II. Functions

Week 4. Functions: graphs, tables and formulas

Problem of the Week: The Farmer's Fence

A field bounded on one side by a river is to be fenced on three sides so as to form a rectangular enclosure. If 197 feet of fencing is to be used, what dimensions will yield an enclosure of the largest possible area?

Functions with graphs that are lines

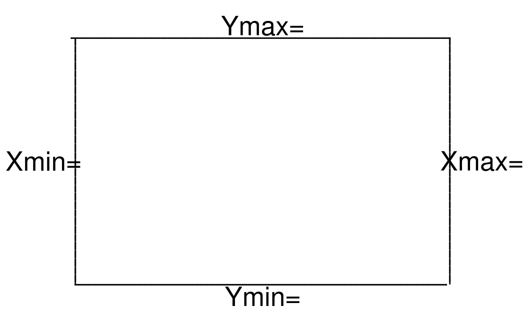
In Algebra I, students learn about equations for straight lines. But lines are geometric shapes and it is also important to understand the geometry of lines. Do problem 59 and problem 62 geometrically. That is, do not use an equation for the line until after you are asked to do so in the problems.

59. On a piece of graph paper with carefully labeled coordinate axes, sketch the line that goes through the two points $(1, 2)$ and $(3, 5)$. Give the exact coordinates for the following points on your graph. Show the points on your graph.

a) Three points on the line: one between the two given points and one on each side of the given points.
b) The point exactly half way between the two points.
c) The two points that divide the interval into three equal segments.
d) A point whose x–coordinate is negative and another point whose y–coordinate is negative.
e) A point whose x–coordinate is 0 and a point whose y–coordinate is 0.

60. Continuing with the line from problem 59, make a table of values, and describe the pattern. Write an equation that describes the line.

61. Find a way to graph the line from the problem 59 on your calculator. Sketch the calculator graph here, including the window values:



Y =

Use your calculator to generate a table of values for the line.

62. On a piece of graph paper with carefully labeled coordinate axes, sketch the line that goes through the the point $(2, 1)$ and has slope equal to $\frac{1}{3}$. Give the exact coordinates for the following points on your graph. Show the points on your graph.

a) A point on the line on each side of the point $(2, 1)$.

b) A point on the line where both the x–coordinate and y–coordinate are negative.

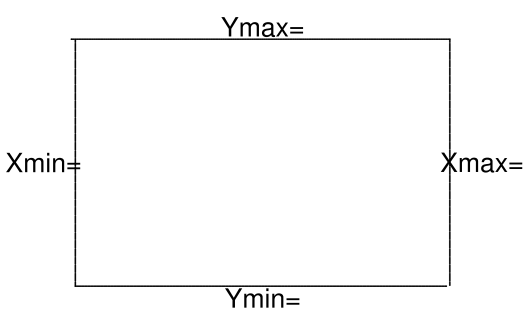
c) The point on the line whose x–coordinate is 0.

d) The point on the line whose y–coordinate is 0.

e) Two points on the line that are exactly $\sqrt{10}$ units from the point $(2, 1)$. What is special about $\sqrt{10}$ for this problem?

63. Continuing with the line from problem 62, make a table of values, and describe the pattern. Write an equation that describes the line.

64. Find a way to graph the line from problem 62 on your calculator. Sketch the calculator graph here, including the window values:

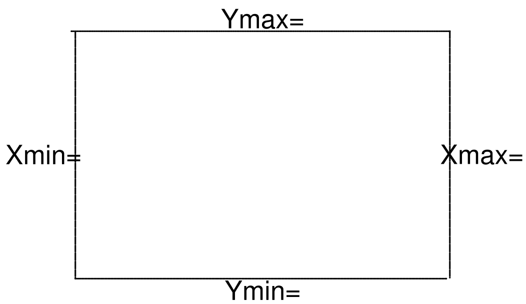


Y =

Use your calculator to generate a table of values for the line.

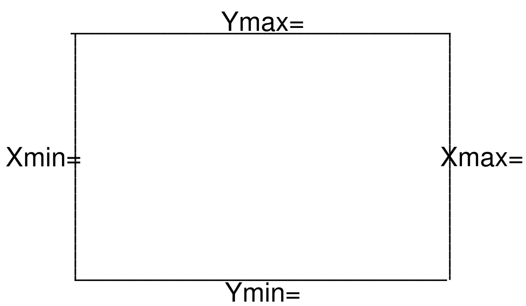
65. Sketch each line, then figure out how to draw each of these lines on your calculator:

a) A line that goes through the points $(-1, 1)$ and has slope, $\frac{2}{5}$.



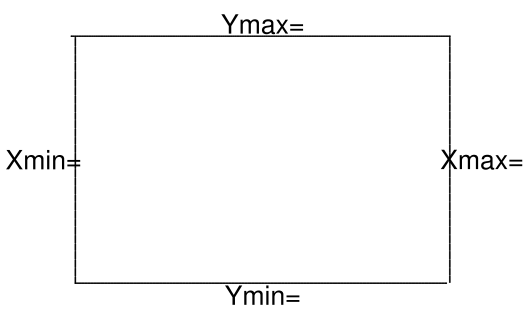
Y =

b) A line that goes through the point $(2, -3)$ and has slope 0.



Y =

c) A line that has slope -0.6 and goes through the point $(1, 1)$



Y =

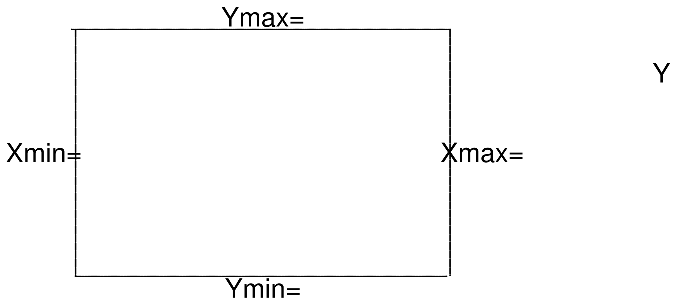
66. Explain why the equation

$$y = m(x - c) + d$$

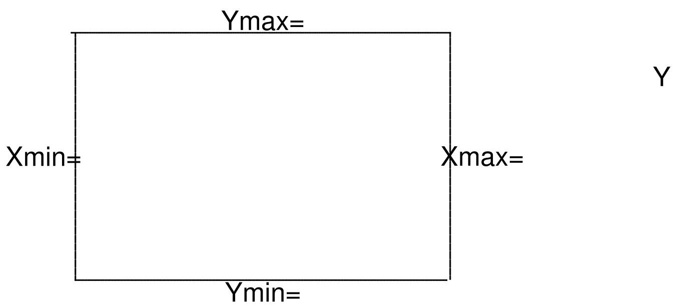
describes the line through the point (c, d) that has slope m. Type this function into your "Y" page, using the letters m, c and d as well as x and use it to graph several different lines by changing the value of those three variables.

67. Figure out how to draw each of these lines on your calculator, then sketch a graph with a reasonable window.

a) A line that has slope 0.03 and goes through the point $(0.1, 1.01)$



b) The line that goes through the two points $(1.5, 7.8)$ and $(-3.1, 10.9)$



68. Sketch the line given by the equation, $y = 3x + 1$. Sketch the line that is perpendicular to this one and intersects it at the point $(0, 1)$. (Wait a minute! Is the point $(0, 1)$ really on the line?) Explain how to use the grid of the graph paper to be very accurate. Find an equation for this perpendicular line.

69. Suppose you are driving a constant speed from Chicago to Detroit, about 275 miles away. Sketch a graph of your distance from Chicago as a function of time. Why is this graph a straight line? What is the slope of the line? What are the units of the slope?

70. A car rental company offers cars at $\$40$ a day and 15 cents a mile. Its competitor's cars are $\$50$ a day and 10 cents a mile. Graph the cost of renting a car as a function of miles driven for each company. For each company, write a formula giving the cost of renting a car for a day as a function of the distance traveled. Explain why 15 cents a mile and 10 cents a mile each represent a slope. On your calculator, graph both functions on one screen. How should you decide which company is cheaper?

71. Consider a graph of Fahrenheit temperature, °F, against Celsius temperature, °C, and *assume that the graph is a straight line*. You know that 212°F and 100°C both represent the temperature at which water boils. Similarly, 32°F and 0°C represent water's freezing point. What is the slope of the graph? What are the units of the slope? What is the equation of the line? Use the equation to find what Fahrenheit temperature corresponds to 20°C. Use the graph to find the same thing. What temperature is the same number of degrees in both Celsius and Fahrenheit? How can you see this on the graph?

Functions with graphs that are parabolas.

Definition: Parabola, quadratic

A *parabola* is the graph of a function which has the form

$$y = ax^2 + bx + c$$

Such functions are called *quadratic* functions. The right hand expression is called a *quadratic polynomial*.

72. Using the above definition, explain why the graph of $y = 3(x-1)(x-2)$ is a parabola. Where does this parabola intersect the x-axis? Graph it to check your answer.

Ymax=

Xmin= Xmax=

Ymin=

Y =

73. Explain why the graph of $y = 3(x-1)^2 + 2$ is a parabola. Explain how to tell that this parabola contains the point (1,2). What do you notice about the point $(1, 2)$? Graph it to check.

Ymax=

Xmin= Xmax=

Ymin=

Y =

74. On your calculator, graph several different parabolas that cross the x-axis at $(1, 0)$ and $(3, 0)$. What do you notice about the vertex of all these parabolas? Is there a pattern? Look at more examples, changing the points where the parabola crosses the x-axis.

Ymax=

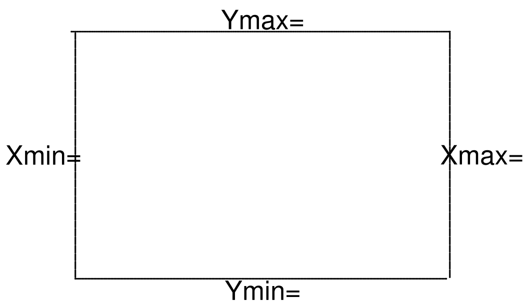
Xmin= Xmax=

Ymin=

Y =

75. Write a formula for the coordinates of the vertex of the parabola given generally by $y = a(x - z_1)(x - z_2)$.

76. On your calculator, graph a parabola that goes through the points $(-1, 0)$, $(0, 3)$ and $(1, 0)$.



Y =

77. Is it possible for a parabola to intersect the x-axis in only one point? If so, describe what would happen geometrically and find the equation for such a parabola.

78. Carefully explain FOIL. That is, explain why

$$(a + b)(c + d) = ac + ad + bc + bd.$$

What is the difference between explaining how to use FOIL and explaining why it works?

79. Where does the parabola $y = x^2 + 4x + 3$ cross the x-axis?
Where does the parabola $y = x^2 - 6x + 8$ cross the x-axis?
Where does the parabola $y = x^2 + x - 2$ cross the x-axis?
Where does the parabola $y = 3x^2 - 2x - 1$ cross the x-axis?

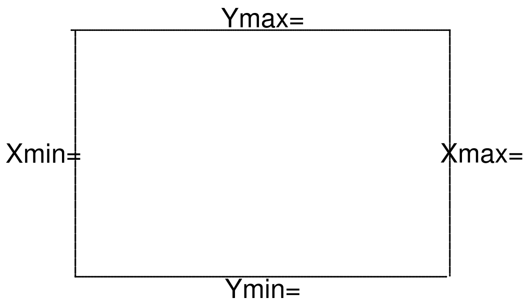
80. Find the points where the parabola $y = x^2 + 2x - 3$ crosses the x-axis by completing the square.
Find the points where the parabola $y = x^2 + 4x + 3$ crosses the x-axis.
Find the points where the parabola $y = x^2 + 4x - 3$ crosses the x-axis.
Find the points where the parabola $y = x^2 - x - 1$ crosses the x-axis.
Find the points where the parabola $y = 3x^2 + 4x - 6$ crosses the x-axis.

81. By completing the square, explain the quadratic formula. That is show why the two solutions to the equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

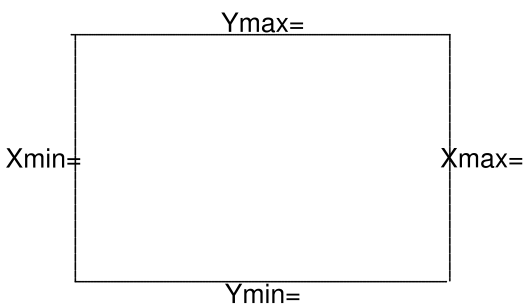
82. At what points does the parabola, $y = x^2 + x + 1$, cross the x-axis? Explain.

83. On your calculator, graph a parabola that has a vertex at the point $(-1, 3)$ and does not cross the x–axis.



Y =

84. Use your calculator to find an equation for the parabola that goes through the points$(-1, 2), (1, 4)$ and $(0, 1)$. Graph this parabola.



Y =

85. Explain the difference between solving the equation $ax^2 + bx + c = 0$ and graphing the equation $y = ax^2 + bx + c$. How does one help with doing the other?

86. Explain what factoring has to do with solving the equation $ax^2 + bx + c = 0$.

87. *The Algebra Nightmare* The two solutions to the equation $x^2 - 4x + 3 = 0$ are $x = 3$ and $x = 1$. And we know that $(x-3)(x-1) = x^2 - 4x + 3$. Something similar is true for all quadratic equations: We know that there are at most two solutions to a quadratic equation, $ax^2 + bx + c = 0$.
The two solutions are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Show that

$$a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = ax^2 + bx + c$$

by multiplying out the left hand side and simplifying to get the right hand side.

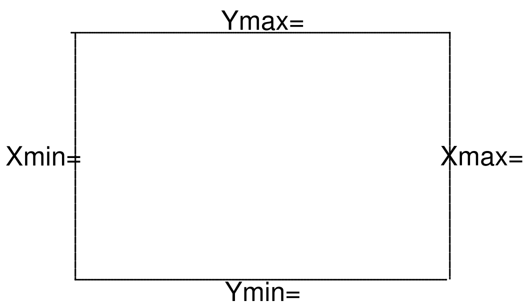
A famous quadratic function

If an object is thrown upward, dropped, or thrown downward and travels in a vertical line subject only to gravity (with wind resistance ignored), then the height s(t) of the object above the ground (in feet) after t seconds is given by

$$s(t) = -16t^2 + Vt + H$$

Where H is the initial height of the object at starting time ($t = 0$) and V is the initial velocity (speed in feet per second) of the object at starting time ($t = 0$).

88. Graph the function for the case when a rock is dropped from a height of 200 feet. Use the calculator to get a graph, then sketch it here. Include the units with your window values.



Y =

Use the graph to answer this question: How long does it take the rock to fall?

89. How high would the rock go if it were given an initial upward velocity of 20 feet per second?

90. How high will a tennis ball go if it is thrown from a height of 4 feet with an initial velocity of 80 feet per second? How long will it take to get there? Do you think you could throw it hard enough to get that initial velocity? What velocity is needed to throw the ball 10 feet in the air? (Surely that can be done.)