Why does a negative × a negative = a positive?

(including how to explain it to your younger brother or sister)

For too many people, mathematics stopped making sense somewhere along the way. Either slowly or dramatically, they gave up on the field as hopelessly baffling and difficult, and they grew up to be adults who—confident that others share their experience—nonchalantly announce, "Math was just not for me" or "I was never good at it."

Usually the process is gradual, but for Ruth McNeill, the turning point was clearly defined. In an article in the Journal of Mathematical Behavior, *she described how it happened:*[1]

> What did me in was the idea that a negative number times a negative number comes out to a positive number. This seemed (and still seems) inherently unlikely—counterintuitive, as mathematicians say. I wrestled with the idea for what I imagine to be several weeks, trying to get a sensible explanation from my teacher, my classmates, my parents, anybody. Whatever explanations they offered could not overcome my strong sense that multiplying intensifies something, and thus two negative numbers multiplied together should properly produce a *very* negative result. I have since been offered a moderately convincing explanation that features a film of a swimming pool being drained that gets run backwards through the projector. At the time, however, nothing convinced me. The most commonsense of all school subjects had abandoned common sense; I was indignant and baffled.
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> Meanwhile, the curriculum kept rolling on, and I could see that I couldn't stay behind, stuck on negative times negative. I would have to pay attention to the next topic, and the only practical course open to me was to pretend to agree that negative times negative equals positive. The book and the teacher and the general consensus of the algebra survivors of society were clearly more powerful than I was. I capitulated. I did the rest of algebra, and geometry, and trigonometry; I did them in the advanced sections, and I often had that nice sense of "aha!" when I could suddenly see how a proof was going to come out. Underneath, however, a kind of resentment and betrayal lurked, and I was not surprised or dismayed by any further foolishness my math teachers had up their sleeves.... Intellectually, I was disengaged, and when math was no longer required, I took German instead.

Happily, Ruth McNeill's story doesn't end there. Thanks to some friendships she formed in college, her interest in math was rekindled. For most of our students, there is no rekindling. This is a tragedy, both for our students and for our country. Part of the reason students give up on math can be attributed to the poor quality of most of the math textbooks used in the United States. Many texts are written with the premise that if they end a problem with the words, "Explain your answer," they are engendering "understanding." However, because these texts do not give students what they would need to enable them to "explain," the books only add to students' mystification and frustration.

Here is an example of how a widely acclaimed contemporary math series handles the topic that baffled Ruth McNeill: After a short set of problems dealing with patterns in multiplication of integers from 5 to 0 times (–4), the student is asked to continue the pattern to predict what (–1)(–4) is and then to give the next four equations in this pattern. There are then four problems, one of them being the product of two negative numbers. In the follow-up problems given next, there are four problems dealing with negative numbers, the last of which is the only one treating multiplication of negative numbers. This is how it reads: "When you add two negative numbers, you get a negative result. Is the same true when you multiply two negative numbers? Explain."

The suggested answer to the "explain" part is: "The product of two negative numbers is a positive." This is not an explanation, but a claim that the stated answer is correct.

Simply asking students to explain something isn't sufficient. They need to be taught enough so that they can explain. And they need to learn what an explanation is and when a statement is not an explanation.

The excerpt that follows is taken from a serious but lively volume entitled Algebra *by I.M.Gelfand and A. Shen, which was originally written to be used in a correspondence school that Gelfand had established. Contrast the inadequate treatment of the multiplication of negative numbers described above to the way Gelfand and Shen handle the topic.*[2] *Although their presentation would need to be fleshed out more if it's being presented to students for the first time, it provides us with a much better model for what "explain" might entail, offering as it does both an accessible explanation and a formal proof.*

—Richard Askey

The multiplication of negative numbers

To find how much three times five is, you add three numbers equal to five:

$$5 + 5 + 5 = 15.$$

The same explanation may be used for the product $1 \cdot 5$ if we agree that a sum having only one term is equal to this term. But it is evidently not applicable to the product $0 \cdot 5$ or $(-3) \cdot 5$: Can you imagine a sum with a zero or with minus three terms?

However, we may exchange the factors:

$$5 \cdot 0 = 0 + 0 + 0 + 0 + 0 = 0,$$

$$5 \cdot (-3) = (-3) + (-3) + (-3) + (-3) + (-3) = -15.$$

So if we want the product to be independent of the order of factors (as it was for positive numbers) we must agree that

$$0 \cdot 5 = 0, \quad (-3) \cdot 5 = -15.$$

Now let us consider the product (–3) · (–5). Is it equal to –15 or to +15? Both answers may have advocates. From one point of view, even one negative factor makes the product negative—so if both factors are negative the product has a very strong reason to be negative. From the other point of view, in the table

3 · 5 = +15	3 · (–5) = –15
(–3) · 5 = –15	(–3) · (–5) = ?

we already have two minuses and only one plus; so the "equal opportunities" policy requires one more plus. So what?

Of course, these "arguments" are not convincing to you. School education says very definitely that minus times minus is plus. But imagine that your small brother or sister asks you, "Why?" (Is it a caprice of the teacher, a law adopted by Congress, or a theorem that can be proved?) You may try to answer this question using the following example:

3 · 5 = 15	Getting five dollars three times is getting fifteen dollars.
3 · (–5) = –15	Paying a five-dollar penalty three times is a fifteen-dollar penalty.
(–3) · 5 = –15	Not getting five dollars three times is not getting fifteen dollars.
(–3) · (–5) = 15	Not paying a five-dollar penalty three times is getting fifteen dollars.

Another explanation. Let us write the numbers

1, 2, 3, 4, 5,...

and the same numbers multiplied by three:

3, 6, 9, 12, 15,...

Each number is bigger than the preceding one by three. Let us write the same numbers in the reverse order (starting, for example, with 5 and 15):

5, 4, 3, 2, 1
15, 12, 9, 6, 3

Now let us continue both sequences:

5, 4, 3, 2, 1, 0, –1, –2, –3, –4, –5,...
15, 12, 9, 6, 3, 0, –3, –6, –9, –12, –15,...

Here –15 is under –5, so 3 · (–5) = –15; plus times minus is minus.

Now repeat the same procedure multiplying 1, 2, 3, 4, 5,... by –3 (we know already that plus times minus is minus):

1, 2, 3, 4, 5
–3, –6, –9, –12, –15

Each number is three units less than the preceding one. Now write the same numbers in the reverse order:

5, 4, 3, 2, 1
–15, –12, –9, –6, –3

and continue:

5, 4, 3, 2, 1, 0, –1, –2, –3, –4, –5,...
–15, –12, –9, –6, –3, 0, 3, 6, 9, 12, 15,...

Now 15 is under –5; therefore (–3) · (–5) = 15.

Probably this argument would be convincing for your younger brother or sister. But you have the right to ask: So what? Is it possible to *prove* that (–3) · (–5) = 15?

Let us tell the whole truth now. Yes, it is possible to prove that (–3) · (–5) *must be* 15 *if* we want the usual properties of addition, subtraction, and multiplication that are true for positive numbers to remain true for any integers (including negative ones).

Here is the outline of this proof: Let us prove first that 3 · (–5) = –15. What is –15? It is a number opposite to 15, that is, a number that produces zero when added to 15. So we must prove that

$$3 \cdot (-5) + 15 = 0.$$

Indeed,

$$3 \cdot (-5) + 15 = 3 \cdot (-5) + 3 \cdot 5 = 3 \cdot (-5 + 5) = 3 \cdot 0 = 0.$$

(When taking 3 out of the parentheses we use the law $ab + ac = a(b + c)$ for $a = 3$, $b = -5$, $c = 5$; we assume that it is true for all numbers, including negative ones.) So 3 · (–5) = –15. (The careful reader will ask why 3 · 0 = 0. To tell you the truth, this step of the proof is omitted—as well as the whole discussion of what zero is.)

Now we are ready to prove that (–3) · (–5) = 15. Let us start with

$$(-3) + 3 = 0$$

and multiply both sides of this equality by –5:

$$((-3) + 3) \cdot (-5) = 0 \cdot (-5) = 0.$$

Now removing the parentheses in the left-hand side we get

$$(-3) \cdot (-5) + 3 \cdot (-5) = 0,$$

that is, (–3) · (–5) + (–15) = 0. Therefore, the number (–3) · (–5) is opposite to –15, that is, is equal to 15. (This argument also has gaps. We should prove first that 0 · (–5) = 0 and that there is only one number opposite to –15.) □

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[1] Ruth McNeill, "A Reflection on When I Loved Math and How I Stopped." *Journal of Mathematical Behavior*, vol. 7 (1988) pp. 45-50.

[2] *Algebra* by I. M. Gelfand and A. Shen. Birkhauser Boston (1995, Second Printing): Cambridge, Mass. © 1993 by I. M. Gelfand. Reprinted with permission.