**Solution** With  $h = 0.01$  and  $h = -0.01$ , we have the difference quotients

$$\frac{f(1.01) - f(1)}{0.01} \approx 0.2061 \quad \text{and} \quad \frac{f(0.99) - f(1)}{-0.01} \approx 0.2075.$$

With  $h = 0.001$  and  $h = -0.001$ ,

$$\frac{f(1.001) - f(1)}{0.001} \approx 0.2067 \quad \text{and} \quad \frac{f(0.999) - f(1)}{-0.001} \approx 0.2069.$$

The values of these difference quotients suggest that the limit is between 0.2067 and 0.2069. We conclude that the value is about 0.207; taking smaller  $h$  values confirms this. So we say

$$\begin{array}{l} f'(1) = \\ \text{Instantaneous rate of change} \\ \text{of radius with respect} \\ \text{to volume at } V = 1 \end{array} \approx 0.207.$$

In this example we found an approximation to the instantaneous rate of change, or derivative, by substituting in smaller and smaller values of  $h$ . Now we see how to visualize the derivative.

### Visualizing the Derivative: Slope of Curve and Slope of Tangent

As with velocity, we can visualize the derivative  $f'(a)$  as the slope of the graph of  $f$  at  $x = a$ . In addition, there is another way to think of  $f'(a)$ . Consider the difference quotient  $(f(a+h) - f(a))/h$ . The numerator,  $f(a+h) - f(a)$ , is the vertical distance marked in Figure 2.12 and  $h$  is the horizontal distance, so

$$\text{Average rate of change of } f = \frac{f(a+h) - f(a)}{h} = \text{Slope of line } AB.$$

As  $h$  becomes smaller, the line  $AB$  approaches the tangent line to the curve at  $A$ . (See Figure 2.13.) We say

$$\begin{array}{l} \text{Instantaneous} \\ \text{rate of change of } f \\ \text{at } a \end{array} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \text{Slope of tangent at } A.$$

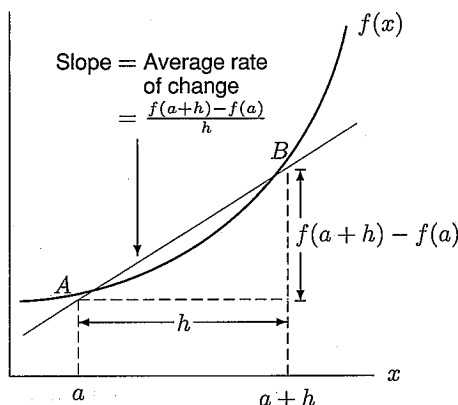


Figure 2.12: Visualizing the average rate of change of  $f$

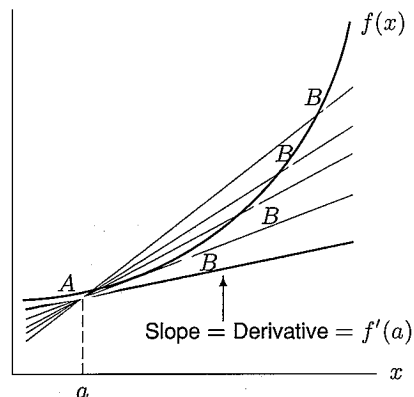


Figure 2.13: Visualizing the instantaneous rate of change of  $f$

The derivative at point  $A$  can be interpreted as:

- The slope of the curve at  $A$ .
- The slope of the tangent line to the curve at  $A$ .

The slope interpretation is often useful in gaining rough information about the derivative, as the following examples show.

**Example 3** Is the derivative of  $\sin x$  at  $x = \pi$  positive or negative?

**Solution** Looking at a graph of  $\sin x$  in Figure 2.14 (remember,  $x$  is in radians), we see that a tangent line drawn at  $x = \pi$  has negative slope. So the derivative at this point is negative.

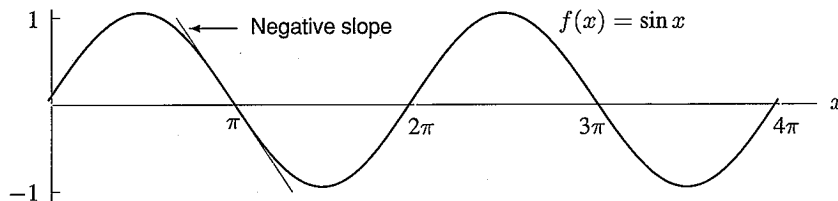


Figure 2.14: Tangent line to  $\sin x$  at  $x = \pi$

Recall that if we zoom in on the graph of a function  $y = f(x)$  at the point where  $x = a$ , we usually find that the graph looks more and more like a straight line with slope  $f'(a)$ .

**Example 4** By zooming in on the point  $(0, 0)$  on the graph of the sine function, estimate the value of the derivative of  $\sin x$  at  $x = 0$ , with  $x$  in radians.

**Solution** Figure 2.15 shows successive graphs of  $\sin x$ , with smaller and smaller scales. On the interval  $-0.1 \leq x \leq 0.1$ , the graph looks like a straight line of slope 1. Thus, the derivative of  $\sin x$  at  $x = 0$  is about 1.

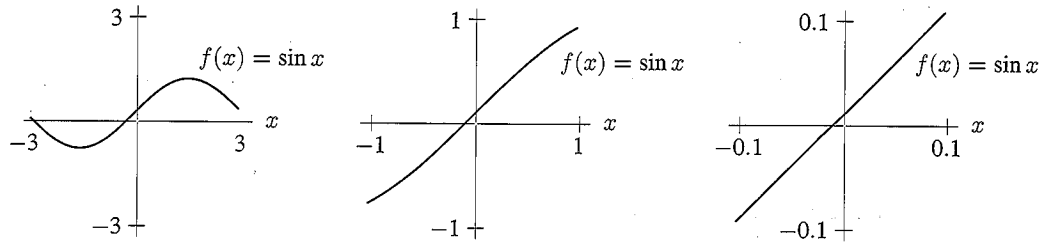


Figure 2.15: Zooming in on the graph of  $\sin x$  near  $x = 0$

Later we will show that the derivative of  $\sin x$  at  $x = 0$  is exactly 1. (See page 216 in Section 4.5.) From now on we will assume that this is so.

**Example 5** Use the tangent line at  $x = 0$  to estimate values of  $\sin x$  near  $x = 0$ .

**Solution** In the previous example we see that near  $x = 0$ , the graph of  $y = \sin x$  looks like the graph of the straight line  $y = x$ ; we can use this line to estimate values of  $\sin x$  when  $x$  is close to 0. For example, the point on the straight line  $y = x$  with  $x$  coordinate 0.32 is  $(0.32, 0.32)$ . Since the line is close to the graph of  $y = \sin x$ , we estimate that  $\sin 0.32 \approx 0.32$ . (See Figure 2.16.) Checking on a calculator, we find that  $\sin 0.32 \approx 0.3146$ , so our estimate is quite close. Notice that the graph suggests that the real value of  $\sin 0.32$  is slightly less than 0.32.

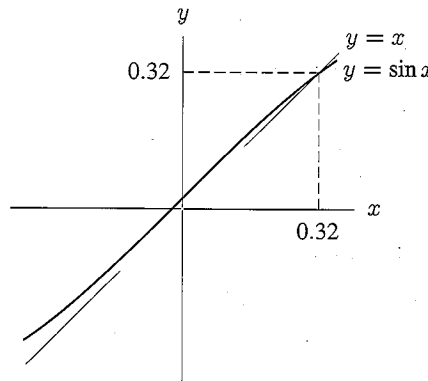


Figure 2.16: Approximating  $y = \sin x$  by  $y = x$

### Why Do We Use Radians and Not Degrees?

After Example 4 we stated that the derivative of  $\sin x$  at  $x = 0$  is 1, when  $x$  is in radians. This is the reason we choose to use radians. If we had done Example 4 in degrees, the derivative of  $\sin x$  would have turned out to be a much messier number. (See Problem 25, page 104.)

## Estimating the Derivative of an Exponential Function

**Example 6** Estimate the value of the derivative of  $f(x) = 2^x$  at  $x = 0$  graphically and numerically.

**Solution** Graphically: Figure 2.17 indicates that the graph is concave up. Assuming this, the slope at  $A$  is between the slope of  $BA$  and the slope of  $AC$ . Since

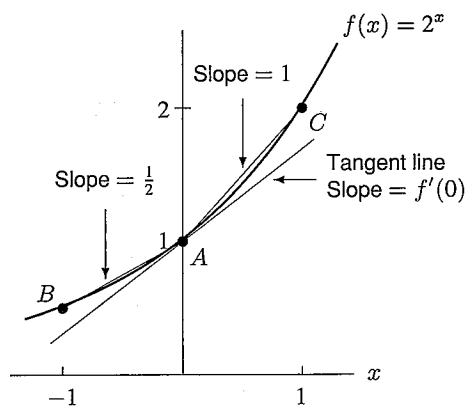
$$\text{Slope of line } BA = \frac{(2^0 - 2^{-1})}{(0 - (-1))} = \frac{1}{2} \quad \text{and} \quad \text{Slope of line } AC = \frac{(2^1 - 2^0)}{(1 - 0)} = 1,$$

we know that the derivative is between  $1/2$  and  $1$ .

Numerically: To estimate the derivative at  $x = 0$ , we need to look at values of the difference quotient

$$\frac{f(0+h) - f(0)}{h} = \frac{2^h - 2^0}{h} = \frac{2^h - 1}{h}$$

for small  $h$ . Table 2.4 shows some values of  $2^h$  together with values of the difference quotients. (See Problem 33 on page 104 for what happens for very small values of  $h$ .)



**Figure 2.17:** Graph of  $y = 2^x$  showing the derivative at  $x = 0$

**TABLE 2.4** Numerical values for difference quotient of  $2^x$  at  $x = 0$

$h$	$2^h$	Difference quotient: $\frac{2^h - 1}{h}$
-0.0003	0.999792078	0.693075
-0.0002	0.999861380	0.693099
-0.0001	0.999930688	0.693123
0	1	
0.0001	1.00006932	0.693171
0.0002	1.00013864	0.693195
0.0003	1.00020797	0.693219

Our concavity assumption tells us that difference quotients calculated with negative  $h$ 's are smaller than the derivative, and those calculated with positive  $h$ 's are larger. From Table 2.4 we see that the derivative is between  $0.693123$  and  $0.693171$ . To three decimal places,  $f'(0) = 0.693$ .

**Example 7** Find an approximate equation for the tangent line to  $f(x) = 2^x$  at  $x = 0$ .

**Solution** From the previous example, we know the slope of the tangent line is about  $0.693$ . Since we also know the line has  $y$ -intercept  $1$ , its equation is

$$y = 0.693x + 1.$$

## Computing the Derivative of $x^2$

In previous examples we got an approximation to the derivative by using smaller and smaller values of  $h$ . We now see how to find the derivative of  $x^2$  exactly.

**Example 8** Find the derivative of the function  $f(x) = x^2$  at the point  $x = 1$ .

**Solution** We need to look at

$$f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

This is the same as

$$\lim_{h \rightarrow 0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h \rightarrow 0} \frac{(1+2h+h^2) - 1}{h} = \lim_{h \rightarrow 0} \frac{2h+h^2}{h}$$

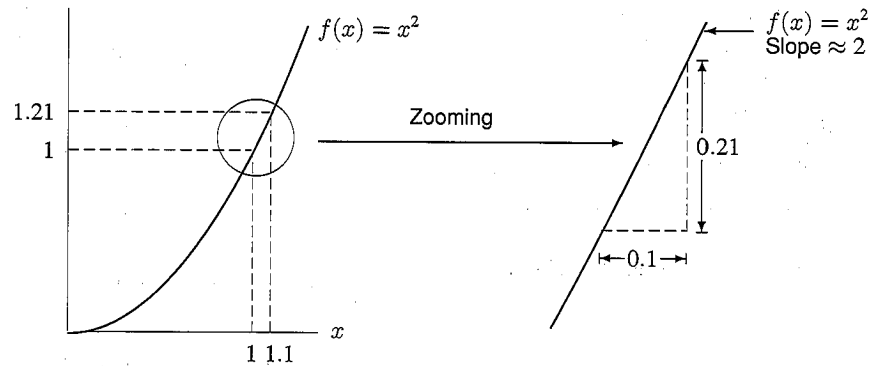
Since the limit only examines values of  $h$  close to, but not equal to zero, we can divide by  $h$  in the expression  $(2h + h^2)/h$ . We get

$$\lim_{h \rightarrow 0} \frac{h(2+h)}{h} = \lim_{h \rightarrow 0} (2+h)$$

This limit is 2, so  $f'(1) = 2$ . At  $x = 1$  the rate of change of  $x^2$  is 2.

Since the derivative is the rate of change,  $f'(1) = 2$  means that for small changes in  $x$  near  $x = 1$ , the change in  $f(x) = x^2$  is about twice as big as the change in  $x$ . As an example, if  $x$  changes from 1 to 1.1, a net change of 0.1, then  $f(x)$  changes by about 0.2. Figure 2.18 shows this geometrically.

Table 2.5 shows the derivative of  $f(x) = x^2$  numerically. Notice that near  $x = 1$ , every time the value of  $x$  increases by 0.001, the value of  $x^2$  increases by approximately 0.002. Near  $x = 1$  the function is approximately linear with slope  $0.002/0.001 = 2$ .



**Figure 2.18:** Graph of  $f(x) = x^2$  near  $x = 1$  has slope  $\approx 2$

**TABLE 2.5** Values of  $f(x) = x^2$  near  $x = 1$

$x$	$x^2$	Difference in successive $x^2$ values
0.998	0.996004	0.001997
0.999	0.998001	0.001999
1.000	1.000000	0.002001
1.001	1.002001	0.002003
1.002	1.004004	
↑		↑
$x$ increments of 0.001		All approximately 0.002

### Problems for Section 2.2

- Sketch a rough graph of  $f(x) = \sin x$ , and use the graph to decide whether the derivative of  $f(x)$  at  $x = 3\pi$  is positive or negative. Give reasons for your decision.
- Make a table of values rounded to two decimal places for the function  $f(x) = e^x$  for  $x = 1, 1.5, 2, 2.5,$  and  $3$ . Then use the table to answer parts (b) and (c).
  - Find the average rate of change of  $f(x)$  between  $x = 1$  and  $x = 3$ .
  - Use average rates of change to approximate the instantaneous rate of change of  $f(x)$  at  $x = 2$ .
- Label points  $A, B, C, D, E,$  and  $F$  on the graph of  $y = f(x)$  in Figure 2.19.
  - Point  $A$  is a point on the curve where the derivative is negative.
  - Point  $B$  is a point on the curve where the value of the function is negative.
  - Point  $C$  is a point on the curve where the derivative is largest.
  - Point  $D$  is a point on the curve where the derivative is zero.
  - Points  $E$  and  $F$  are different points on the curve where the derivative is about the same.

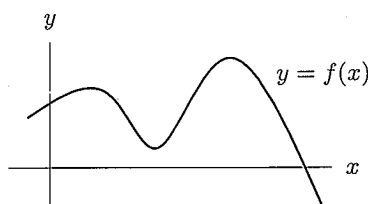


Figure 2.19

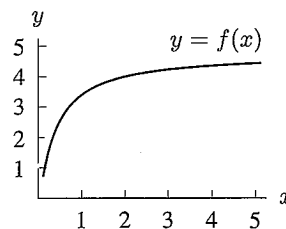


Figure 2.20

- The graph of  $y = f(x)$  is shown in Figure 2.20. Which is larger in each of the following pairs?
  - Average rate of change: Between  $x = 1$  and  $x = 3$ ? Or between  $x = 3$  and  $x = 5$ ?
  - $f(2)$  or  $f(5)$ ?
  - $f'(1)$  or  $f'(4)$ ?
- On a copy of Figure 2.21, mark lengths that represent the quantities in parts (a)–(d). (Pick any convenient  $x$ , and assume  $h > 0$ .)
  - $f(x)$
  - $f(x+h)$
  - $f(x+h) - f(x)$
  - $h$

(e) Using your answers to parts (a)–(d), show how the quantity  $\frac{f(x+h) - f(x)}{h}$  can be represented as the slope of a line on the graph.

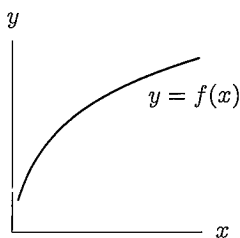


Figure 2.21

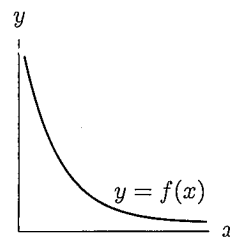


Figure 2.22

- On a copy of Figure 2.22, mark lengths that represent the quantities in parts (a)–(d). (Pick any convenient  $x$ , and assume  $h > 0$ .)
  - $f(x)$
  - $f(x+h)$
  - $f(x+h) - f(x)$
  - $h$

(e) Using your answers to parts (a)–(d), show how the quantity  $\frac{f(x+h) - f(x)}{h}$  can be represented as the slope of a line on the graph.



7. Show how you can represent the following on a copy of Figure 2.23.
- (a)  $f(4)$       (b)  $f(4) - f(2)$       (c)  $\frac{f(5) - f(2)}{5 - 2}$       (d)  $f'(3)$
8. Consider the function  $y = f(x)$  shown in Figure 2.23. For each of the following pairs of numbers, decide which is larger. Explain your answer.
- (a)  $f(3)$  or  $f(4)$ ?      (b)  $f(3) - f(2)$  or  $f(2) - f(1)$ ?
- (c)  $\frac{f(2) - f(1)}{2 - 1}$  or  $\frac{f(3) - f(1)}{3 - 1}$ ?      (d)  $f'(1)$  or  $f'(4)$ ?
9. With the function  $f$  given by Figure 2.23, arrange the following quantities in ascending order:
- 0, 1,  $f'(2)$ ,  $f'(3)$ ,  $f(3) - f(2)$
10. Suppose  $y = f(x)$  graphed in Figure 2.23 represents the cost of manufacturing  $x$  kilograms of a chemical. Then  $f(x)/x$  represents the average cost of producing 1 kilogram when  $x$  kilograms are made. This problem asks you to visualize these averages graphically.
- (a) Show how to represent  $f(4)/4$  as the slope of a line.
- (b) Which is larger,  $f(3)/3$  or  $f(4)/4$ ?

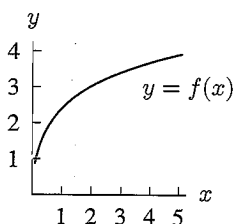


Figure 2.23

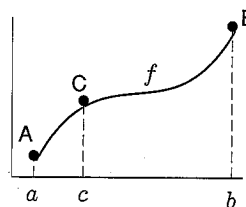


Figure 2.24

11. Consider the function shown in Figure 2.24.
- (a) Write an expression involving  $f$  for the slope of the line joining  $A$  and  $B$ .
- (b) Draw the tangent line at  $C$ . Compare its slope to the slope of the line in part (a).
- (c) Are there any other points on the curve at which the slope of the tangent line is the same as the slope of the tangent line at  $C$ ? If so, mark them on the graph. If not, why not?
12. Table 2.6 shows values of  $f(x) = x^3$  near  $x = 2$  (to three decimal places). Use it to estimate  $f'(2)$ .

TABLE 2.6

$x$	1.998	1.999	2.000	2.001	2.002
$x^3$	7.976	7.988	8.000	8.012	8.024

Find the derivatives in Problem 13–18 algebraically.

13.  $g(t) = 3t^2 + 5t$  at  $t = -1$       14.  $f(x) = 5x^2$  at  $x = 10$       15.  $f(x) = x^3$  at  $x = -2$
16.  $f(x) = x^3 + 5$  at  $x = 1$       17.  $g(x) = 1/x$  at  $x = 2$       18.  $g(z) = z^{-2}$ , find  $g'(2)$

For Problems 19–22, find the equation of the line tangent to the function at the given point.

19.  $f(x) = x^3$  at  $x = -2$       20.  $f(x) = 5x^2$  at  $x = 10$
21.  $f(x) = x$  at  $x = 20$       22.  $f(x) = 1/x^2$  at  $(1, 1)$
23. Find the value of the derivative of  $f(x) = x^2 + 1$  at  $x = 3$  algebraically. Find the equation of the tangent line to  $f$  at  $x = 3$ .
24. Find the equation of the tangent line to  $f(x) = x^2 + x$  at  $x = 3$ . Sketch a graph of the function and this tangent line.

25. (a) Estimate  $f'(0)$  if  $f(x) = \sin x$ , with  $x$  in degrees.  
 (b) In Example 4 on page 99, we found that the derivative of  $\sin x$  at  $x = 0$  was 1. Why do we get a different result here? (This problem shows why radians are almost always used in calculus.)
26. Estimate the derivative of  $f(x) = x^x$  at  $x = 2$ .
27. For  $y = f(x) = 3x^{3/2} - x$ , use your calculator to construct a graph of  $y = f(x)$ , for  $0 \leq x \leq 2$ . From your graph, estimate  $f'(0)$  and  $f'(1)$ .
28. Let  $f(x) = \ln(\cos x)$ . Use your calculator to approximate the instantaneous rate of change of  $f$  at the point  $x = 1$ . Do the same thing for  $x = \pi/4$ . (Note: Be sure that your calculator is set in radians.)
29. There is a function called the error function,  $y = \operatorname{erf}(x)$ . Suppose your calculator has a button for  $\operatorname{erf}(x)$  that gives the following values:

$$\operatorname{erf}(0) = 0 \quad \operatorname{erf}(1) = 0.84270079 \quad \operatorname{erf}(0.1) = 0.11246292 \quad \operatorname{erf}(0.01) = 0.01128342.$$

- (a) Use all this information to determine your best estimate for  $\operatorname{erf}'(0)$ . (Give only those digits of which you feel reasonably certain.)  
 (b) Suppose you find that  $\operatorname{erf}(0.001) = 0.00112838$ . How does this extra information change your answer to part (a)?
30. (a) Use your calculator to approximate the derivative of the hyperbolic sine function (written  $\sinh x$ ) at the points 0, 0.3, 0.7, and 1.  
 (b) Can you find a relation between the values of this derivative and the values of the hyperbolic cosine (written  $\cosh x$ )?
31. The population,  $P$ , of China, in billions, can be approximated by the function

$$P = 1.15(1.014)^t,$$

where  $t$  is the number of years since the start of 1993. According to this model, how fast is the population growing at the start of 1993 and at the start of 1995? Give your answers in millions of people per year.

32. (a) Sketch graphs of the functions  $f(x) = \frac{1}{2}x^2$  and  $g(x) = f(x) + 3$  on the same set of axes. What can you say about the slopes of the tangent lines to the two graphs at the point  $x = 0$ ?  $x = 2$ ? Any point  $x = x_0$ ?  
 (b) Explain why adding a constant value,  $C$ , to any function does not change the value of the slope of its graph at any point. [Hint: Let  $g(x) = f(x) + C$ , and calculate the difference quotients for  $f$  and  $g$ .]
33. Suppose Table 2.4 on page 100 is continued with smaller values of  $h$ . A particular calculator gives the results in Table 2.7. (Your calculator may give slightly different results.) Comment on the values of the difference quotient in Table 2.7. In particular, why is the last value of  $(2^h - 1)/h$  zero? What do you expect the calculated value of  $(2^h - 1)/h$  to be when  $h = 10^{-20}$ ?

**TABLE 2.7** Questionable values of difference quotients of  $2^x$  near  $x = 0$

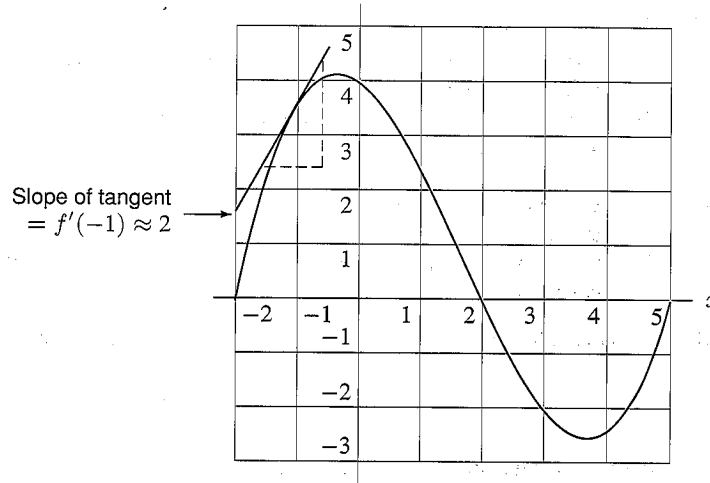
$h$	Difference quotient: $(2^h - 1)/h$
$10^{-4}$	0.6931712
$10^{-6}$	0.693147
$10^{-8}$	0.6931
$10^{-10}$	0.69
$10^{-12}$	0

## 2.3 THE DERIVATIVE FUNCTION

In the last section we looked at the derivative of a function at a fixed point. Now we consider what happens at a variety of points. The derivative generally takes on different values at different points and is itself a function.

First, remember that the derivative of a function at a point tells us the rate at which the value of the function is changing at that point. Geometrically, if we “zoom in” on a point in the graph until it looks like a straight line, the slope of that line is the derivative at that point. Equivalently, we can think of the derivative as the slope of the tangent line at the point, because as we “zoom in,” the curve and the tangent line become indistinguishable.

**Example 1** Estimate the derivative of the function  $f(x)$  graphed in Figure 2.25 at  $x = -2, -1, 0, 1, 2, 3, 4, 5$ .



**Figure 2.25:** Estimating the derivative graphically as the slope of the tangent line

**Solution** From the graph we estimate the derivative at any point by placing a straightedge so that it forms the tangent line at that point, and then using the grid squares to estimate the slope of the straightedge. For example, the tangent at  $x = -1$  is drawn in Figure 2.25, and has a slope of about 2, so  $f'(-1) \approx 2$ . Notice that the slope at  $x = -2$  is positive and fairly large; the slope at  $x = -1$  is positive but smaller. At  $x = 0$ , the slope is negative, by  $x = 1$  it has become more negative, and so on. Some estimates of the derivative are listed in Table 2.8. You should check these values. Are they reasonable? Is the derivative positive where you expect? Negative?

**TABLE 2.8** Estimated values of derivative of function in Figure 2.25

$x$	-2	-1	0	1	2	3	4	5
$f'(x)$	6	2	-1	-2	-2	-1	1	4

The important point to notice is that for every  $x$ -value, there's a corresponding value of the derivative. Therefore, the derivative is itself a function of  $x$ .

For any function  $f$ , we define the **derivative function**,  $f'$ , by

$$f'(x) = \text{Rate of change of } f \text{ at } x = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

For every  $x$ -value for which this limit exists, we say  $f$  is *differentiable at that  $x$ -value*. If the limit exists for all  $x$  in the domain of  $f$ , we say  $f$  is *differentiable everywhere*. The functions we shall deal with will be differentiable at every point in their domain, except perhaps for a few isolated points.

## The Derivative Function: Graphically

**Example 2** Sketch the graph of the derivative of the function shown in Figure 2.25.

**Solution** We plot the values of this derivative given in Table 2.8. We obtain Figure 2.26, which shows a graph of the derivative (the black curve), along with the original function (color).

You should check for yourself that this graph of  $f'$  makes sense. The values of  $f'$  are positive where  $f$  is increasing ( $x < -0.3$  or  $x > 3.8$ ) and negative where  $f$  is decreasing. Notice that at the points where  $f$  has large positive slope, such as  $x = -2$ , the graph of the derivative is far above the  $x$ -axis, as it should be, since the value of the derivative is large there. On the other hand, at points where the slope is gentler, such as  $x = -1$ , the graph of  $f'$  is closer to the  $x$ -axis, since the derivative is smaller.

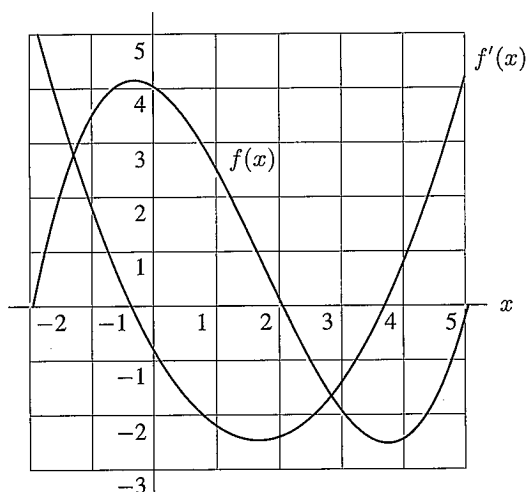


Figure 2.26: Function (colored) and derivative (black) from Example 1

## What Does the Derivative Tell Us Graphically?

When  $f'$  is positive, the tangent line to  $f$  is sloping up; when  $f'$  is negative, the tangent line to  $f$  is sloping down. If  $f' = 0$  everywhere, then the tangent line to  $f$  is horizontal everywhere, and so  $f$  is constant. We see that the sign of  $f'$  tells us whether  $f$  is increasing or decreasing.

If  $f' > 0$  on an interval, then  $f$  is *increasing* over that interval.  
If  $f' < 0$  on an interval, then  $f$  is *decreasing* over that interval.

Moreover, the magnitude of the derivative gives us the magnitude of the rate of change; so if  $f'$  is large (positive or negative), then the graph of  $f$  is steep (up or down), whereas if  $f'$  is small the graph of  $f$  slopes gently. With this in mind, we can deduce a lot about the behavior of a function from the behavior of its derivative.

## The Derivative Function: Numerically

If we are given a table of function values instead of a graph of the function, we can estimate values of the derivative.

**Example 3** Table 2.9 gives values of  $c(t)$ , the concentration ( $\mu\text{g}/\text{cm}^3$ ) of a drug in the bloodstream at time  $t$  (min). Construct a table of estimated values for  $c'(t)$ , the rate of change of  $c(t)$  with respect to time.

**TABLE 2.9** Concentration as a function of time

$t$ (min)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$c(t)$ ( $\mu\text{g}/\text{cm}^3$ )	0.84	0.89	0.94	0.98	1.00	1.00	0.97	0.90	0.79	0.63	0.41

**Solution** We want to estimate values of  $c'$  using the values in the table. To do this, we have to assume that the data points are close enough together that the concentration doesn't change wildly between them. From the table, we can see that the concentration is increasing between  $t = 0$  and  $t = 0.4$ , so we expect a positive derivative there. However, the increase is quite slow, so we expect the derivative to be small. The concentration doesn't change between 0.4 and 0.5, so we expect the derivative to be roughly 0 there. From  $t = 0.5$  to  $t = 1.0$ , the concentration starts to decrease, and the rate of decrease gets larger and larger, so we expect the derivative to be negative and of greater and greater magnitude.

Using the data in the table, we can estimate the derivative using the difference quotient:

$$c'(t) \approx \frac{c(t+h) - c(t)}{h}$$

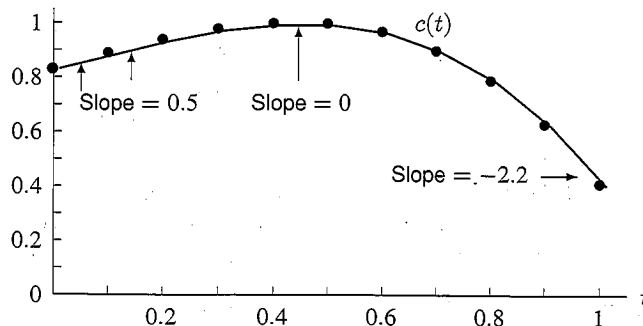
Since the data points are 0.1 apart, we use  $h = 0.1$ . We estimate that,

$$\begin{aligned} c'(0) &\approx \frac{c(0.1) - c(0)}{0.1} = \frac{0.89 - 0.84}{0.1} = 0.5 \mu\text{g}/\text{cm}^3/\text{min} \\ c'(0.1) &\approx \frac{c(0.2) - c(0.1)}{0.1} = \frac{0.94 - 0.89}{0.1} = 0.5 \mu\text{g}/\text{cm}^3/\text{min} \\ c'(0.2) &\approx \frac{c(0.3) - c(0.2)}{0.1} = \frac{0.98 - 0.94}{0.1} = 0.4 \mu\text{g}/\text{cm}^3/\text{min} \\ c'(0.3) &\approx \frac{c(0.4) - c(0.3)}{0.1} = \frac{1.00 - 0.98}{0.1} = 0.2 \mu\text{g}/\text{cm}^3/\text{min} \\ c'(0.4) &\approx \frac{c(0.5) - c(0.4)}{0.1} = \frac{1.00 - 1.00}{0.1} = 0.0 \mu\text{g}/\text{cm}^3/\text{min} \end{aligned}$$

and so on. These values are tabulated in Table 2.10. Notice that the derivative has small positive values up until  $t = 0.4$ , where it is roughly 0, and then it gets more and more negative, as we expected. The slopes are shown on the graph of  $c(t)$  in Figure 2.27.

**TABLE 2.10**  
Estimated  
derivative of  
concentration

$t$	$c'(t)$
0	0.5
0.1	0.5
0.2	0.4
0.3	0.2
0.4	0.0
0.5	-0.3
0.6	-0.7
0.7	-1.1
0.8	-1.6
0.9	-2.2



**Figure 2.27:** Graph of concentration as a function of time

### Improving Numerical Estimates for the Derivative

In the previous example, the estimate for the derivative at 0.2 used the interval to the right; we found the average rate of change between  $t = 0.2$  and  $t = 0.3$ . However, we could equally well have gone to the left and used the rate of change between  $t = 0.1$  and  $t = 0.2$  to approximate the derivative at 0.2. For a more accurate result, we could average these slopes and say

$$c'(0.2) \approx \frac{1}{2} \left( \begin{array}{c} \text{Slope to left} \\ \text{of 0.2} \end{array} + \begin{array}{c} \text{Slope to right} \\ \text{of 0.2} \end{array} \right) = \frac{0.5 + 0.4}{2} = 0.45.$$

In general, averaging the slopes leads to a more accurate answer.

### Derivative Function: From a Formula

If we are given a formula for  $f$ , can we come up with a formula for  $f'$ ? We often can, as shown in the next example. Indeed, much of the power of calculus depends on our ability to find formulas for the derivatives of all the functions we described earlier. This is done systematically in Chapter 4.

#### Derivative of a Constant Function

The graph of a constant function  $f(x) = k$  is a horizontal line, with a slope of 0 everywhere. Therefore, its derivative is 0 everywhere. (See Figure 2.28.)

If  $f(x) = k$ , then  $f'(x) = 0$ .

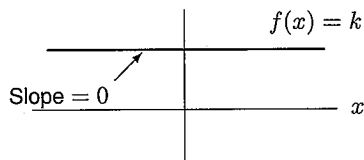


Figure 2.28: A constant function

#### Derivative of a Linear Function

We already know that the slope of a straight line is constant. This tells us that the derivative of a linear function is constant.

If  $f(x) = b + mx$ , then  $f'(x) = \text{Slope} = m$ .

#### Derivative of a Power Function

**Example 4** Find a formula for the derivative of  $f(x) = x^2$ .

**Solution** Before computing the formula for  $f'(x)$  algebraically, let's try to guess the formula by looking for a pattern in the values of  $f'(x)$ . Table 2.11 contains values of  $f(x) = x^2$  (rounded to three decimals), which we can use to estimate the values of  $f'(1)$ ,  $f'(2)$ , and  $f'(3)$ .

**TABLE 2.11** Values of  $f(x) = x^2$  near  $x = 1$ ,  $x = 2$ ,  $x = 3$  (rounded to three decimals)

$x$	$x^2$	$x$	$x^2$	$x$	$x^2$
0.999	0.998	1.999	3.996	2.999	8.994
1.000	1.000	2.000	4.000	3.000	9.000
1.001	1.002	2.001	4.004	3.001	9.006
1.002	1.004	2.002	4.008	3.002	9.012

Near  $x = 1$ , the value of  $x^2$  increases by about 0.002 each time  $x$  increases by 0.001, so

$$f'(1) \approx \frac{0.002}{0.001} = 2.$$

Similarly, near  $x = 2$  and  $x = 3$ , the value of  $x^2$  increases by about 0.004 and 0.006, respectively, when  $x$  increases by 0.001. So

$$f'(2) \approx \frac{0.004}{0.001} = 4 \quad \text{and} \quad f'(3) \approx \frac{0.006}{0.001} = 6.$$

Knowing the value of  $f'$  at specific points can never tell us the formula for  $f'$ , but it certainly can be suggestive: Knowing  $f'(1) \approx 2$ ,  $f'(2) \approx 4$ ,  $f'(3) \approx 6$  suggests that  $f'(x) = 2x$ .

The derivative is calculated by forming the difference quotient and taking the limit as  $h$  goes to zero. The difference quotient is

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h}.$$

Since  $h$  never actually reaches zero, we can divide it out in the last expression to get  $2x + h$ . The limit of this as  $h$  goes to zero is  $2x$ , so

$$f'(x) = \lim_{h \rightarrow 0} (2x + h) = 2x.$$

**Example 5** Calculate  $f'(x)$  if  $f(x) = x^3$ .

**Solution** We look at the difference quotient

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h}.$$

Multiplying out gives  $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$ , so

$$f'(x) = \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h}.$$

Since in taking the limit as  $h \rightarrow 0$ , we consider values of  $h$  near, but not equal to, zero, we can divide by  $h$  giving

$$f'(x) = \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2).$$

As  $h \rightarrow 0$ , the value of  $(3xh + h^2) \rightarrow 0$  so

$$f'(x) = \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) = 3x^2.$$

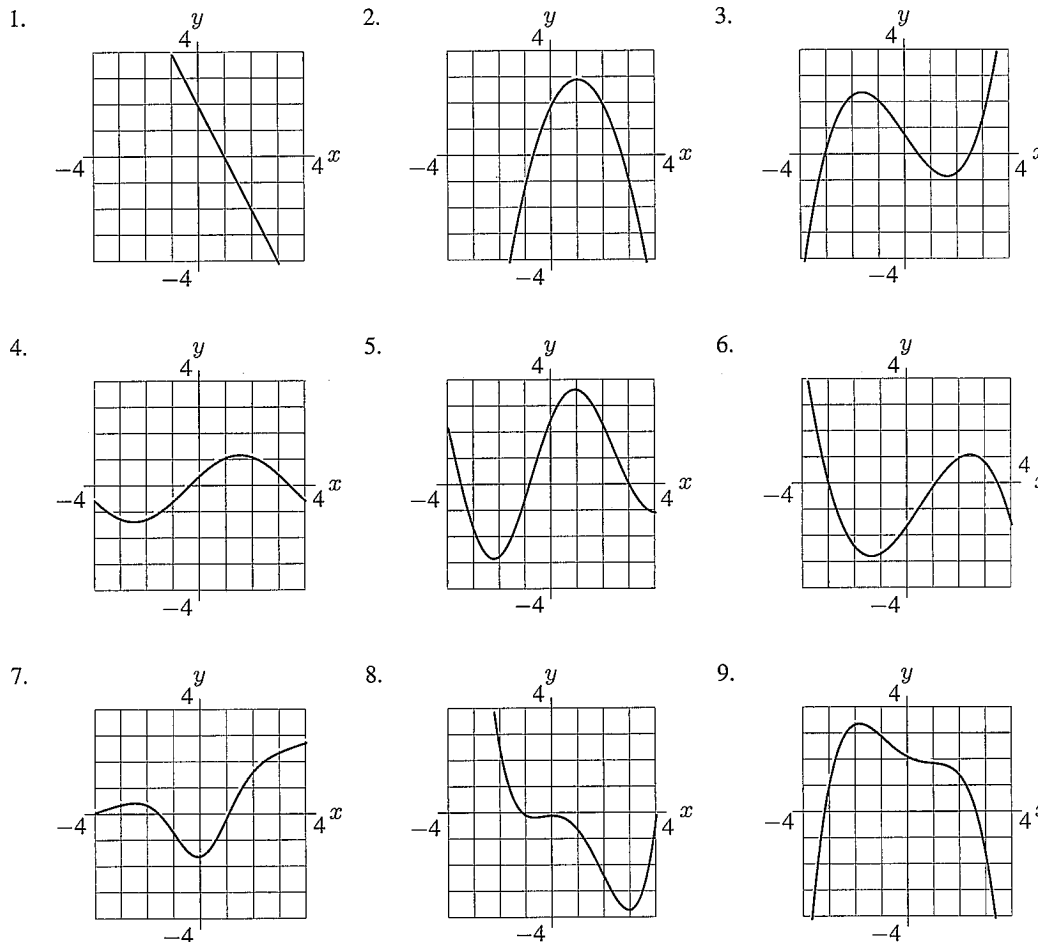
The previous two examples show how to compute the derivatives of power functions of the form  $f(x) = x^n$ , when  $n$  is 2 or 3. We can use the Binomial Theorem (from the Focus on Theory section on page 78) to show that, for  $n$  a positive integer,

$$\text{If } f(x) = x^n \text{ then } f'(x) = nx^{n-1}.$$

This result is in fact valid for any real value of  $n$ .

**Problems for Section 2.3**

For Problems 1–9, sketch a graph of the derivative function of each of the given functions.



10. (a) Sketch a smooth curve whose slope is both everywhere positive and increasing gradually.  
 (b) Sketch a smooth curve whose slope is both everywhere positive and decreasing gradually.  
 (c) Sketch a smooth curve whose slope is both everywhere negative and increasing gradually (i.e., becoming less and less negative).  
 (d) Sketch a smooth curve whose slope is both everywhere negative and decreasing gradually (i.e., becoming more and more negative).
11. Given the numerical values shown, find approximate values for the derivative of  $f(x)$  at each of the  $x$ -values given. Where is the rate of change of  $f(x)$  positive? Where is it negative? Where does the rate of change of  $f(x)$  seem to be greatest?

$x$	0	1	2	3	4	5	6	7	8
$f(x)$	18	13	10	9	9	11	15	21	30

12. The values of  $x$  and the corresponding values of  $g(x)$  are given in the table. For what value of  $x$  is  $g'(x)$  closest to 3?

$x$	2.7	3.2	3.7	4.2	4.7	5.2	5.7	6.2
$g(x)$	3.4	4.4	5.0	5.4	6.0	7.4	9.0	11.0



Find a formula for the derivatives of the functions in Problems 13–16 algebraically.

13.  $g(x) = 2x^2 - 3$     14.  $k(x) = 1/x$     15.  $l(x) = 1/x^2$     16.  $m(x) = 1/(x + 1)$

17. Draw the graph of a continuous function  $y = f(x)$  that satisfies the following three conditions.

- $f'(x) > 0$  for  $x < -2$ ,
- $f'(x) < 0$  for  $-2 < x < 2$ ,
- $f'(x) = 0$  for  $x > 2$ .

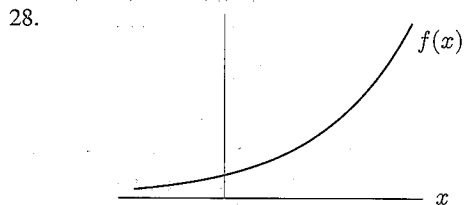
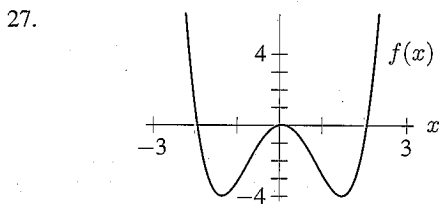
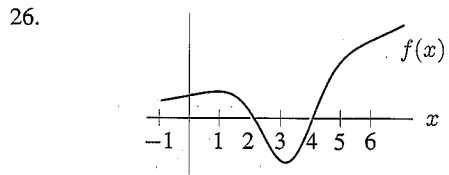
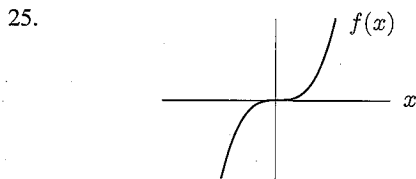
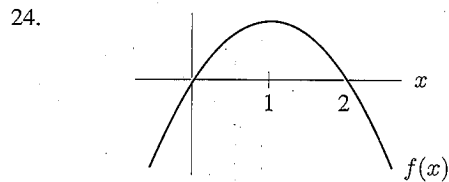
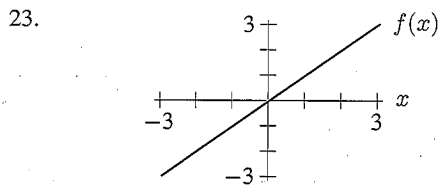
18. Draw the graph of a continuous function  $y = f(x)$  that satisfies the following three conditions.

- $f'(x) > 0$  for  $-\frac{\pi}{2} < x < \frac{\pi}{2}$ ,
- $f'(x) < 0$  for  $-\pi < x < -\frac{\pi}{2}$  and  $\frac{\pi}{2} < x < \pi$ ,
- $f'(x) = 0$  at  $x = -\frac{\pi}{2}$  and  $x = \frac{\pi}{2}$ .

For Problems 19–22, sketch the graph of  $f(x)$ , and use this graph to sketch the graph of  $f'(x)$ .

19.  $f(x) = x^2$     20.  $f(x) = x(x - 1)$     21.  $f(x) = \cos x$     22.  $f(x) = \log x$

For Problems 23–28, sketch the graph of  $y = f'(x)$  for the function given.



29. In the graph of  $f$  in Figure 2.29, at which of the labeled  $x$ -values is

- |                       |                    |
|-----------------------|--------------------|
| (a) $f(x)$ greatest?  | (b) $f(x)$ least?  |
| (c) $f'(x)$ greatest? | (d) $f'(x)$ least? |

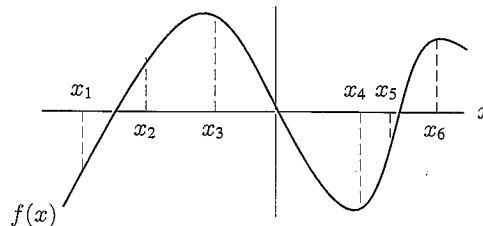


Figure 2.29

30. Consider a vehicle moving along a straight road. Suppose  $f(t)$  gives the vehicle's distance from its starting point at time  $t$ . Which of the graphs in Figure 2.30 could be  $f'(t)$  for the following scenarios? (Assume the scales on the vertical axes are all the same.)
- A bus on a popular route, with no traffic
  - A car with no traffic and all green lights
  - A car in heavy traffic conditions

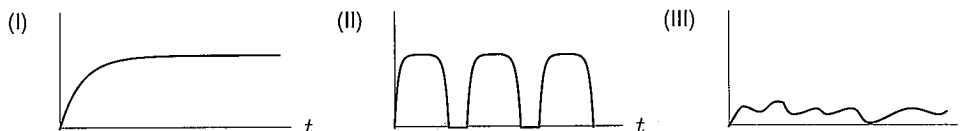


Figure 2.30

31. A child inflates a balloon, admires it for a while and then lets the air out at a constant rate. If  $V(t)$  gives the volume of the balloon at time  $t$ , then Figure 2.31 shows  $V'(t)$  as a function of  $t$ . At what time does the child:
- Begin to inflate the balloon?
  - Finish inflating the balloon?
  - Begin to let the air out?
  - What would the graph of  $V'(t)$  look like if the child had alternated between pinching and releasing the open end of the balloon, instead of letting the air out at a constant rate?

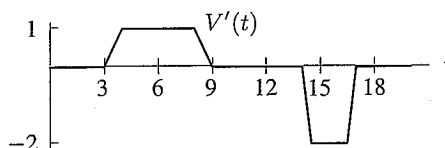


Figure 2.31

32. The population of a herd of deer is modeled by

$$P(t) = 4000 + 500 \sin\left(2\pi t - \frac{\pi}{2}\right)$$

where  $t$  is measured in years from January 1.

- How does this population vary with time? Sketch a graph of  $P(t)$  for one year.
  - Use the graph to decide when in the year the population is a maximum. What is that maximum? Is there a minimum? If so, when?
  - Use the graph to decide when the population is growing fastest. When is it decreasing fastest?
  - Estimate roughly how fast the population is changing on the first of July.
33. Looking at the graph, explain why if  $f(x)$  is an even function, then  $f'(x)$  is odd.
34. Looking at the graph, explain why if  $g(x)$  is an odd function, then  $g'(x)$  is even.

## 2.4 INTERPRETATIONS OF THE DERIVATIVE

We have already seen how the derivative can be interpreted as a slope and as a rate of change. In this section, we will see examples of other interpretations. The purpose of these examples is not to make a catalog of interpretations but to illustrate the process of obtaining them.

## An Alternative Notation for the Derivative

So far we have used the notation  $f'$  to stand for the derivative of the function  $f$ . An alternative notation for derivatives was introduced by the German mathematician Wilhelm Gottfried Leibniz (1646–1716) when calculus was first being developed. If the variable  $y$  depends on the variable  $x$ , that is, if

$$y = f(x),$$

then he wrote  $dy/dx$  for the derivative, so

$$\frac{dy}{dx} = f'(x).$$

Leibniz's notation is quite suggestive, especially if we think of the letter  $d$  in  $dy/dx$  as standing for "small difference in . . ." The notation  $dy/dx$  reminds us that the derivative is a limit of ratios of the form

$$\frac{\text{Difference in } y\text{-values}}{\text{Difference in } x\text{-values}}$$

The notation  $dy/dx$  is useful for determining the units for the derivative: The units for  $dy/dx$  are the units for  $y$  divided by the units for  $x$ . The separate entities  $dy$  and  $dx$  officially have no independent meaning: they are all part of one notation. In fact, a good formal way to view the notation  $dy/dx$  is to think of  $d/dx$  as a single symbol meaning "the derivative with respect to  $x$  of . . .". So  $dy/dx$  can be viewed as

$$\frac{d}{dx}(y), \quad \text{meaning "the derivative with respect to } x \text{ of } y."$$

On the other hand, many scientists and mathematicians really do think of  $dy$  and  $dx$  as separate entities representing "infinitesimally" small differences in  $y$  and  $x$ , even though it is difficult to say exactly how small "infinitesimal" is. Although not formally correct, it can be very helpful to think intuitively of  $dy/dx$  as a very small change in  $y$  divided by a very small change in  $x$ .

For example, recall that if  $s = f(t)$  is the position of a moving object at time  $t$ , then  $v = f'(t)$  is the velocity of the object at time  $t$ . Writing

$$v = \frac{ds}{dt}$$

reminds us that  $v$  is a velocity, since the notation suggests a distance,  $ds$ , over a time,  $dt$ , and we know that distance over time is velocity. Similarly, we recognize

$$\frac{dy}{dx} = f'(x)$$

as the slope of the graph of  $y = f(x)$  by remembering that slope is vertical rise,  $dy$ , over horizontal run,  $dx$ .

The disadvantage of Leibniz's notation is that it is awkward to specify the  $x$ -value at which we are evaluating the derivative. To specify  $f'(2)$ , for example, we have to write

$$\left. \frac{dy}{dx} \right|_{x=2}$$

## Using Units to Interpret the Derivative

The following examples illustrate how useful units can be in suggesting interpretations of the derivative.

For example, suppose  $s = f(t)$  gives the distance, in meters, of a body from a fixed point as a function of time,  $t$ , in seconds. Then knowing that

$$\left. \frac{ds}{dt} \right|_{t=2} = f'(2) = 10 \text{ meters/sec}$$

tells us that when  $t = 2$  sec, the body is moving at an instantaneous velocity of 10 meters/sec. This means that if the body continued to move at this speed for a whole second, it would cover 10 meters. In practice, however, the velocity of the body may be changing and so may not remain 10 meters/sec for long. Notice that the units of instantaneous velocity and of average velocity are the same. The units of the instantaneous and the average rate of change are *always* the same.

**Example 1** The cost  $C$  (in dollars) of building a house  $A$  square feet in area is given by the function  $C = f(A)$ . What is the practical interpretation of the function  $f'(A)$ ?

**Solution** In the alternative notation,

$$f'(A) = \frac{dC}{dA}.$$

This is a cost divided by an area, so it is measured in dollars per square foot. You can think of  $dC$  as the extra cost of building an extra  $dA$  square feet of house. Then you can think of  $dC/dA$  as the additional cost per square foot. So if you are planning to build a house roughly  $A$  square feet in area,  $f'(A)$  is the cost per square foot of the *extra* area involved in building a slightly larger house, and is called the *marginal cost*. The marginal cost is probably smaller than the average cost per square foot for the entire house, since once you are already set up to build a large house, the cost of adding a few square feet is likely to be small.

**Example 2** The cost of extracting  $T$  tons of ore from a copper mine is  $C = f(T)$  dollars. What does it mean to say that  $f'(2000) = 100$ ?

**Solution** In the alternative notation,

$$f'(2000) = \left. \frac{dC}{dT} \right|_{T=2000} = 100.$$

Since  $C$  is measured in dollars and  $T$  is measured in tons,  $dC/dT$  must be measured in dollars per ton. So the statement  $f'(2000) = 100$  says that when 2000 tons of ore have already been extracted from the mine, the cost of extracting the next ton is approximately \$100.

**Example 3** If  $q = f(p)$  gives the number of pounds of sugar produced when the price per pound is  $p$  dollars, then what are the units and the meaning of

$$f'(3) = 50?$$

**Solution** Since  $f'(3)$  is the limit as  $h \rightarrow 0$  of

$$\frac{f(3+h) - f(3)}{h}$$

and  $f(3+h) - f(3)$  is in pounds, whereas  $h$  is in dollars, the units of the difference quotient are pounds/dollar. Since  $f'(3)$  is the limit of the difference quotient, its units are also pounds/dollar. The statement

$$f'(3) = 50 \text{ pounds/dollar}$$

tells us that the rate of change of  $q$  with respect to  $p$  is 50 when  $p = 3$ . Rephrasing, this means that when the price is \$3, the quantity produced is increasing at 50 pounds/dollar. This is an instantaneous rate of change, meaning that, if the rate were to remain 50 pounds/dollar, and if the price were to increase by a whole dollar, the quantity produced would increase by approximately 50 pounds. In fact, the rate probably doesn't remain constant and so the quantity produced would probably not be exactly 50 pounds. Notice that the units of the derivative and of the average rate of change are again the same.

**Example 4** You are told that water is flowing through a pipe at a constant rate of 10 cubic feet per second. Interpret this rate as the derivative of some function.

**Solution** You might think at first that the statement has something to do with the velocity of the water, but in fact a flow rate of 10 cubic feet per second could be achieved either with very slowly moving water through a large pipe, or with very rapidly moving water through a narrow pipe. If we look at the units—cubic feet per second—we realize that we are being given the rate of change of a quantity measured in cubic feet. But a cubic foot is a measure of volume, so we are being told the rate of change of a volume. One way to visualize this is to imagine all the water that is flowing through the pipe ending up in a tank somewhere. Let  $V(t)$  be the volume of water in the tank at time  $t$ . Then we are being told that the rate of change of  $V(t)$  is 10, or

$$V'(t) = \frac{dV}{dt} = 10.$$

**Example 5** Suppose  $P = f(t)$  is the population of Mexico in millions, where  $t$  is the number of years since 1980. Explain the meaning of the statements:

(a)  $f'(6) = 2$                       (b)  $f^{-1}(95.5) = 16$                       (c)  $(f^{-1})'(95.5) = 0.46$

**Solution** (a) The units of  $P$  are millions of people, the units of  $t$  are years, so the units of  $f'(t)$  are millions of people per year. Therefore the statement  $f'(6) = 2$  tells us that at  $t = 6$  (that is, in 1986), the population of Mexico was increasing at 2 million people per year.

(b) The statement  $f^{-1}(95.5) = 16$  tells us that the year when the population was 95.5 million was  $t = 16$  (that is, in 1996).

(c) The units of the derivative  $(f^{-1})'(P)$  are years per million of population. The statement  $(f^{-1})'(95.5) = 0.46$  tells us that when the population was 95.5 million, it took about 0.46 years for the population to increase by 1 million.

### Problems for Section 2.4

- Let  $f(x)$  be the elevation in feet of the Mississippi river  $x$  miles from its source. What are the units of  $f'(x)$ ? What can you say about the sign of  $f'(x)$ ?
- The temperature,  $T$ , in degrees Fahrenheit, of a cold yam placed in a hot oven is given by  $T = f(t)$ , where  $t$  is the time in minutes since the yam was put in the oven.
  - What is the sign of  $f'(t)$ ? Why?
  - What are the units of  $f'(20)$ ? What is the practical meaning of the statement  $f'(20) = 2$ ?
- Suppose  $P(t)$  is the monthly payment, in dollars, on a mortgage which will take  $t$  years to pay off. What are the units of  $P'(t)$ ? What is the practical meaning of  $P'(t)$ ? What is its sign?
- Suppose  $C(r)$  is the total cost of paying off a car loan borrowed at an annual interest rate of  $r\%$ . What are the units of  $C'(r)$ ? What is the practical meaning of  $C'(r)$ ? What is its sign?
- After investing \$1000 at an annual interest rate of 7% compounded continuously for  $t$  years, your balance is  $\$B$ , where  $B = f(t)$ . What are the units of  $dB/dt$ ? What is the financial interpretation of  $dB/dt$ ?
- Investing \$1000 at an annual interest rate of  $r\%$ , compounded continuously, for 10 years gives you a balance of  $\$B$ , where  $B = g(r)$ . What is a financial interpretation of the statements
  - $g(5) \approx 1649$ ?
  - $g'(5) \approx 165$ ? What are the units of  $g'(5)$ ?
- An ice cream company knows that the cost,  $C$  (in dollars), to produce  $g$  quarts of cookie dough ice cream is a function of  $g$ , so  $C = f(g)$ .
  - If  $f(200) = 70$ , what are the units of the 200? What are the units of the 70? Explain clearly what this equation is telling you.
  - If  $f'(200) = 3$ , what are the units of the 200? What are the units of the 3? Explain clearly what this equation is telling you.

8. An economist is interested in how the price of a certain item affects its sales. Suppose that at a price of  $\$p$ , a quantity,  $q$ , of the item is sold. If  $q = f(p)$ , explain the meaning of each of the following statements:
- (a)  $f(150) = 2000$  (b)  $f'(150) = -25$
9. Let  $p(h)$  be the pressure in dynes per  $\text{cm}^2$  on a diver at a depth of  $h$  meters below the surface of the ocean. What do each of the following quantities mean to the diver? Give units for the quantities.
- (a)  $p(100)$  (b)  $h$  such that  $p(h) = 1.2 \cdot 10^6$   
 (c)  $p(h) + 20$  (d)  $p(h + 20)$   
 (e)  $p'(100)$  (f)  $h$  such that  $p'(h) = 20$
10. Let  $f(t)$  be the number of centimeters of rainfall that has fallen since midnight, where  $t$  is the time in hours. Interpret the following in practical terms, giving units.
- (a)  $f(10) = 3.1$  (b)  $f^{-1}(10) = 16$  (c)  $f'(8) = 0.4$  (d)  $(f^{-1})'(5) = 2$
11. Let  $W$  be the amount of water, in gallons, in a bathtub at time  $t$ , in minutes.
- (a) What are the meaning and units of  $dW/dt$ ?  
 (b) Suppose the bathtub is full of water at time  $t_0$ , so that  $W(t_0) > 0$ . Subsequently, at time  $t_p > t_0$ , the plug is pulled. Is  $dW/dt$  positive, negative, or zero:  
 (i) For  $t_0 < t < t_p$ ?  
 (ii) After the plug is pulled?  
 (iii) When all the water has drained from the tub?
12. Suppose  $w$  is the weight of a stack of papers in an open container, and  $t$  represents time. A lighted match is thrown into the stack at time  $t = 0$ .
- (a) What is the sign of  $dw/dt$  for  $t$  in the period of time while the stack of paper is in the process of burning?  
 (b) What behavior of  $dw/dt$  indicates that the fire is no longer burning?  
 (c) If the fire starts small but increases in vigor over a certain time interval, then does  $dw/dt$  increase or decrease over that interval? What about  $|dw/dt|$ ?
13. If  $g(v)$  is the fuel efficiency, in miles per gallon, of a car going at  $v$  miles per hour, what are the units of  $g'(55)$ ? What is the practical meaning of the statement  $g'(55) = -0.54$ ?
14. (a) If you jump out of an airplane without a parachute, you fall faster and faster until wind resistance causes you to approach a steady velocity, called the *terminal* velocity. Sketch a graph of your velocity against time.  
 (b) Explain the concavity of your graph.  
 (c) Assuming wind resistance to be negligible at  $t = 0$ , what natural phenomenon is represented by the slope of the graph at  $t = 0$ ?
15. A company's revenue from car sales,  $C$  (measured in thousands of dollars), is a function of advertising expenditure,  $a$ , also measured in thousands of dollars. Suppose  $C = f(a)$ .
- (a) What does the company hope is true about the sign of  $f'$ ?  
 (b) What does the statement  $f'(100) = 2$  mean in practical terms? How about  $f'(100) = 0.5$ ?  
 (c) Suppose the company plans to spend about \$100,000 on advertising. If  $f'(100) = 2$ , should the company spend more or less than \$100,000 on advertising? What if  $f'(100) = 0.5$ ?
16. Let  $P(x) =$  the number of people in the US of height  $\leq x$  inches. What is the meaning of  $P'(66)$ ? What are its units? Estimate  $P'(66)$  (using common sense). Is  $P'(x)$  ever negative? [Hint: You may want to approximate  $P'(66)$  by a difference quotient, using  $h = 1$ . Also, you may assume the US population is about 250 million, and that 66 inches = 5 feet 6 inches.]

## 2.5 THE SECOND DERIVATIVE

Since the derivative is itself a function, we can consider its derivative. For a function  $f$ , the derivative of its derivative is called the *second derivative*, and written  $f''$  (read “ $f$  double-prime”). If  $y = f(x)$ , the second derivative can also be written as  $\frac{d^2y}{dx^2}$ , which means  $\frac{d}{dx} \left( \frac{dy}{dx} \right)$ , the derivative of  $\frac{dy}{dx}$ .