Workshop Geometry:

Mathematics for Teaching Elementary School

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Workshop Geometry

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Unit #1 Geometry Revisited

INTRODUCTION

The goals for this unit are three-fold

1. To review and expand our basic **knowledge** of geometry. Every elementary school teacher should have a basic understanding of high school geometry.

2. To **construct** as many geometric objects with as many different tools as we can. Hands-on activities not only help us learn but also dramatically help us understand the role of the concrete and the visual in children's learning.

3. Continue to improve our **explanation** skills. Understanding why things work helps everyone learn mathematics.

The importance of geometry in the elementary school mathematics is recognized by the National Council of Teachers of Mathematics in the 2002 Standards for School Mathematics*. Geometry is one of the five Content Standards listed in that document. The standard is stated in this way:

> Instructional programs from prekindergarten through grades 6–8 should enable all students to—
>
> Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships
> - precisely describe, classify, and understand relationships among types of two- and three-dimensional objects using their defining properties;
> - understand relationships among the angles, side lengths, perimeters, areas, and volumes of similar objects;
> - create and critique inductive and deductive arguments concerning geometric ideas and relationships, such as congruence, similarity, and the Pythagorean relationship.
>
> Specify locations and describe spatial relationships using coordinate geometry and other representational systems
> - use coordinate geometry to represent and examine the properties of geometric shapes;
> - use coordinate geometry to examine special geometric shapes, such as regular polygons or those with pairs of parallel or perpendicular sides.
>
> Apply transformations and use symmetry to analyze mathematical situations
> - describe sizes, positions, and orientations of shapes under informal transformations such as flips, turns, slides, and scaling;
> - examine the congruence, similarity, and line or rotational symmetry of objects using transformations.
>
> Use visualization, spatial reasoning, and geometric modeling to solve problems
> - draw geometric objects with specified properties, such as side lengths or angle measures;
> - use two-dimensional representations of three-dimensional objects to visualize and solve problems such as those involving surface area and volume;
> - use visual tools such as networks to represent and solve problems;
> - use geometric models to represent and explain numerical and algebraic relationships;
> - recognize and apply geometric ideas and relationships in areas outside the mathematics classroom, such as art, science, and everyday life.

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* http://standards.nctm.org

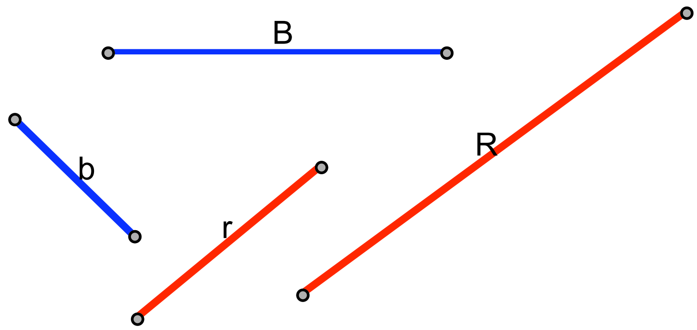
Table of contents for Unit #1

Lesson 1 – Geometry Revisited

While you work on the following problem, think about what you learned in high school geometry and other mathematics classes. What words help you communicate with your fellow problem solvers? What geometry facts are involved? What problem solving techniques do you choose?

Class Activity: Geostrip Triangles

Given these four different line segments, how many different triangles can you make?



Discussion of problem with two solutions

The first thing is to tackle what we mean by "different." In geometry, we say that two figures are *congruent* if one can be moved onto the other. This will be our first definition.

Definition: Two shapes are *congruent* if one can be moved onto the other so that they coincide completely.

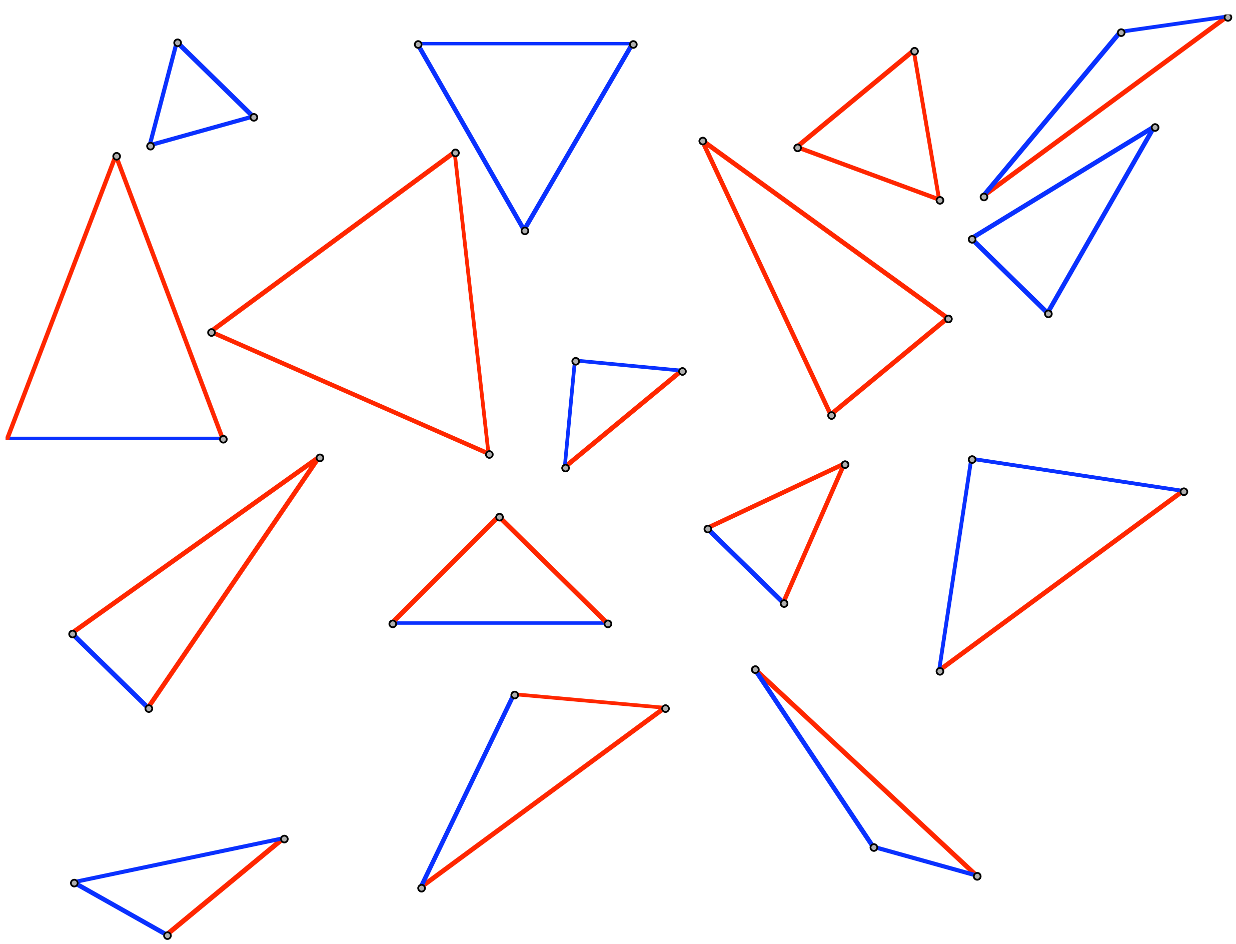
This is what is frequently meant by being "the same" in geometry. It is always good in mathematics to be as precise as possible. So now let's restate the problem:

"How many non-congruent triangles can be made from line segments of the given four lengths?"

In one classroom, two different solutions came up. Let's see if we can find the connections between the two different solutions.

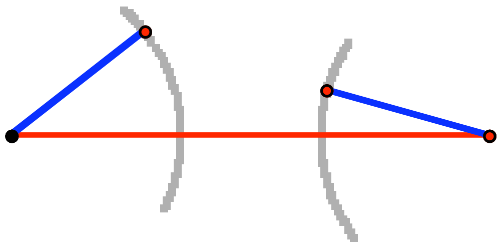
The first solution: Group A worked with red and blue Geostrips which are in the same *proportions* as the lengths indicated. This group made all of the triangles shown on this page.

(Please note: this is a scale drawing of the possible triangles. Lengths are different but the proportions are the same.)



1. Assignment: Are there triangles missing? Which one(s)? Are any triangles repeated? Cross out any duplicates.

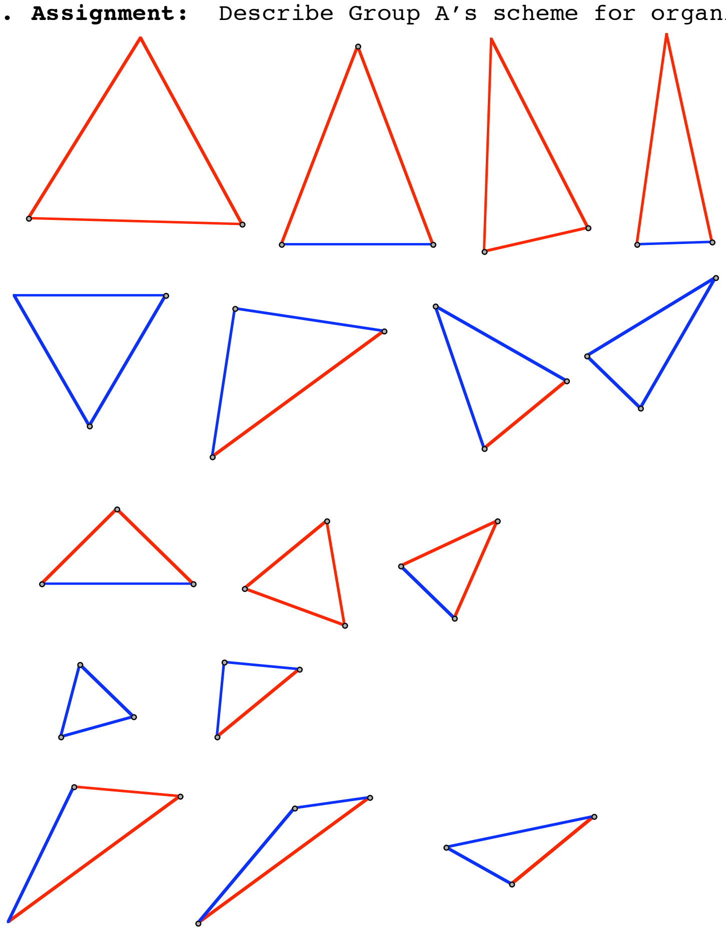
One interesting geometry fact that Group A noticed is that it is not always possible to make a triangle with three given lengths. For example, taking two of the short blue and a long red put them in this position:



No matter how you rotate the two blue strips, you cannot make them meet to form a triangle.

When Group A couldn't find any more triangles they began to think they had them all but were also a bit worried there might be repetitions in their collection. They decided to organize their collection to get a better idea of the situation. In this way, they found a missing isosceles triangle and also noticed a repeated triangle. This *organized list* convinced them that they had all possible triangles and no repeats. Counting the triangles, they found they had 16 triangles.

2. Assignment: Describe Group A's scheme for organizing the triangles.



(Please note: this is a scale drawing of the possible triangles. Lengths are different but the proportions are the same)

3. Assignment: Why are there four triangles in each of the first and second rows, but not in the third or fourth rows?

Why are there only three triangles in the last row?

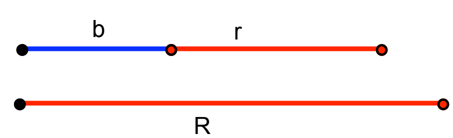
The second solution: Group B decided to make a list of all possible triangles. They used R to denote the longer red line segment and r to denote the shorter red line segment. Similarly they used B and b for the blue line segments. They made a list of all possible combinations of the letters. There are different ways to do this but they chose to list all possible arrangements using only one letter first, then to list all possible using exactly two different letters and finally to list all the possibilities using three different letters.

Three congruent			
RRR	BBB	rrr	Bbb
Two R's	***Two B's***	***Two r's***	***Two b's***
RRB	BBR	rrR	bbR
RRr	BBr	rrB	bbB
RRb	BBb	rrb	Bbr
No R	***No B***	***No r***	***No b***
Bbr	Rbr	BRb	BRr

4. Assignment: Someone says, "Look! RBB isn't on your list!" Does this mean that the list is incomplete? Explain. Convince a skeptic that all possible triangles appear on this list.

Then someone recalled that some of the combinations might not actually make a triangle. For example, rbR isn't possible because, if the letters represent the length of each segment, r+b < R

5. Assignment: Measure the lengths of r, b, and R and confirm arithmetically(by adding and comparing numbers) that the inequality is true.

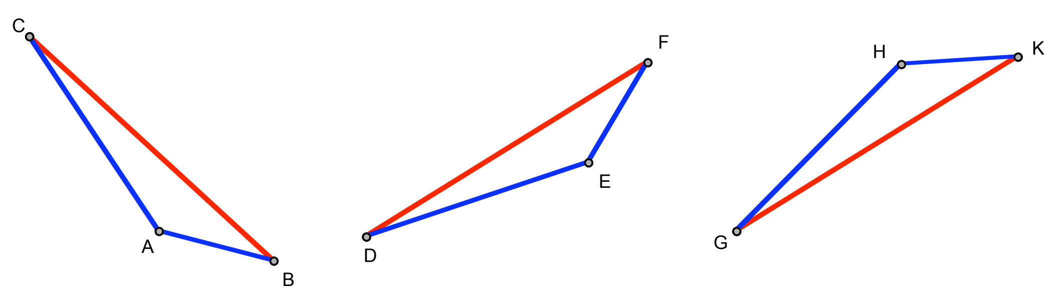


After carefully investigating each possibility, they modified their list:

Three congruent			
RRR	BBB	rrr	bbb
Two R's	***Two B's***	***Two r's***	***Two b's***
RRB	BBR	rrR - not possible	bbR – not possible
RRr	BBr	rrB	bbB – not possible
RRb	BBb	rrb	bbr
No R	***No B***	***No r***	***No b***
Bbr	Rbr – not possible	BRb	BRr

6. **Assignment:** Check that all of the combinations listed as "not possible" are indeed not possible because two of the sides sum to less than the third side, or at best sum to exactly the measure of the third side. Conversely, check that all the remaining possibilities are, in fact, possible because any two lengths always sum to more than the third.

Someone else thought there might be another problem. Maybe it would be possible to make two different triangles using the same three sides. They were all trying to do just this by putting three sides together in different arrangements when someone recalled from high school geometry: If you have two triangles with all sides congruent, the triangles must be congruent. So any three lengths makes only one unique triangle.



$\Delta ABC \cong \Delta EFD \cong \Delta HKG$

7. **Assignment:** Trace triangle ABC on a piece of tracing paper. Show that it is congruent to the other two triangles by moving onto each one. You may need to flip the paper over.

In the end Group B also concluded that there are 20—4 = 16 possible triangles.

When the two groups came together they noticed some remarkable things. Although both groups started with very different ideas of how to solve the problem, they arrived at the same answer, 16 triangles. Moreover, both groups had employed similar geometric facts -- even though they thought about them differently. Finally, they noticed a striking correspondence in how each group chose to classify the triangles. Group A's collection of triangles corresponds to Group B's organized list of possibilities.

8. Assignment: Go back to page 4 and list the letter combinations from the second solution next to each triangle that the first group made.

9. Assignment: Go back to page 1. Is your solution closer to the first or closer to the second solution or is it completely different? What more needs to be added for your solution to be completely convincing? Make changes and addition on that page until you are satisfied that you have a complete explanation shown -- with words, pictures and symbols -- on the page.

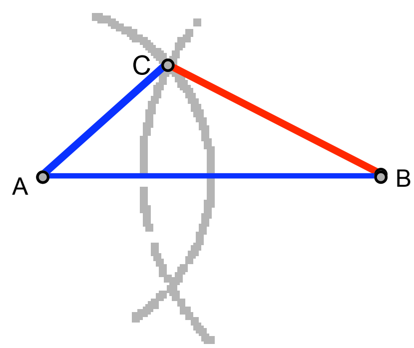
CONCRETE, SYMBOLIC, ABSTRACT

Often students find mathematics difficult because it is so abstract. Using concrete models, like geostrips, can help get a handle on abstractions, and describing the model with symbols provides a method of communicating mathematics ideas with others. Group A used a very concrete approach to the problem. Group B preferred to use symbols to solve the problem. Both methods provided help in understanding several abstract geometric facts. In this section we will examine these facts.

Let's look more closely at the geometric facts both groups recalled from high school geometry. The first two facts are so well known that they have special names: "The Triangle Inequality" and "Side-Side-Side" of just "SSS".

> **Triangle Inequality:** The sum of the lengths of two sides of any triangle is always greater than the length of the third side.

Here is a picture justification of the Triangle Inequality:

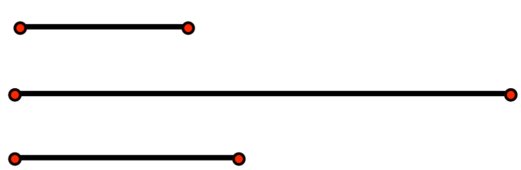


10. Explain how this picture justifies the triangle inequality.

The line segments, $\overline{AC}$ and $\overline{BC}$, when rotated onto the third side of the triangle will overlap showing that the sum of those two lengths are greater than the length of $\overline{AB}$.

Another way to think about The Triangle Inequality is to recall the expression "The shortest path between two points is along a straight line." Walking from point A to point B by going first through point C means more steps than going "as the crow flies," straight from A to B.

11. Assignment: Show that these three lengths cannot form a triangle.



The second fact is SSS.

> **Side-Side-Side (SSS)**: If the three sides of a triangle are equal in length to the three sides of another triangle, then the two triangles are congruent.

Sometimes this fact is stated: "Triangles are rigid." Rigid means that one cannot change the angles without changing the lengths of the sides.

12. Think of examples of triangles in architecture, carpentry, art or elsewhere and examine how the rigidity of the triangle is important.

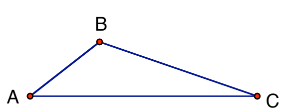
A brief justification of SSS: Why is it not possible to make two different triangles with three given sides? When working with Geostrips, most students automatically accept SSS, but it may not be obvious to everyone or in every situation. After all a triangle consists of six parts: three sides AND three angles. One thing that SSS tells us is that one only needs to know <u>three</u> of the six parts – namely the lengths of the three sides -- to make the triangle.

In the following justification of SSS, we need the defining characteristics of circles:

> **Definition:** A *circle* is a plane curve everywhere equidistant from a given fixed point, the center.

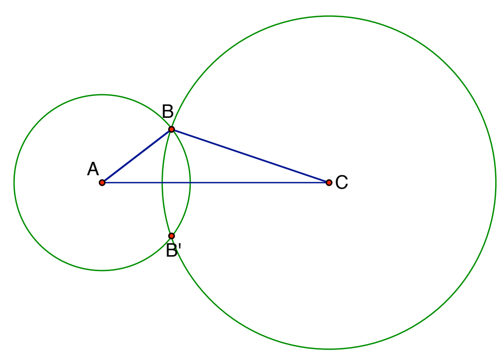
As you read this justification, think about how this definition is being used.

We start with any triangle.



We imagine any other triangle with sides congruent to AB, BC, AC and imagine that you move that triangle so that side congruent to AC coincides with the side AC. Now you may have to flip your triangle over so that the side that is congruent to AB is on the left, and the side that is congruent to BC is on the right. Now, where is the third vertex of our imagined triangle? If it is right on top of point B, the two triangles are congruent. Next, we imagine two circles to locate that vertex.

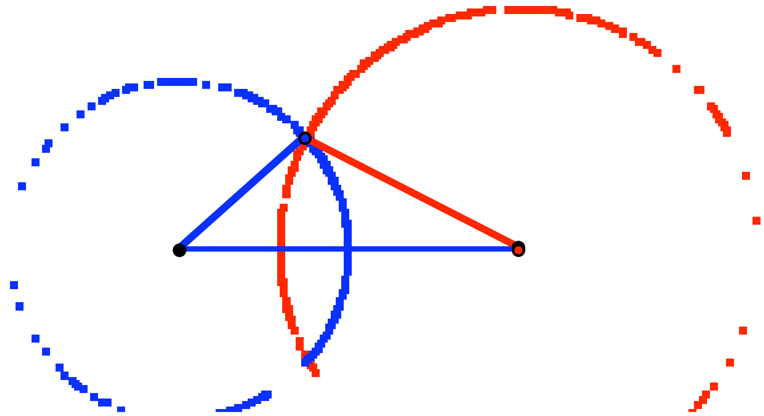
Construct a circle at A that has a radius equal to the length of the segment AB. This circle contains the point B, but it must also contain the third vertex on our imagined triangle, since this vertex has to be a distance, equal to the length of side AB, from A. (see the definition of a circle) Repeat on the other side, so you have constructed a circle centered at C that has a radius equal to the length of BC and must also contain the third vertex of our triangle.



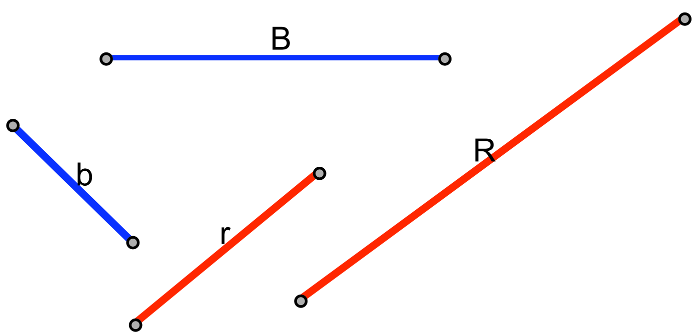
Since the third vertex of our triangle is contained in both circles there is only two possible places for it. One is right on top of point B. In this case, our imagined triangle coincides with the original triangle and we are done, the two triangles are congruent.

The other case is when the third vertex of our triangle coincides with point B'. In this case, just fold the paper on the line AC. This fold, folds the circles on top of themselves, point B' now coincides with point B, and again we are done because our imagined triangle coincides with the original triangle. Now we know that our imagined triangle must be congruent to the original triangle. This completes the justification.

Construction of triangles using SSS. It is not easy to construct a triangle, given three side lengths with just a ruler, but with a compass, it is easy. Just follow the steps as listed in the previous argument. Once you have one side drawn, use your compass to draw the two circles with radii equal to the other two lengths. Then choose which point you want for the third vertex. In either case you are guaranteed a triangle that has the three given lengths.



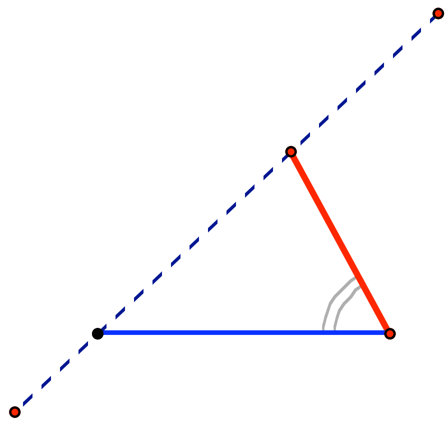
13. Assignment: Practice making triangles with your compass: using the lengths from the first activity, construct triangles RBr and BBb.



There are two other "congruency" facts which you may recall from High School Geometry.

Side-Angle-Side (SAS): If two sides of a triangle are equal in length to the two sides of another triangle, and if the angle BETWEEN the two sides has the same measure for both triangles, then the two triangles are congruent.

14. Assignment: Attach two Geostrips and hold them so as to fix the angle between them. Describe your choices to complete a triangle.



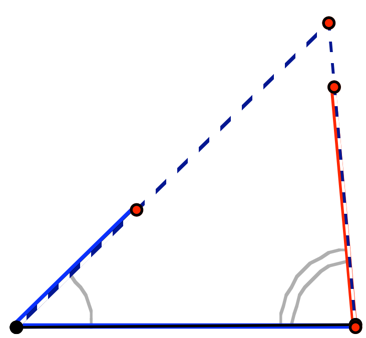
The red line segment doesn't change length.
The blue line segment doesn't change length.
And the angle doesn't change.

There is only one way to connect the other two points with a starightline

Angle-Side-Angle (ASA): If two angles of a triangle are equal in measure to two angles of another triangle, and if the side BETWEEN the two angles has the same measure for both triangles, then the two triangles are congruent.

15. Assignment: Attach three Geostrips in a line and hold them so as to fix the angles at the two pins. Describe what choices there are to complete a triangle.

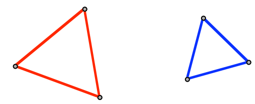
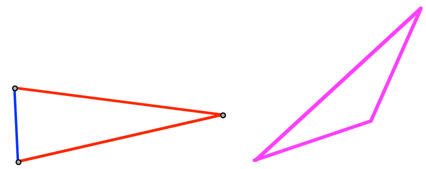
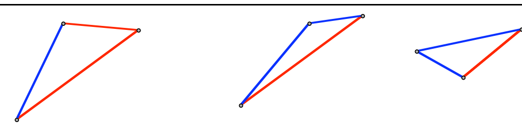
You may have to extend one or both line segments but if the angles do not add up to 180°or more there is one and only one way for them to meet in the third vertex.



This blue line segment and the two angles are fixed.

There are three common ways to think about triangles: by sides, by angles, and by vertices. We will discuss a little about each view.

CLASSIFICATION OF TRIANGLES BY SIDES: In solving the opening problem, both groups used an organizing method to help analyze the situation. In fact both used the same classification scheme and a scheme that has made sense to a lot of people. You may be familiar with the words defined in this table:

Equilateral triangle	All three sides of the triangle are congruent.	
Isosceles triangle	At least two sides of the triangle are congruent. (includes equilateral triangles)	
Scalene triangle	No two sides are congruent	

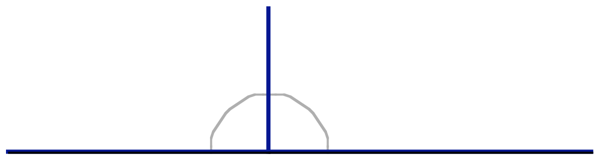
Carefully notice the difference in the three definitions. Every word is important.

16. Assignment: Explain to a friend why it is that an equilateral triangle is also an isosceles triangle but an equilateral triangle is not also a scalene triangle. Write that explanation here.

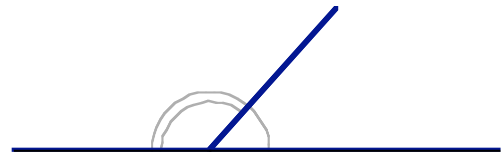
17. Assignment: Sketch a Venn diagram that shows the correct relationship among these three types of triangles.

CLASSIFICATION OF TRIANGLES BY ANGLES: There is another classification scheme that refers to the **angles** of the triangle. To understand this scheme we must first define what we mean by a right angle. Often someone will say a right angle is 90° which, of course, is correct but it's not much of a definition if we don't know what is meant by one degree. In this lesson, we define right angle *without using degree measurements* and wait for the next lesson before defining a *degree.*

Definition: If a straight line is divided into two angles and if the two angles are congruent, then we say that each angle is a *right angle*.



Two congruent angles – each one a *right angle.*



Two non-congruent angles. Neither is a *right angle*

Examples:

1) Draw a straight line on a piece of paper and fold the paper so that the line folds on top of itself. The fold line and the original line forms two angles – because one angle folds on top of the other angle the two angles are congruent and hence right angles.

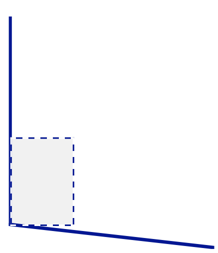
2) Line up two 3X5 cards to make a straight line, but if you put one of the cards on top of the other you confirm that the two corner angles are congruent. Since you can line up the two angles to form a straight line, the corner angle is a right angle.

18. Assignment: List some other examples of right angles. How do you know each example is a right angle?

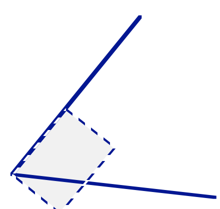
Definition: An *acute angle* is an angle that is smaller than a right angle.

That is, an angle is *acute* if it can be moved to fit inside a right angle. An *obtuse angle* is an angle that is larger than a right angle. That is, a right angle can be moved to fit inside of it.

Definition: An *obtuse angle* is an angle that is larger than a right angle.



obtuse

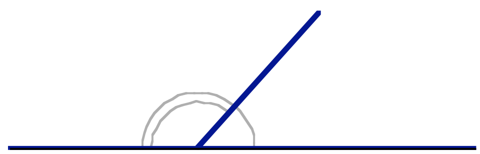


acute

An easy way to judge if an angle is right, acute or obtuse is to hold the corner of a 3X5 card (or some other convenient right angle) to the figure – on the vertex of the angle and the edge on one of the rays of the angle.

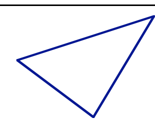
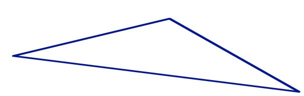
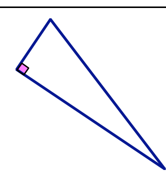
Group discussion:

If a line is divided into two angles that are NOT congruent, one angle is acute and one angle is obtuse. With your group, make a convincing argument why this is the case. What are the important points to include in your explanation?



Now we are ready to classify triangles by the type of angles it contains. We have already defined what is meant by an acute, an obtuse and a right *angle.* Now we define what is meant by an acute, an obtuse and a right *triangle.*

CLASSIFICATION OF TRIANGLES BY ANGLES:

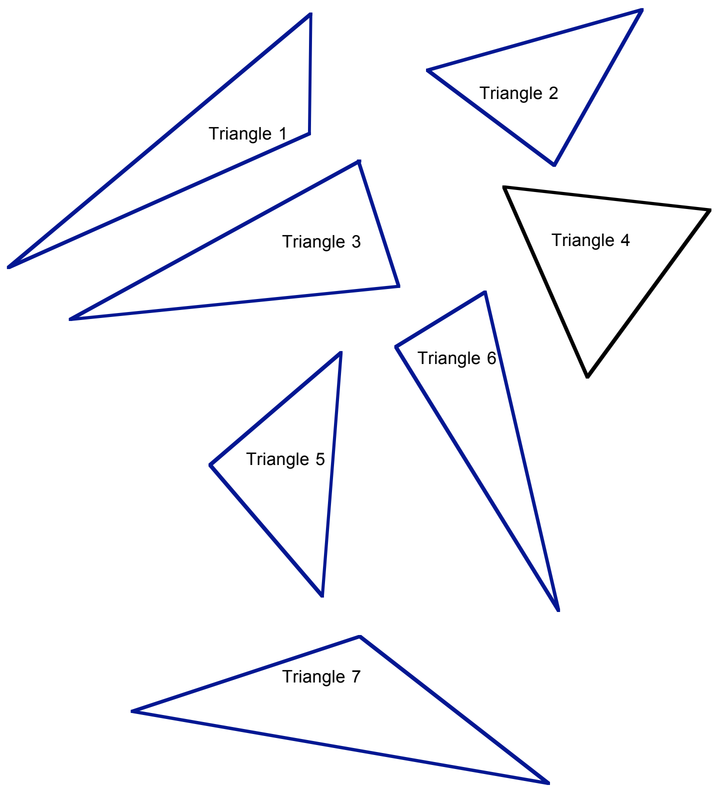
Acute triangle	<u>All</u> angles are acute angles.	
Obtuse triangle	<u>One</u> angle is obtuse.	
Right triangle	<u>One</u> angle is a right angle.	

The little square in the triangle denotes a right angle.

19. Assignment: Explain to a friend what difference the words “All” and “One” make in the definitions in the table.

We can combine the two classifications to get 9 different categories for triangles.

20.Assignment: Match the triangles with the classification:



A. Acute scalene
B. Acute isosceles
C. Acute equilateral

D. Obtuse scalene
E. Obtuse isosceles
F. Obtuse equilateral

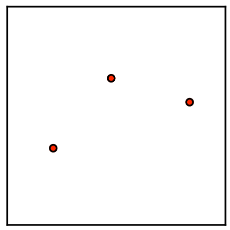
G. Right scalene
H. Right isosceles
I. Right equilateral

There are only seven triangles at the left but nine classifications. Which types of triangles are missing?

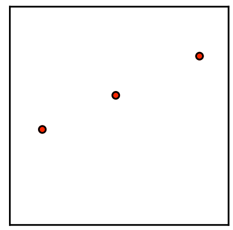
Is it possible to construct the missing ones?

So far we have discussed triangles by looking at the lengths of the sides and by looking at the three angles of a triangle. **A third way to think about triangles** is to think about the three points that make the vertices of the triangle. Under what conditions do three points make a triangle? Two points determine a line and the third point may be on that line – in which case you don't get a triangle. But if the third point is not on the line connecting the three points constructs a triangle. We say that three *non-collinear* points determine a triangle.

21. Assignment: Which set makes a triangle when you connect the dots?



SET 1

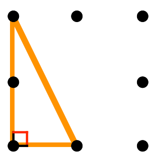


SET 2

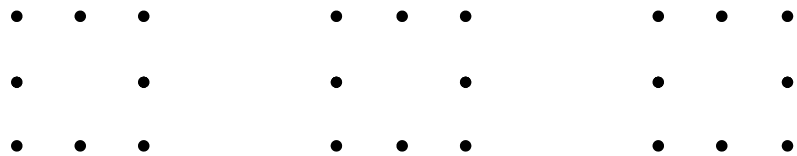
Group Project: Triangles on a square

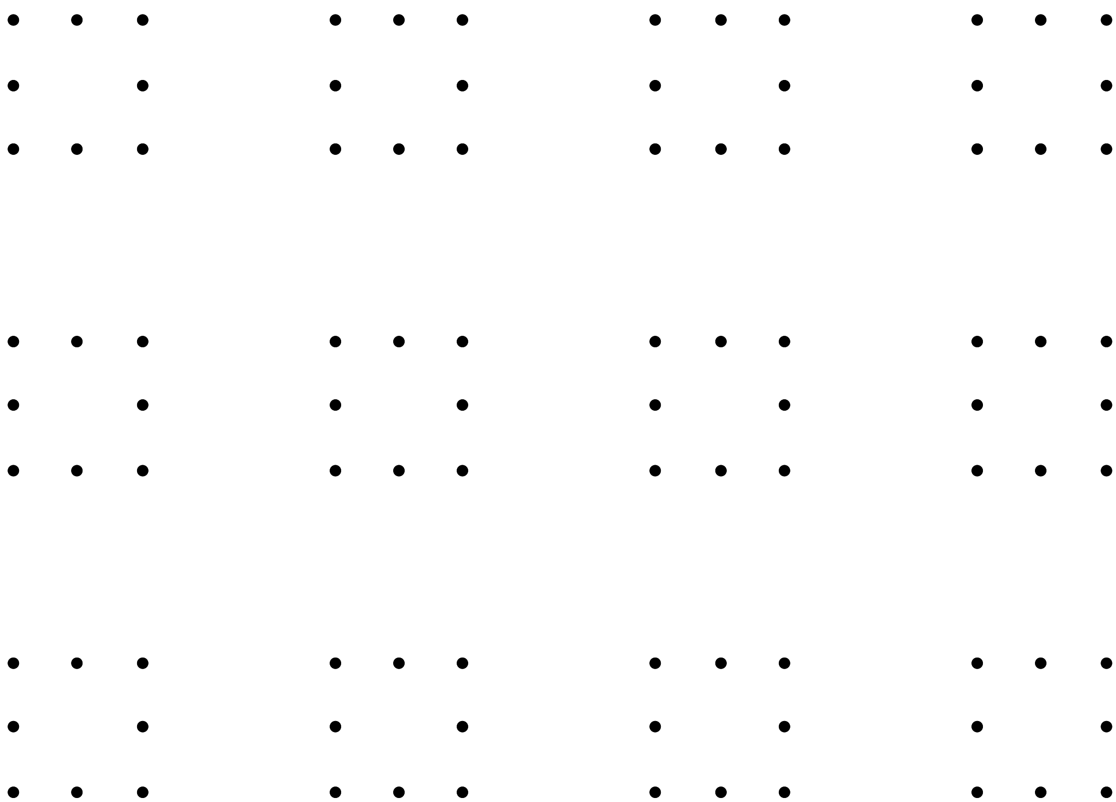
How many different (non-congruent) triangles can you find using the corners and the midpoints of the sides of a square? Classify each triangle by sides and angles. Be prepared to justify your answers. One solution is given. Show them all here. How do you know that you have them all? How do you know that they are all "different"?

Work with your group to find as many different triangles as you can. Make sure that no two of your triangles are congruent. BE CAREFUL. Two triangles may look different but by rotating, reflecting or shifting them on the grid they may coincide. *Tracing the triangle onto tracing paper is an easy way to compare two triangles.*



This is a right scalene triangle. The square makes the right angle and the side lengths are 1 unit, 2 units, and $\sqrt{5}$ units

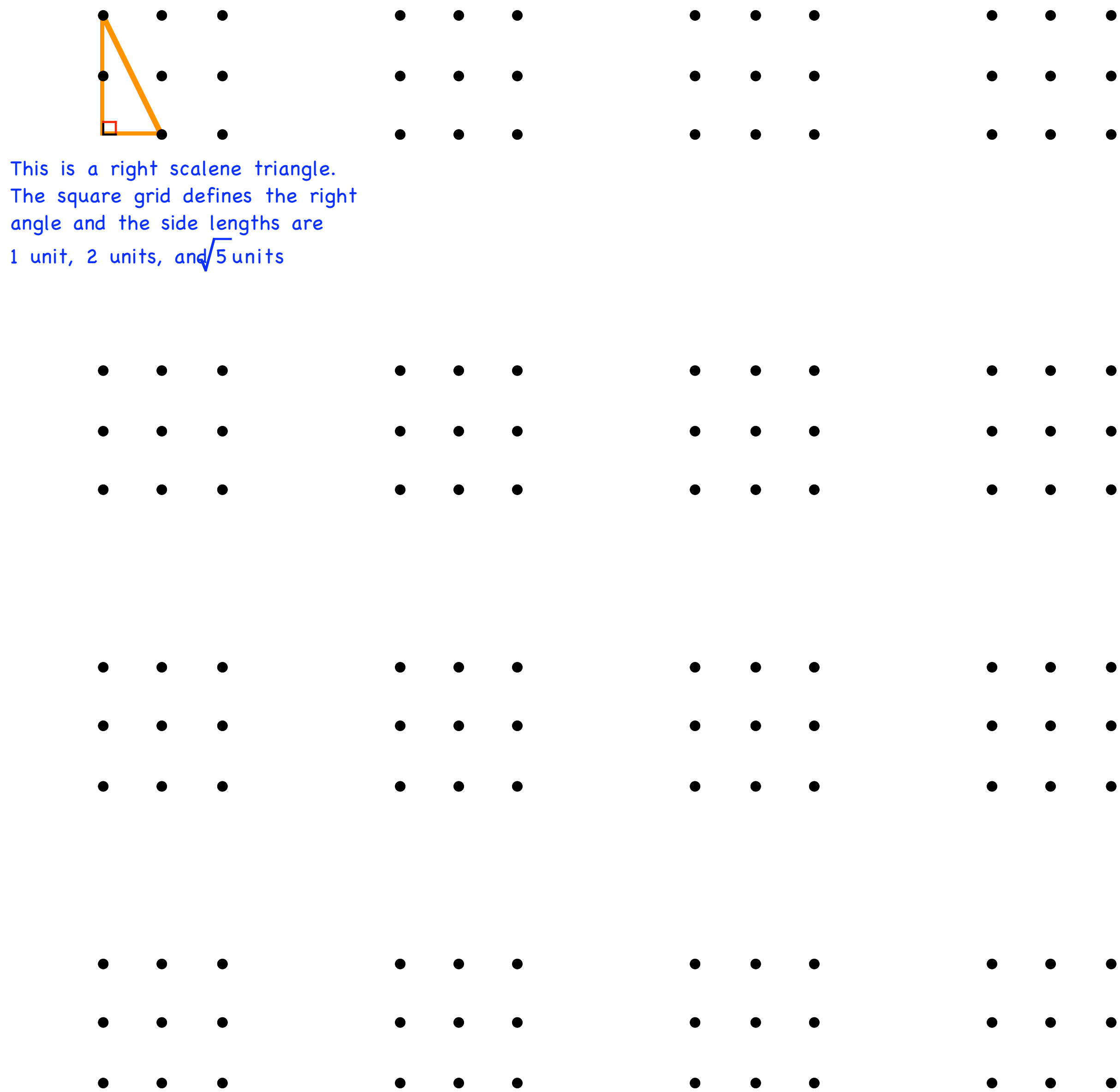




Another questions to consider: How many right isosceles triangles can be made? (This time count ones that are congruent but in different positions as being different triangles.)

Challenge: Triangles on a square grid

How many different (non-congruent) triangles can you find using three dots on a 3X3 dot grid? Classify each triangle by sides and angles. We know it has a right angle because the grid is designed to give us right angles. Be prepared to justify your answers. One solution is given.



22. Challenge: Find a way to organize a list of all the triangles you found so that you can determine whether or not you've found them all.

Definitions are an important part of any kind of mathematics, and geometry is no exception. The following exercise, which you may want to repeat on your own, indicates how tricky it can be to get started with good definitions.

Class Activity: An Experiment in Circular Definitions

Look up a definition of the word "circle" and make a tree of words and definitions used until common words are found.

Results: Here is the definition we found in the on-line version of *The American Heritage® Dictionary of the English Language* - Fourth Edition

Circle: A plane curve everywhere equidistant from a given fixed point, the center.

Suppose we do not understand what is meant in this definition, we decide to look up "plane", "curve", "point", "center". We know that "equidistant" means equal distance so we might just look up "distance". "Equal" might be another word to look up but we decide not to follow that line of definitions, although it may be interesting to do so. Here are the definitions we found. We follow through by looking up each of the bold face words.

Plane: *Mathematics* A surface containing all the straight **line**s that connect any two **point**s on it.
Curve: A **line** that deviates from **straight**ness in a smooth, continuous fashion.
Distance: *Mathematics* The length or numerical value of a straight **line** or curve.
Point: *Mathematics* a) A **dimension**less **geometric object** having no properties except **location**.
b) An element in a geometrically described **set**
Center: A **point equidistant** from all **point**s on the circumference of a **circle** or on the surface of a sphere.

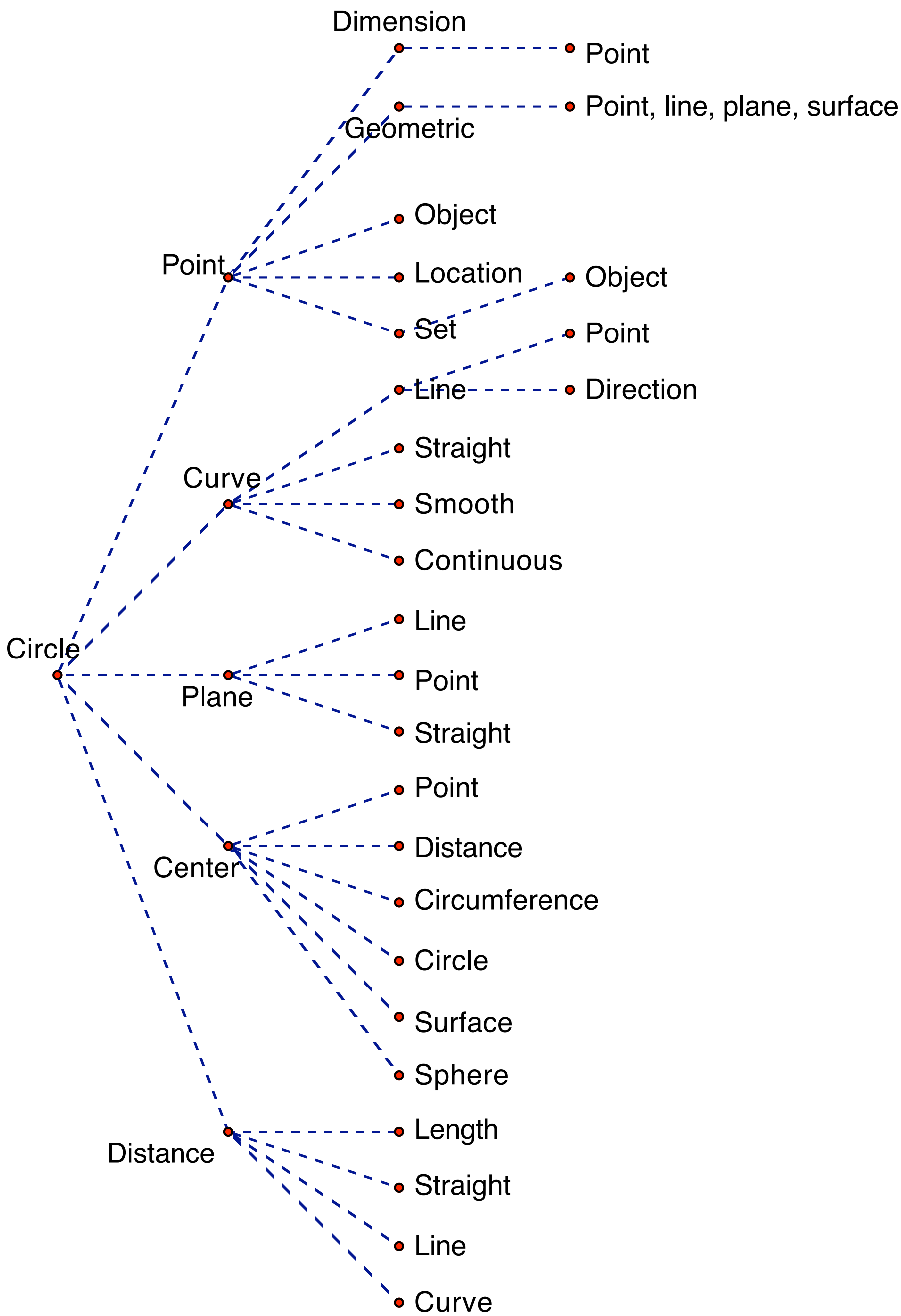
And continue looking up new words as they appear:

Line: *Mathematics* A geometric figure formed by a **point** moving along a fixed **direction** and the reverse direction.
Dimension: *Mathematics.* The least number of independent coordinates required to specify uniquely the **point**s in a space.
Geometric: The mathematics of the properties, measurement, and relationships of **point**s, **line**s, angles, surfaces, and solids.
Location: A place where something is or could be **located**; a site.
Set: *Mathematics A* **collection** of entities, called **element**s of the set, that may be real **object**s or conceptual entities. *(from the Columbia dictionary)*

And more words,

Collection: A group of **object**s or works to be seen, studied, or kept together.
Group: An assemblage of persons or **object**s gathered or located together. OR
Mathematics A **set** with a **binary associative operation** such that the operation admits an **identity element** and each element of the set has an **inverse** element for the operation.

Here is the tree we made as we looked up more and more words.



Analysis: We notice three things happening.

1) After a while we don't find mathematical definitions for the words and, as such, lose the precision necessary for doing mathematics, as with the words, *set, collection, group* (first definition).
2) Also, the opposite happens, words begin to have more and more technical meanings, so it looks like we need to know more and more to understand the word. The mathematical definition of *group*, for example, has really taken us into an unknown realm.
3) Certain words appear repeatedly: *point, line, object, set, element.*.

In the end, we determine there really is no end to this process. We can't get to the bottom of things. Nonetheless, the definition of circle is clear because we have a strong intuitive idea of most of the words.

Further, to study geometry we must come with a good idea of what is meant by “point” and “line” and “plane” because no one can precisely define them for us. Or if they could something else would be left with no definition.

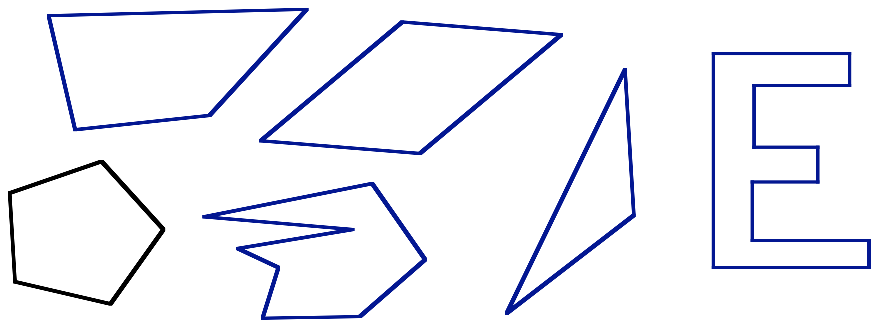
Another way to learn definitions for geometric objects is to see a lot of examples. Children learn definitions from the context in which they are used. To learn correctly in this way a child must experience lots of different examples and assimilate the common features. Of course, even when one is able to read definitions, one needs to see examples to make words more meaningful.

The following pages attempt to define three words, *polygon* and *convex* and *regular polygon*, by using multiple examples of each one. In each case you are asked to state a definition with your own words. You will also want to look up a definition in a dictionary, but make a stab at your own definition first.

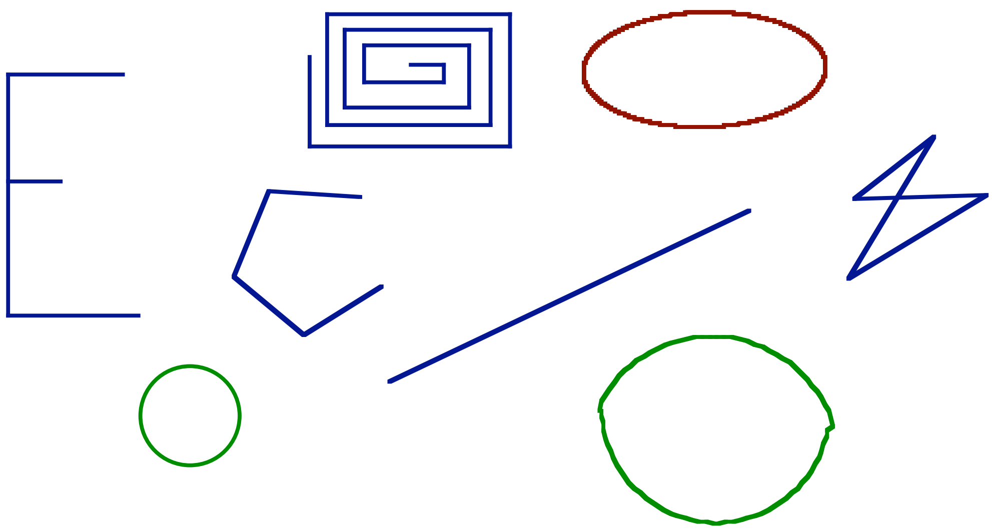
23. Assignment: After completing the three pages concerning the definition of polygon, convex and regular polygon, make your own worksheet to help children understand the definition of triangle. Include enough examples of figures that are triangles AND figures that are NOT triangles, so that a child will know exactly what a triangle is.

Definition: *Polygon*

These are polygons:



These are NOT polygons:



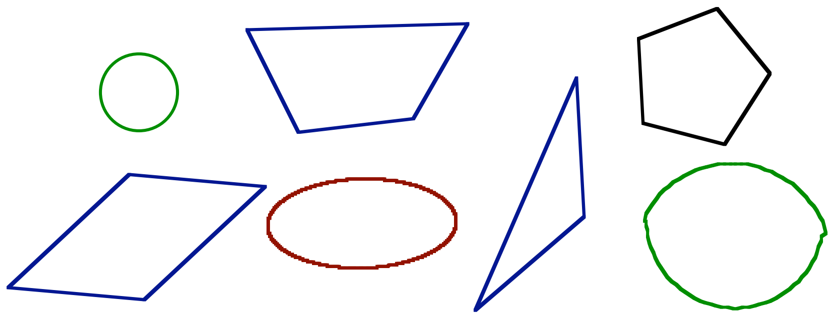
Sketch three more figures that are polygons:

Sketch three more figures that are NOT polygons:

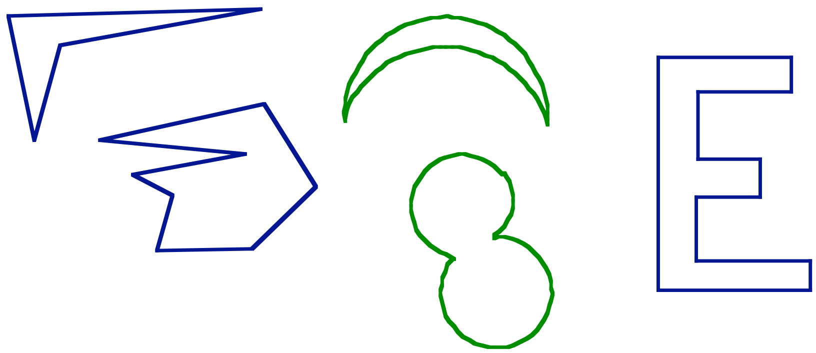
In your own words, what is a polygon? Look up the definition of polygon in a dictionary or by doing a web search. Compare the technical definition you found to your own.

Definition: ***Convex***

These shapes are convex:



These are NOT convex:



Sketch three more figures that are convex figures:

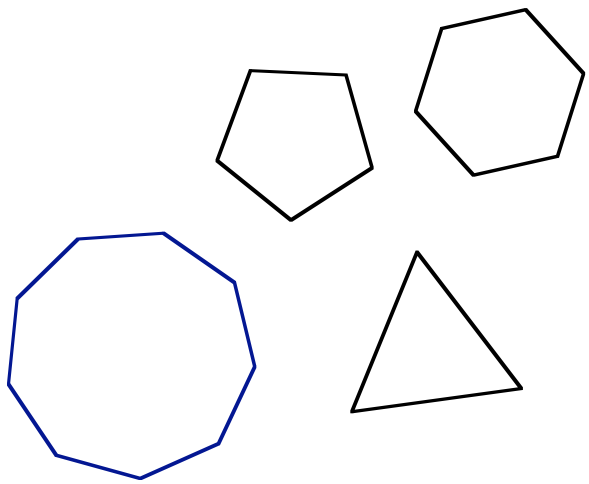
Sketch three more figures that are NOT convex:

In your own words, what does it mean for a figure to be convex? Look up the definition of convex in a dictionary or by doing a web search. Compare the technical definition you found to your own.

Definition: ***Regular Polygon***

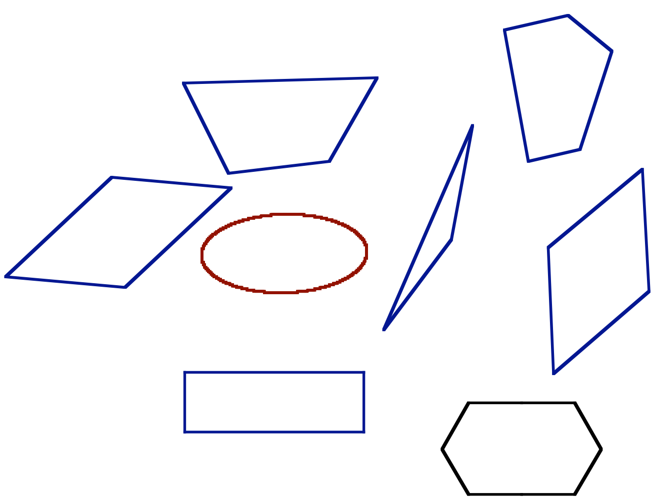
These are regular polygons:

Draw two more regular polygons:



These are NOT regular polygons:

Draw three more polygons that are not regular polygons:



In your own words, what is a regular polygon?

24. Assignment: What are some advantages to learning words from context in this way? What are some disadvantages?

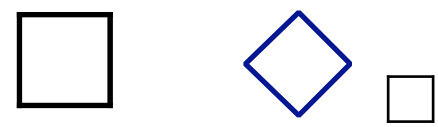
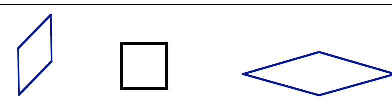
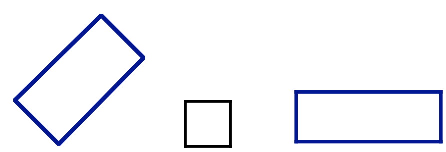
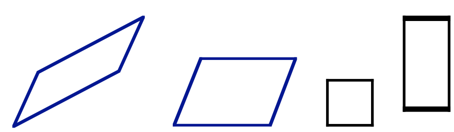
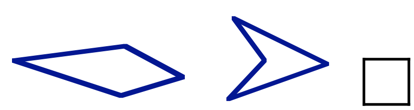
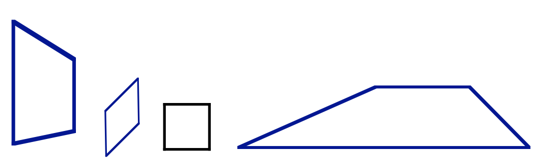
Definition: A *quadrilateral* is a polygon that has exactly four sides.

25. Assignment: How many vertices does a quadrilateral have? Explain your reasoning. Do any four points determine a quadrilateral? Give examples to explain your answer.

Classification of quadrilaterals by sides and one angle:

Quadrilaterals are trickier than triangles. Here is one familiar way to classify quadrilaterals using both sides and angles as conditions. Unlike the classification schemes for triangles that we saw earlier, this one does not include ALL quadrilaterals. That is, there are quadrilaterals that do not fit into any of these six categories.

Both conditions must hold.

Square	All sides congruent	One right angle	
Rhombus	All sides congruent	No other angle restrictions*	
Rectangle	Opposite sides congruent	One right angle	
Parallelogram	Opposite sides congruent	No other angle restrictions*	
Kite	Two distinct pairs of adjacent sides are congruent.	No other angle restrictions*	
Trapezoid	At least one pair of opposite sides are parallel.[1]	No other angle restrictions*	

[1] Sometimes defined to exclude parallelograms

* As we will see in Lesson 4, the sum of the four angles must always be 360˚

Assignment: Make a Venn diagram that shows the relationship among the squares, rhombuses, parallelograms and rectangles.

26. **Assignment:** Sketch a quadrilateral that is not any of the above types of quadrilaterals. Explain your sketch.

27. **Assignment:** Explain why these definitions tell us that each statement below is true. The first one is done for you.

A square is also a rectangle.
Since all sides of a square are congruent, it must be true that opposites sides are congruent. This satisfies the first condition to be a rectangle. Since a square has one right angle, a square also satisfies the second condition to be a rectangle. Hence a square is also a rectangle.

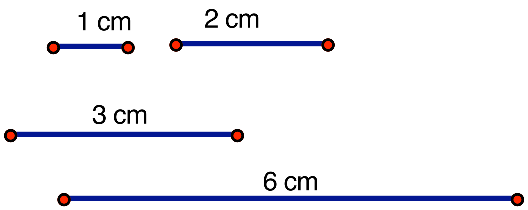
A rectangle is also a parallelogram.

A square is also a kite.

A parallelogram is also a trapezoid.

Lesson 1 Problems:

1. How does the problem in the first Class Activity change if the lengths of the line segments are changed? For example, how many different (non-congruent triangles) can be made using these lengths?

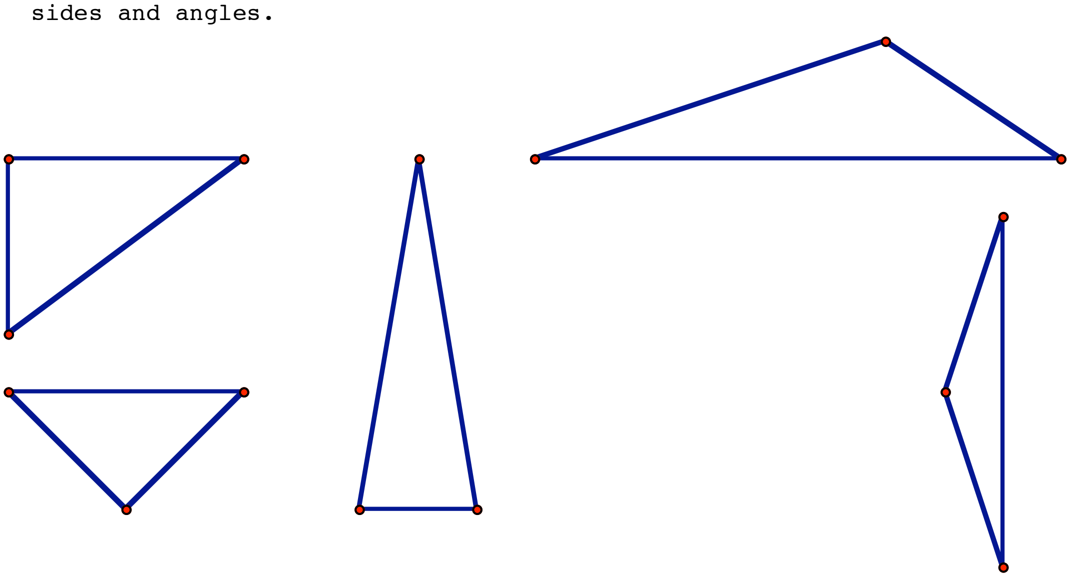


2. Which of the following sets of numbers can be sides of a triangle?

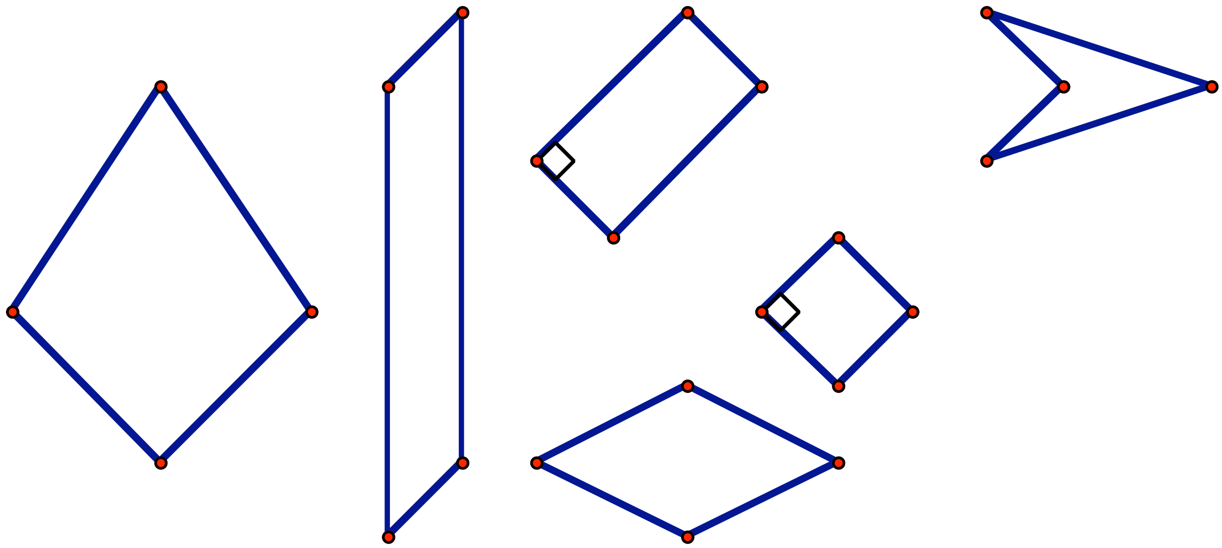
a) 12 cm, 12 cm, 30 cm
b) 5 cm, 5 cm, 5 cm
c) 10 cm, 3 cm, 6 cm
d) 3 cm, 4 cm, 5 cm

Circle all possible correct answers. Use a ruler and a compass to make an accurate drawing of each possibility.

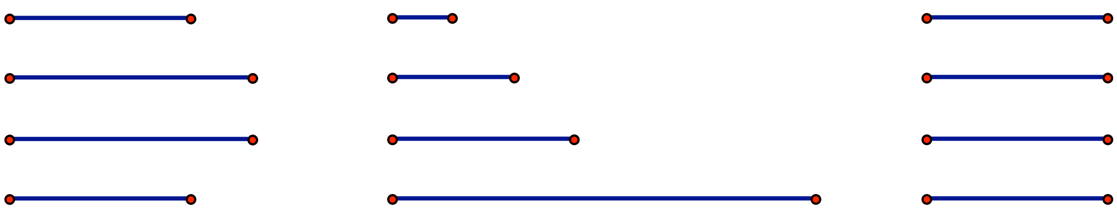
3. Compare the sides and angles of these triangles and classify the triangles by both sides and angles.



4. Compare the sides of these quadrilaterals. Indicate which sides are congruent in each figure, then classify the quadrilaterals. Right angles are shown. Explain your answer in terms of the definitions given in this book.

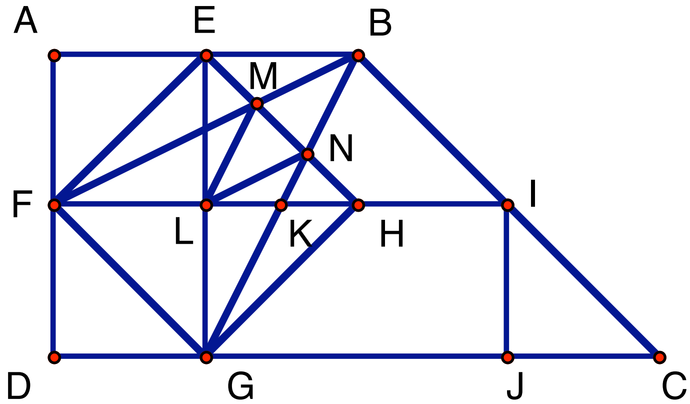


5. For each group, circle all quadrilateral types that can be made using the four lengths (Use each length only once.)



Group 1	Group 2	Group 3
Parallelogram	Parallelogram	Parallelogram
Rhombus	Rhombus	Rhombus
Rectangle	Rectangle	Rectangle
Square	Square	Square
Kite	Kite	Kite

6. Find the following shapes in the figure. List them by giving the vertices. Be prepared to justify your answers.



a. 3 squares

1-

2-

3-

b. 2 rectangles that are not squares

1-

2-

c. a parallelogram that is not a rectangle

d. 7 congruent right isosceles triangles

1- 4- 7-

2- 5-

3- 6-

e. 2 isosceles triangles that are not congruent to those in part d)

1-

2-

f. a rhombus that is not a square

g. a scalene triangle with no right angles

h. a right scalene triangle

i. a trapezoid that is not isosceles

j. an isosceles trapezoid

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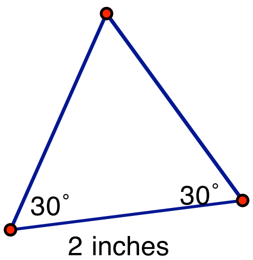
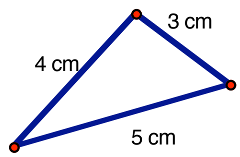
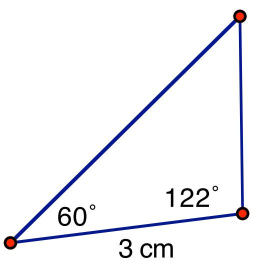
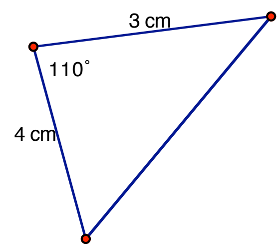
7. These triangles are not drawn to scale. Construct an accurate version of each one, then measure the remaining angles and sides. In each case explain why there is only one possible triangle to construct from the given information using SSS, SAS, or ASA.

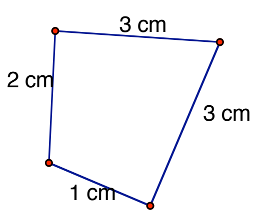


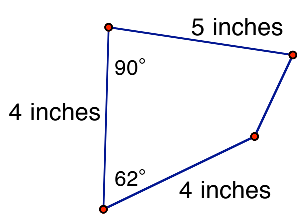




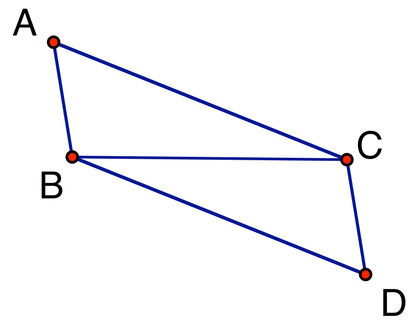


8. These two quadrilaterals are not drawn to scale. Carefully construct an accurate version. Measure the remaining angles and sides. In each case, is it possible to construct a quadrilateral, satisfying the same conditions, but not congruent to the one you constructed?





9. Trace this diagram to show (by moving on to be on top of the other) that the two triangles formed by adding a diagonal to a parallelogram are congruent. Which of these statements is correct?



$\Delta ABC \cong \Delta BCD$

$\Delta ABC \cong \Delta BDC$

$\Delta ABC \cong \Delta DCB$

$\Delta ABC \cong \Delta DBC$

$\Delta ABC \cong \Delta CBD$

$\Delta ABC \cong \Delta CDB$

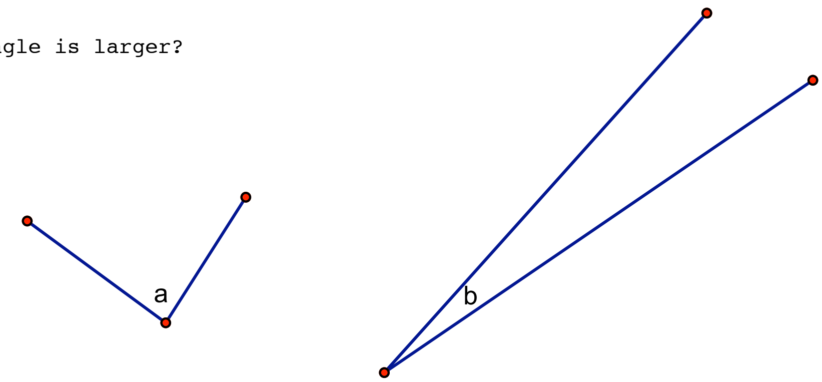
Is the correct congruency confirmed by SSS, SAS or ASA? Explain.

10. If I had 4 Geostrips with lengths 1 inch, 2 inches, 3 inches and 5 inches, I could not make any triangles. Explain why not. What is the shortest strip, longer than 5 inches, that you could add to this collection and still not be able to construct a triangle?

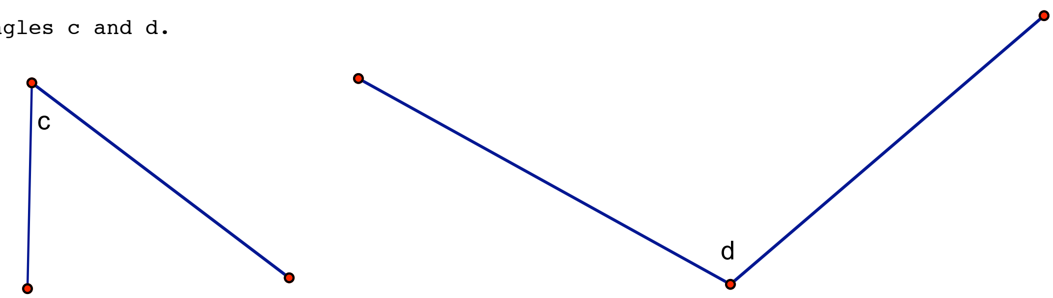
11. Challenge: Suppose someone has 7 strips but cannot make even one triangle. Is this possible? Is this possible if the longest length is 21 inches? List the seven lengths if possible or provide an explanation why not?

Practice using a protractor to measure angles

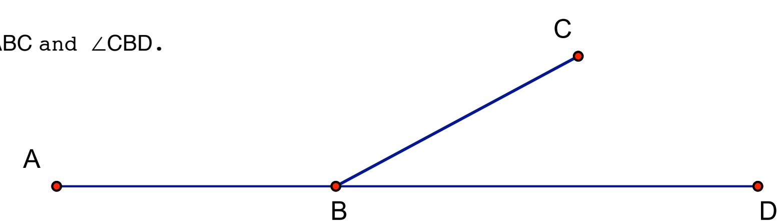
Which angle is larger?



Measure angles c and d.



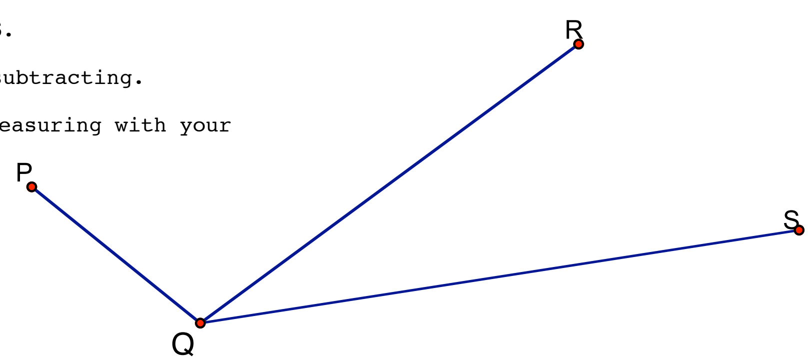
Measure angles ∠ABC and ∠CBD.



Measure angles ∠PQR and ∠PQS.

Find the measure of ∠RQS by subtracting.

Confirm your computation by measuring with your protractor.



Other ways to measure angles

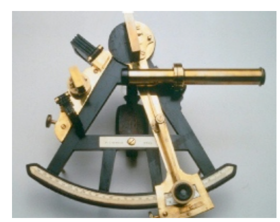
The measurement of angles has many real life applications. Here are three examples. All three come with their own set of complications when attempting to measure the proposed angles. Angles do not always lie flat on a plane where a protractor could easily measure them. So special instruments, as shown below, need to be developed.

Example from Engineering: The **angle of repose** is the maximum angle of stability for a pile of granular material, such as sand. The angle of repose changes with the material as determined by friction, cohesion and the shape of individual particles. Any child with experience at the beach or in a sandbox is familiar with this property -- one can pile up sand only so high, then the sand will start to slip down the sides. The only way to build taller is to build wider as well. Wallace Stegner uses the concept as a metaphor for life's calamities in his Pulitzer Prize winning novel, "Angle of Repose" (1971).

Many patents have been filed for instruments that could measure the angle of repose. Here is a quote from the abstract for the 1976 patent #3940997[1]:

". . . a device and method for measuring the angle of repose of a granular material. The device is composed of a rectangular body having an open ended partition dividing the body into two cells. A sample of granular material is placed in one cell and allowed to flow from one cell into the other cell under the influence of gravity. Both the static and dynamic angle of repose can be determined."

Example from Navigation: A sextant is an ancient instrument used to measure the angle of the sun (or a planet or a star) above the horizon. This information, in turn, is used to determine one's location on the Earth. You can see young seaman learning to use one in the Peter Weir movie, "Master and Commander: The Far Side of the World" (2003).



" A very handsome example by H. Limbach of Hull of a sextant with an ebony frame."
– From The History of the Sextant by Peter Ifland, Talk given at the Physics Museum, University of Coimbra, October 3, 2000.

The web page,
http://academic.brooklyn.cuny.edu/geology/leveson/core/linksa/findlat.html
contains an entertaining description of how measuring the angle from the horizon can be done with a sextant.

Example from Carpentry: Roof pitch is one of the most used terms in roofing. A typical roof pitch measurement might be 7/12. This fraction is just like the "rise over run" slope of a line. It means that the roof rises 7" for every 12" it runs. The example to right shows one way to measure the angle that the roof makes with a horizontal line. The pitch can be determined from the angle (and vice versa) using trigonometry, but the tool shown here measures the angle and has tables on the back that help the carpenter convert between angle and slope measurement.

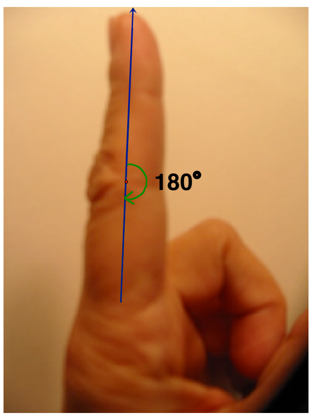
[1] Picture from Wikipedia, the free encyclopedia, article on Angle of Repose

Example from hand therapy: When bones, joints and tendons are not working properly (for whatever reason) the angle of rotation of a bone at a joint may be diminished. Measuring the maximum angle a bone makes at a joint can help the patient and therapist track progress during rehabilitation.

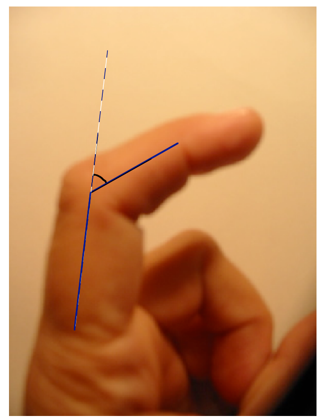
Definition: *Goniometry* is the measuring of angles created by the bones of the body at the joints. The words come from two Greek roots: "gonia", meaning angle (so this is the same "gon" as in polygon) and "metron", meaning measure. Hence, gonometry means the measurement of angles. A *gonometer* is a devise to measure angles between bones.

Group Project: Making a gonometer

We want to measure the angle between the fully extended finger and the first joint when that joint is bent as much as possible.



Fully extended



Bent

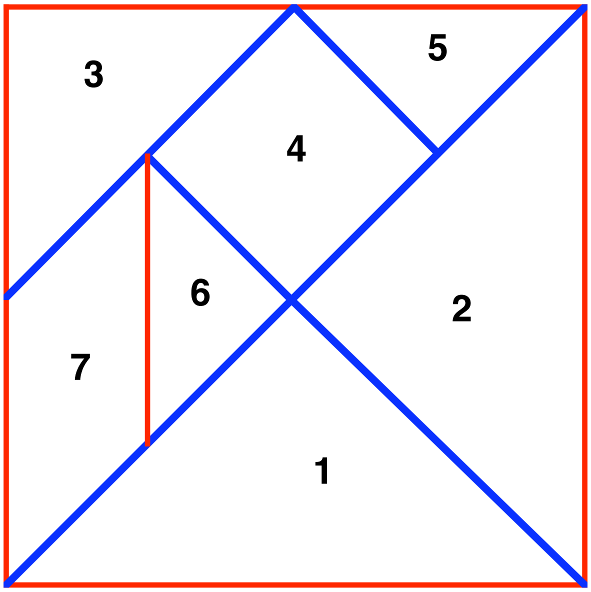
Working in pairs, figure out a way to measure this angle on your own hand. Describe the tools you used to make the measurements and how you used the tools. What might be done to increase the accuracy of your measurements?

Use your method to measure the angles made at your knuckle and at each joint of your index finger when you make a tight fist. Make a model scale drawing of your finger and label the angles measured.

28. Assignment: Do research to find a picture of a real gonometer. How does it differ from your method?

Lesson 2 – Puzzles, tessellations and angle basics

Activity: Tangrams



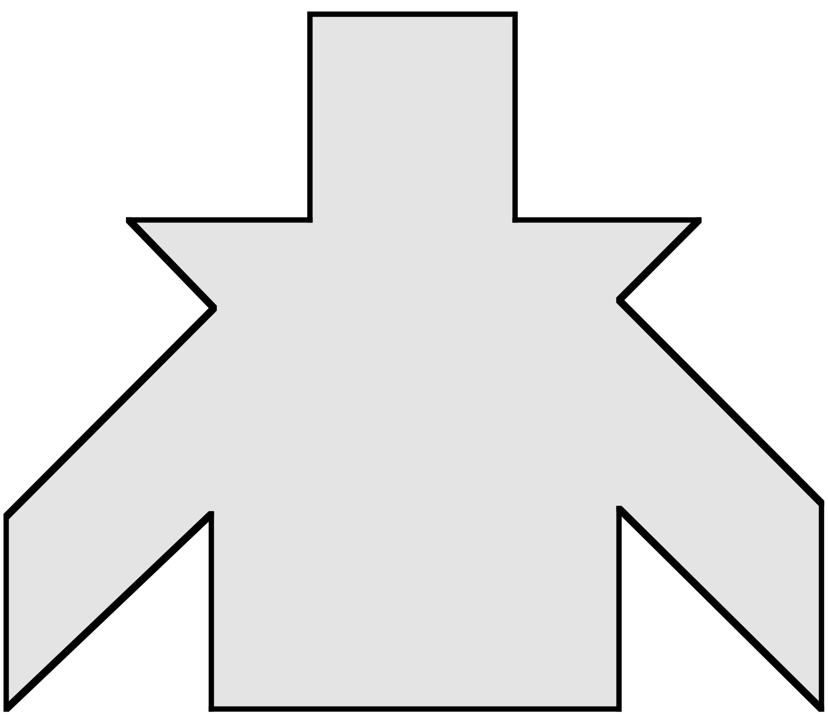
Make four tangram sets from 3 inch by 3 inch squares. A 3x5 note card makes good sets this size.

Before you cut out the pieces, color every vertical or horizontal line red. Color all the diagonal lines blue. This color code may help you do the puzzles on the next two pages.

Use these pieces to show your solutions to the problems in this activity. Save the pieces in an envelope or small ziplock bag for later activities.

Arrange the tangram pieces to make a familiar shape like a house or an animal. Use all the pieces from your set and trace the figure on a piece of paper.

Arrange the pieces of one tangram set to completely tile this figure. Show the solution on this figure.



Tangram Problems

1. Trace the solutions on this paper:

Use the two smallest right isosceles triangles(pieces 5 and 6) to make a square. Is it congruent to piece 4?

Use the same pieces to make a triangle? Describe the triangle. Is this triangle congruent to piece 3?

Use the same pieces to make a parallelogram? Is this parallelogram congruent to piece 7? Is it a rhombus?

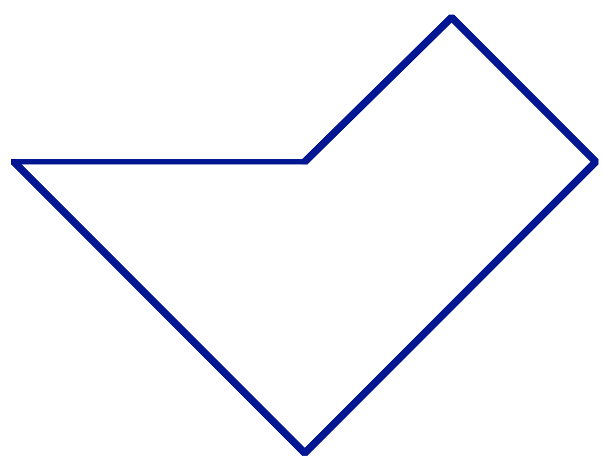
2. In the previous problem you made a square using just 2 pieces from the set. Is it possible to make a square using just 3 pieces? 4 pieces? 5 pieces? 6 pieces? Show each possible square on another sheet of paper.

3. I made a fancy tangram set this size from wood. The square (piece 4) weighs 1 oz. How much does each of these pieces weigh?

piece 1 _____ piece 3 _____ piece 5 _____ piece 7 _____

How much does the entire set (all 7 pieces) weigh? _____

4. Use your tangram pieces to cover this shape and decide how much it would weigh if the square weighs 1 oz.?



5. Use your tangram pieces to make a square that would weigh twice as much as the square, piece 4.

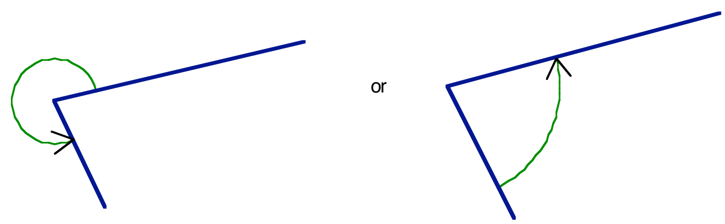
Angle Basics

In this lesson we will explore different ways of determining angles. We start with basic definitions. One thing that can be confusing about angles is that there are different ways to think about them. An *angle* may be defined in two ways:

> **Definition:** ***Angle***
> a. The figure formed by two rays diverging from a common point. The common point is called the *vertex* of the angle.
> b. The rotation required to superimpose either of two such lines on the other.

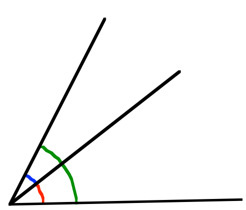
When there are two definitions for the same thing, you need to think about why both definitions define the same thing.

The first definition may be confusing because two rays actually form two different angles. The figure described in part a. makes two different angles. Which angle is meant can be determined by whether the rotation, mentioned in part b, is c*lockwise* or *counterclockwise*.



Unless otherwise stated in this book we will mean the *smaller* of the two angles.

Adding and subtracting angles. We add two angles by adjoining them along a common ray and putting one angle after the other. We subtract two angles by adjoining them along a common ray and putting the smaller inside the larger. Because we are not defining negative angles (this is done in the study of trigonometry), we do not talk about subtracting a larger angle from a smaller one.

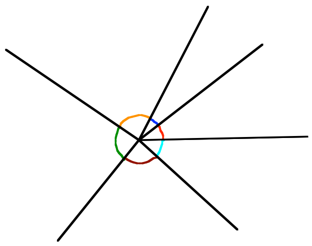


The green angle is the red angle plus the blue angle.
The red angle is the green angle minus the blue angle.
The blue angle is the green angle minus the red angle.

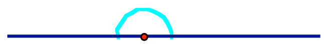
Comparing the size of angles. We compare the size of angles by placing both vertices and one ray together to see which one sweeps out more of an angle. In the above picture, both the red and the blue angles are less than the green angle.

In order to define a degree and use numbers to describe the size of an angle, we start by defining one particular angle:

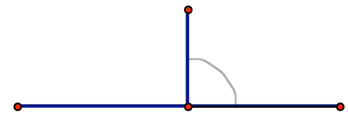
> **Definition**: An angle that sweeps out a full circle is 360 *degrees*, often written as 360˚.



These six angles sum to 360˚



A straight line makes a 180˚ angle because two of them make a full circle or 360˚



A right angle measures 90˚ because two of them make a straight line or 180˚

All other angle measurements are made to make standard addition and subtraction of positive numbers work out while adding and subtracting angles.

> **Definition:** One *degree* is the measure of one angle if it takes 360 of them to make a full circle.

A degree is a standard unit for measuring angles. We define *one degree* proportionally. A full circle angle measures 360 degrees and zero angle measures 0 degrees. If both rays of an angle coincide, the angle is either 0 or 360˚

The Babylonians first decided that a full circle should be 360˚ and that convention has stuck and become standard throughout the world despite attempts to try for 400 degrees which would have made a right angle 100 degrees and hence more agreeable to the decimal number system. Another way to measure angles is based on the length of the circumference of a unit circle that is swept out by the angle. In this system 2π measures the full circle. These units are called radians, but we will not use them in this course.

Why did the Babylonians choose 360˚? This is what Cecil Adams, author of *The Straight Dope* newspaper column (http://www.straightdope.com/) has to say about it.

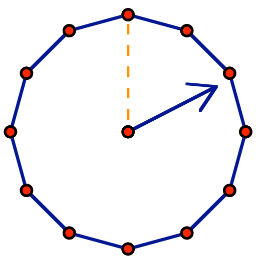
> Dear Cecil: Why are there 360 degrees in a circle? --Listener, NPR
>
> Cecil replies: My assistant, little Ed, got this question the other day on a radio talk show and predictably had no clue. However, from long experience we have learned that when in doubt, blame it on the Babylonians. Sure enough, when we looked up "degree" in our *Oxford English Dictionary*, we read, "this division of the circle is very ancient, and appears to have been originally applied to the circle of the Zodiac, a *degree* being the space or distance travelled by the sun each day according to ancient Babylonian and Egyptian computation, just as a *sign* represented the space passed through in a month."
>
> But wait, you say. The year has 365 days, not 360. We seem to be five degrees short.
>
> Well, yeah. Standards of scientific measurement in those days were a little more relaxed. Three hundred sixty was also readily divisible into thirds, fourths, fifths, sixths, etc.--no small advantage. You think you would have enjoyed trigonometry more if the number of degrees in a right angle had been 91 1/4?
>
> Cecil Adams, The Straight Dope, http://www.straightdope.com/

6. Assignment: Explain how Cecil got $91\frac{1}{4}$.

Clocks are an excellent way to estimate angles: As the minute hand sweeps out five minutes it travels one-twelfth of the way around the clock, hence the angle between twelve o'clock and one o'clock is one-twelfth of 360˚ or 30˚.

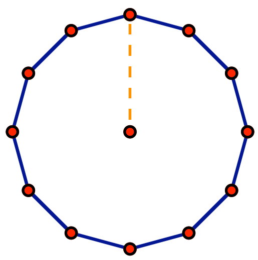
Example: The first clock shows only the hour hand of the clock. What is the angle between straight up twelve o'clock and the hour hand? The answer is shown below the picture.

7. Assignment: On the other clocks show where the hour hand would be at the given time and find the angle from 12:00 to the hour hand.

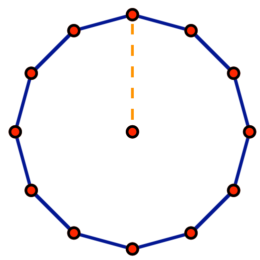


2: 00 60°

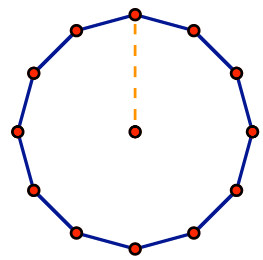
because the hand has traveled 2/12 or 1/6 of the way around the clock and 1/6 of 360° is 60°



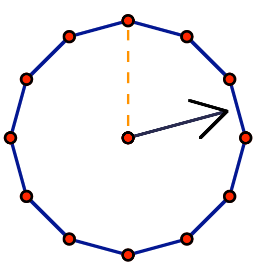
4:00 ______˚



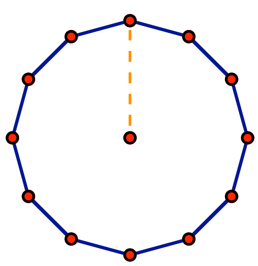
8:00 ______˚



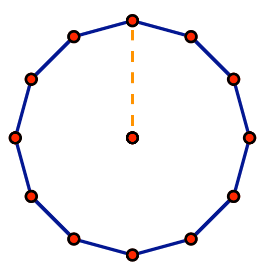
6:00 ______˚



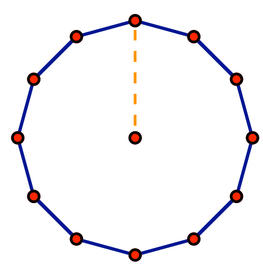
2: 30 ______˚



7:30 ______˚



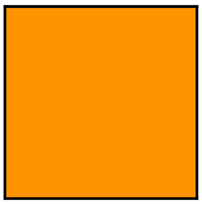
1:45 ______˚



6:54 ______˚

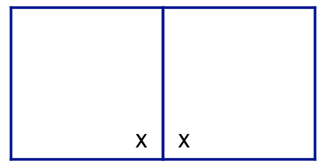
Class Activity: Pattern Block Angles

Trace each pattern block onto this page. Make arrangements of pattern blocks that allow you to compute each angle. Label the measures of each angle in each of the pattern blocks. The square is done for you.



Put the square pattern block on top of this copy of one and rotate it to see that each angle has the same measurement. Call that measurement x.

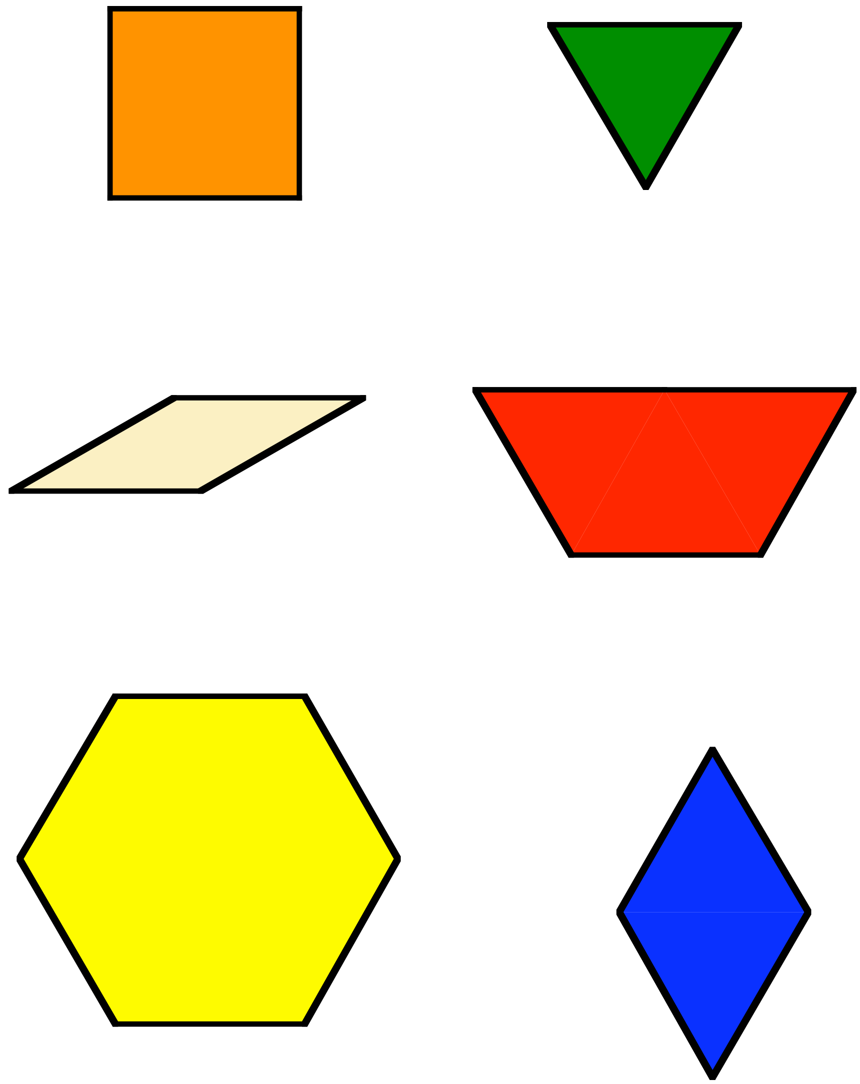
Put two orange pattern blocks together to see that two of the angles make a straight line so that so that twice the angle, x, is 180°. Which means that x=90°



$2x=180$
$x=90$

8. When you have found all the angles – there are nine different ones – label your work with the numbers 1 through 9 indicating the order in which you did your work. The square will be number 1. Draw arrows from one number to any other number that used that angle in the calculation. The arrows should always go from smaller numbers to larger numbers, indicating that you did not use an angle measurement before you "proved" what it is.

Make copies of pattern blocks to help complete the previous activity and the following two activities at home if the real pattern blocks are not available. If you used pattern blocks in class, you will need to do something to record your work in this book.



Group Project: Tessellating with one pattern block

Pick one of the pattern blocks and cover this page with copies of it in such a way that

1 – There are no gaps – the blocks cover the entire space.
2 – There are no overlaps.
3 – Every side of a one block coincides exactly with a side of the adjoining block.

You may not have enough blocks to cover this entire page. However, the goal is to make a pattern that you could continue in all directions forever as long as you were supplied with more and more blocks.

Each person in your group should use a different pattern block.

NOTE: If you chose the triangle, the square or the hexagon, you made a regular tessellation of the plane.

9. Look at the tessellations made by each person in your group. For each different shape discuss whether or not a different (not congruent) pattern could be constructed using that shape.

Square

Triangle

Rhombus

Hexagon

Trapezoid

Group Project: Tessellating with two pattern blocks

Using just the square and the triangle, cover this page with copies of them in such a way that

1 – There are no gaps – the blocks cover the entire space.
2 – There are no overlaps
3 – Every side of a one block coincides exactly with a side of the adjoining pattern block

You may notice that there are two ways you can join squares and triangles at one vertex to make a full circle. Show each way here:

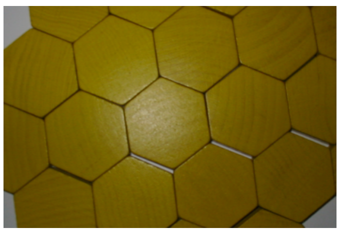
If only one of these ways appears in your tessellation, you have made a *semi-regular tessellation*.

Definition: *A tessellation of the plane* is a collection of shapes that cover an entire plane in such a way that none of the shapes overlap and there are no gaps.

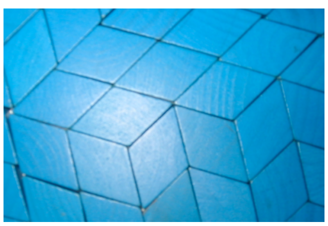
Definition: *A regular tessellation of the plane* is a tessellation of the plane is which the shapes are all the same regular polygon. The regular polygons must be arranged so that each edge coincides exactly with an edge of the adjoining polygon.

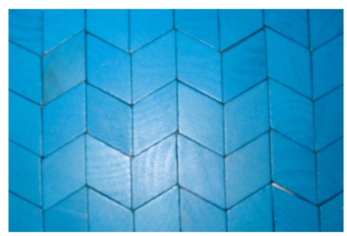
Definition: *A semi-regular tessellation of the plane* is a tessellation of the plane is which the shapes are two or more regular polygons. The regular polygons must be arranged so that each edge coincides exactly with an edge of the adjoining polygon. Each vertex in the tessellation must be congruent to every other vertex.

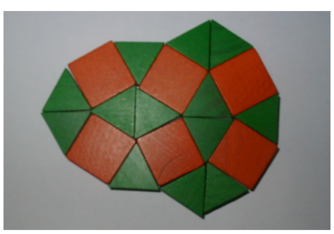
Examples and counterexamples: Here are some examples of configurations of pattern blocks:

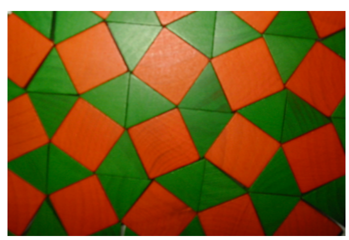

A regular tessellation

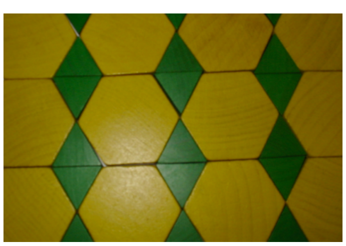

Not a tessellation of pattern blocks

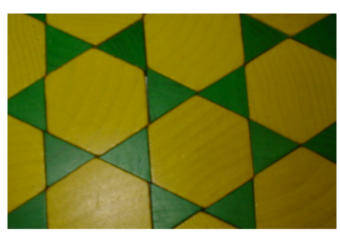

A tessellation of plane with rhombuses, but no pattern is apparent


A tessellation of the plane with rhombuses, and there is a pattern


A nice shape, but not a tessellation of the plane


A semi-regular tessellation of the plane. It contains the shape at the left.


Not a semi-regular tessellation, because not all vertices are congruent


A semi-regular tessellation with regular hexagons and equilateral triangles

Group Project: Tessellating with a scalene triangle.

Cut out a scalene triangle that fits on a 3X5 card. Label the angles a, b, c and color each angle a different color. Tessellate this entire page with your triangle in such a way that every side of a copy of the triangle coincides exactly with a side of the adjoining copy of the triangle. As you copy each triangle label the angles with the correct letter and color.

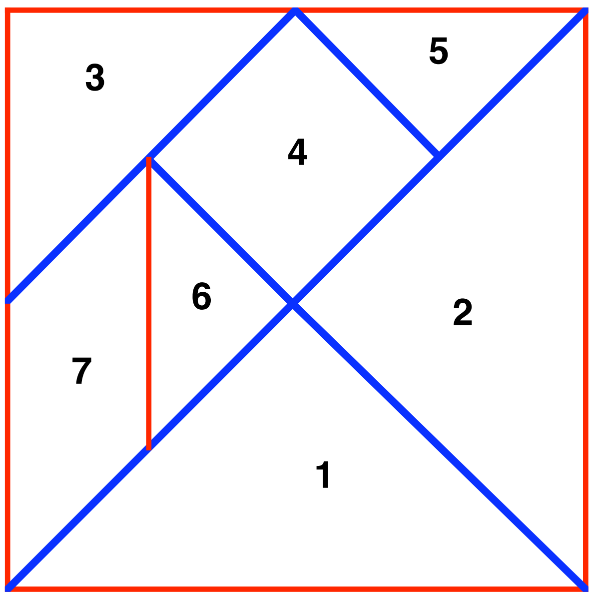
One can learn many things from this tessellation. For these two facts, trace the parts of your tessellation that demonstrate it:

The sum of the measures of the interior angles of a triangle is equal to 180˚.

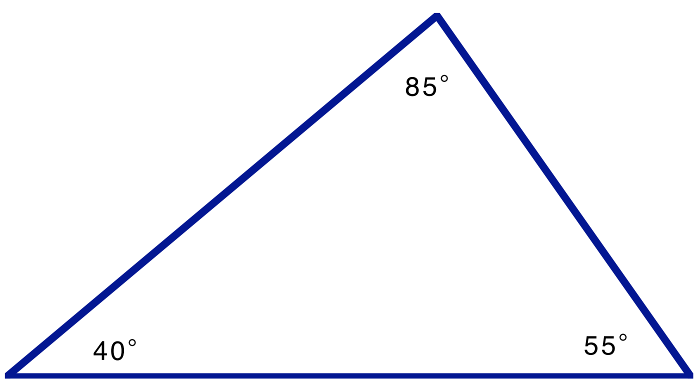
Given two lines cut by a transversal, corresponding angles are congruent if, and only if, the lines are parallel

Problems:

1. Determine the angle measurements of each of the tangram pieces without using a protractor? Describe your methods.

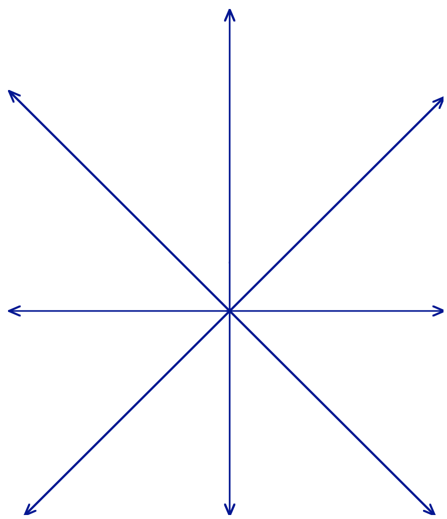


2. Use your protractor to check that the angles, 40° 55° and 85°, are correct. Cut out a copy of this triangle and, by adding or subtracting angles, construct other angles that measure 15° 30° 70° 95° and 100°. Can you construct an angle that measure 75°? Explain how or say why not.



3. The measure of angle X is 9° more than twice the measure of angle Y. If angle X and angle Y are supplementary angles, find the measure of angle X. Include a sketch with your solution.

4. Make your own protractor by folding a piece of paper in half many times. What is the smallest angle shown on this protractor? Mark 90˚ 180˚ 270˚ on your protractor. Also mark 45˚ 135˚ 225˚ and 315˚. Notice that 30˚ is not on your protractor but draw it in the approximate correct location. Which two rays does it fall between?

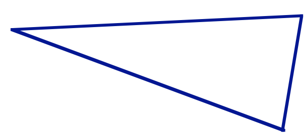


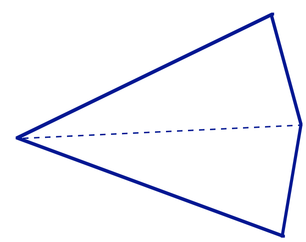
Your foldlines will look something like this after you have folded the paper three times.

Class Activity: Puzzles

Puzzles are among the best ways to help children acquire geometric intuition. Simple puzzles can be made using just triangles. For example, make two copies of the triangle shown, then see how many different quadrilaterals you can make by fitting the two triangles together.

How many different quadrilaterals can you make by joining two sides of two copies of this isosceles triangle? One, a kite, is done for you.

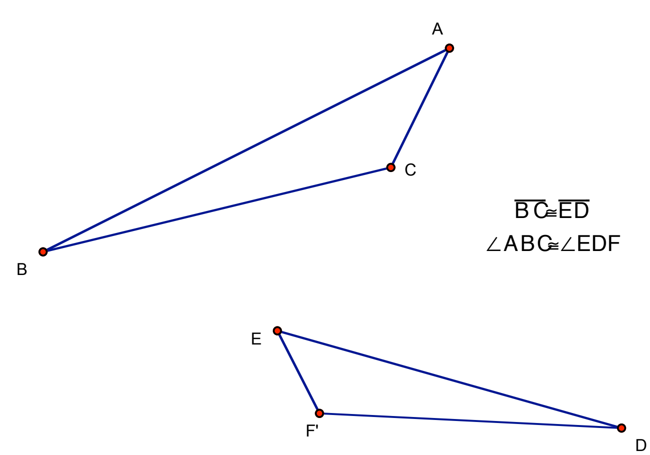




Make a puzzle of two triangles that fit together to make a square. What other shapes can you make with your puzzle?

Make a puzzle of two triangles that fit together to make a rectangle. What other shapes can you make with this puzzle?

Here are two triangles that have an angle and a side in common. Show how to join them to make a trapezoid.



Lesson 3 – Isometries, Transformations that Preserve Lengths and Angles

We have said that two shapes are *congruent* if one can be moved onto the other. To understand congruency between geometric object we want to carefully analyze what is meant by "move."

Definition: An *isometry* is a change that preserves lengths and angles.

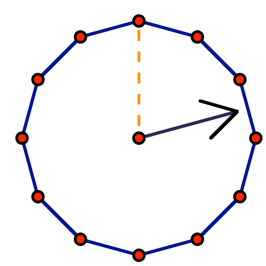
The word "isometry" is from the Greek roots: "iso-" meaning "the same" and "metri-" meaning measurement. It is used here because we are interested in only those changes or transformations that preserve length and angle measures. There can be no stretching, bending or breaking.

Every isometry can be realized as a series of *simple isometries*: a rotation, a reflection, and a translation. For example, to move the left smiley face unto the right smiley face, first translate the face to the right until it coincides with the other circle, then rotate the face ninety degrees so that the smile is to the right. Finally, reflect the face so that the eyes coincide.

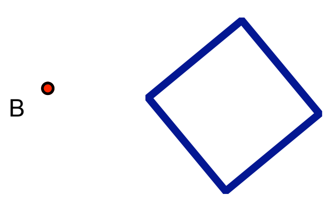
It is possible to do this in other ways. For example, first rotate, then reflect and finally translate.

Rotation
of an object through an <u>angle</u> about a <u>center</u> point

Example: As we considered in Lesson 2, every hour, the hour hand rotates, clockwise, through an angle of 30° about the point that is the center of the clock. This clock shows an hour hand that has rotated through an angle of 75° since 12:00.



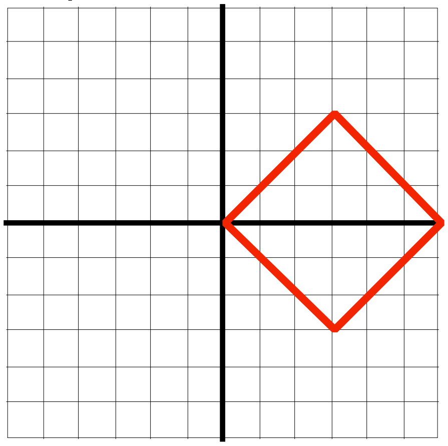
One way to construct a rotation is using the clock and tracing paper.
Example: Rotate the square about the point **B** through the angle **60**$^\circ$, counterclockwise**.**



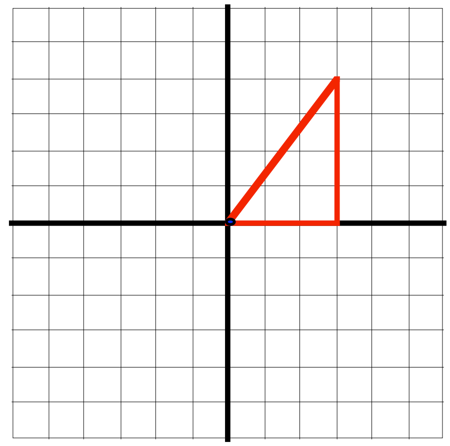
1 - Draw a 12 o'clock hand from the center of rotation.
2 - Copy the clock to a piece of tracing paper
3 - Put the tracing paper on the picture with the center of the clock at the center of rotation.
4 - Trace the square.
5- Rotate the tracing paper until 2 o'clock gets to 12 o'clock.
6- Copy the square to the paper.

Another way to construct a rotation is using grid paper:

Examples:

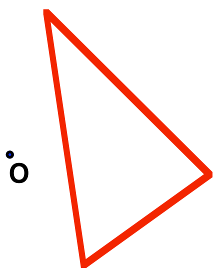


Rotate the parallelogram 90˚ counterclockwise about the point **(0,0)**.



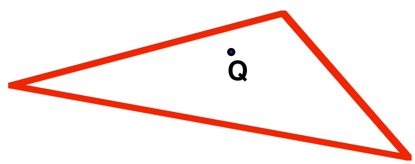
Rotate the triangle 135˚ counterclockwise about the point **(3,0)**.

1. **Assignment:** Draw EACH image under the prescribed rotation:

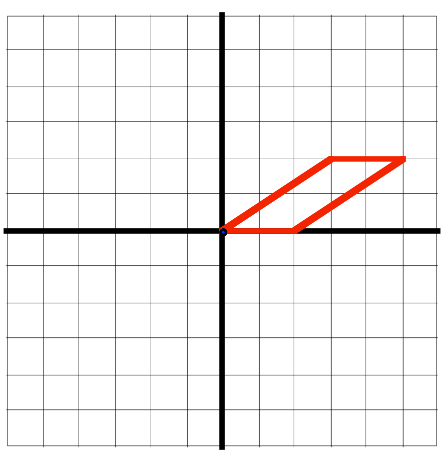


1) Rotate the triangle 45˚ clockwise about the point **O**.

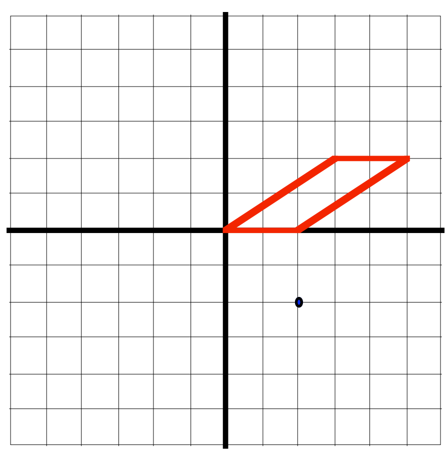
2) Rotate the triangle 75˚ counterclockwise about the point **Q**



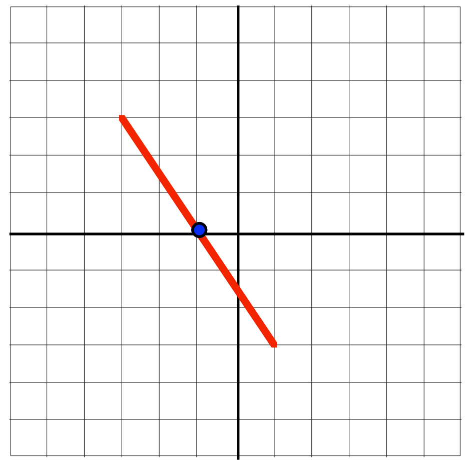
3) Rotate the parallelogram 90˚ clockwise about the point **(0,0)**.



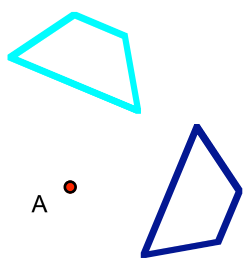
4) Rotate the parallelogram 90˚ counterclockwise about the point **(2,-2)**



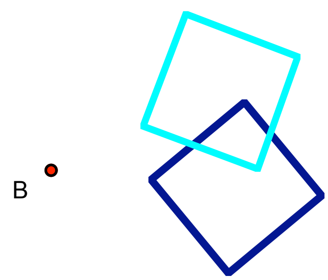
2. **Assignment:** Find the line through the point that is perpendicular to the line shown. (Hint: rotate the line 90 degrees about the point) What is the slope of each line?



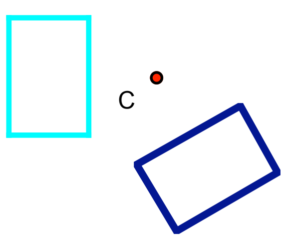
3. **Assignment:** Fill in the blanks and be prepared to explain your reasoning.



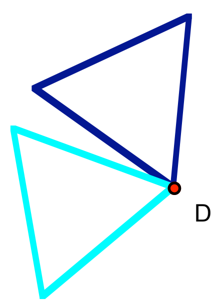
The trapezoid has been rotated about the point A through an angle of _______ °



Thesquare has been rotated about the point B through an angle of _______ °



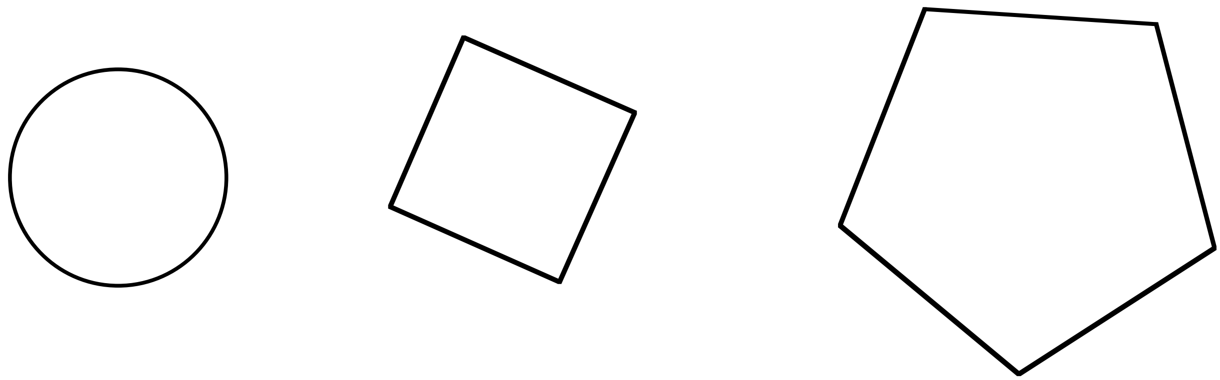
The rectangle has been rotated about the point C through an angle of _______ °



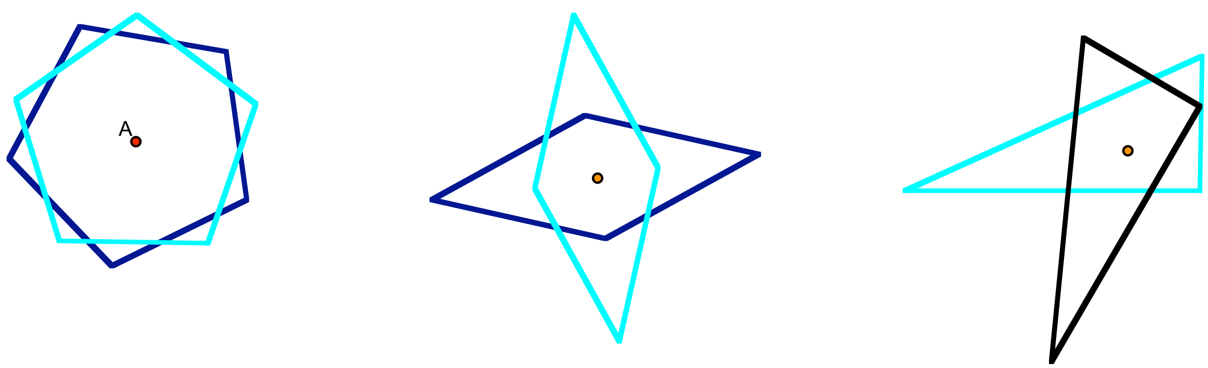
The equilateral triangle has been rotated about the point D through an angle of _______ °

ASIDE: Often in geometry, we want to rotate an object about a point at the ***center*** of the figure, like a square or a triangle.

4. **Assignment:** Find the center of each figure. Explain your method.

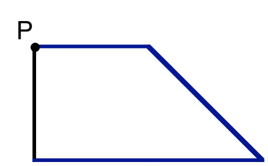


Example: Find the angle of rotation in these three examples.



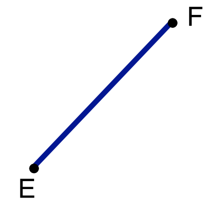
Definition: A rotation through 180˚ has is called a *half-turn*.

Example: Rotate this figure 180˚ about the point P.



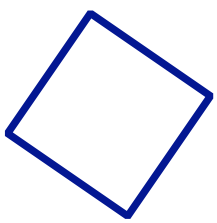
Notice how the image of each line segment that contains the point of rotation moves into a straight line extension of itself, under a half-turn,.

5. **Assignment:** Rotate this line segment 180˚ about the point E.

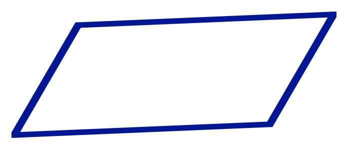


Rotational symmetry: If an object is rotated about a point through some angle and if the rotated figure coincides with its preimage, we say the object has rotation symmetry.

Examples:



A square has rotational symmetry through angles 90°, 180° and 270° about the center point.

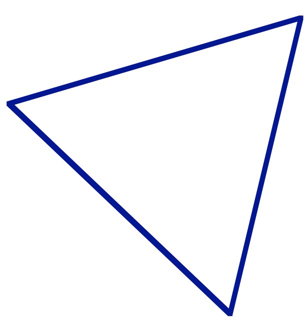


A parallelogram has rotational symmetry through angle of 180°. The center of the rotation is the intersection of the diagonals.

6. **Assignment:** Find the all angles, less than 360˚, of rotational symmetry for each of these figures. Clearly indicate the center and the angles of rotation on the figure.

NOTE: Because every figure returns to itself when rotated (about any point!) through an angle of 360˚, we do not include 360˚ as an angle of rotational symmetry.

#1



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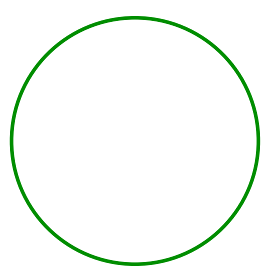

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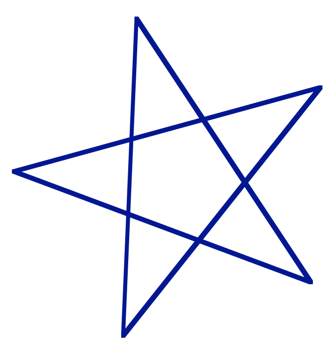

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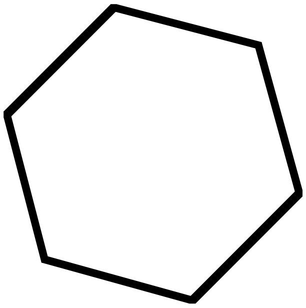

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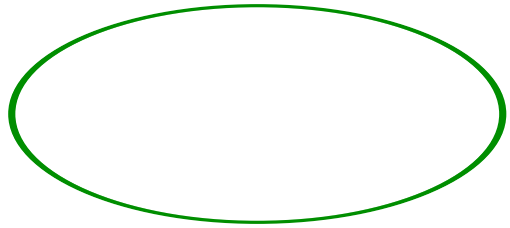

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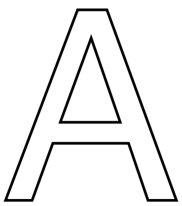

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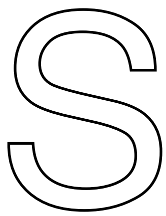

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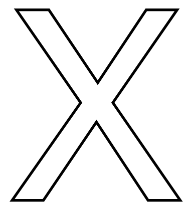

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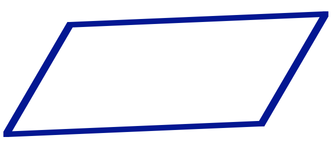

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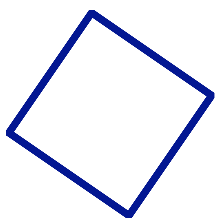


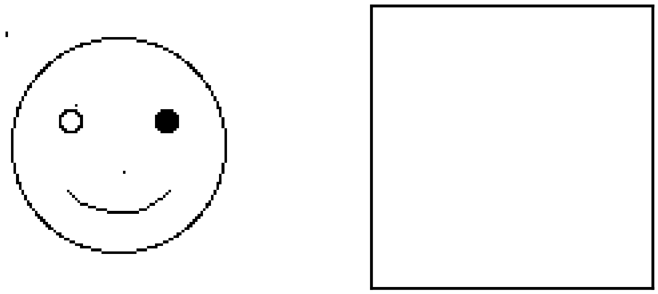
#11

Definition: If a figure has symmetry through an angle of 180˚ we say it has *point* symmetry.

7. **Assignment:** Mark all the figures in the previous problem that have point symmetry.

Reflection
through a line.

The image of an object in a mirror is a reflection of that object through a vertical line. Draw what the smiley face sees in the mirror.

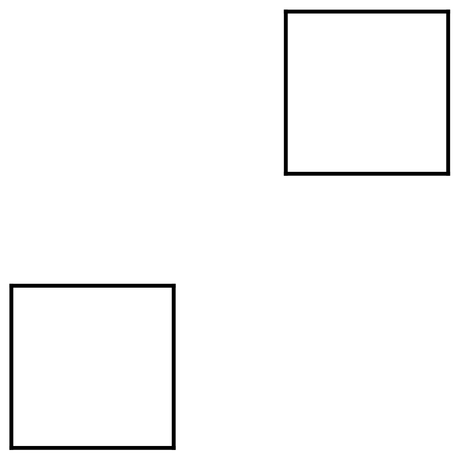


Mirror

One way to find the image would be to fold this sheet so that the face moves inside the mirror.

Definition: The fold line is called the *line of reflection.*

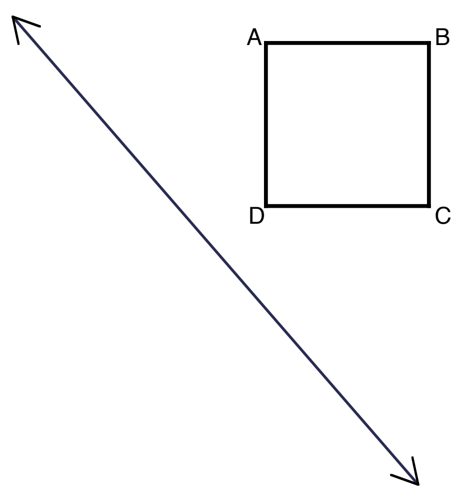
Example: Find the line of reflection through which one square is the image of the other square.



1 - folding: Fold the paper so that the two squares coincide. The folded line is the line of reflection.

2 - using a Mira: Put the Mira on the paper with the beveled side down. Move it until the image through the Mira coincides with the reflected image. Draw the line on the beveled side of the Mira.

Example: Find the image of square ABCD under a reflection through the line by



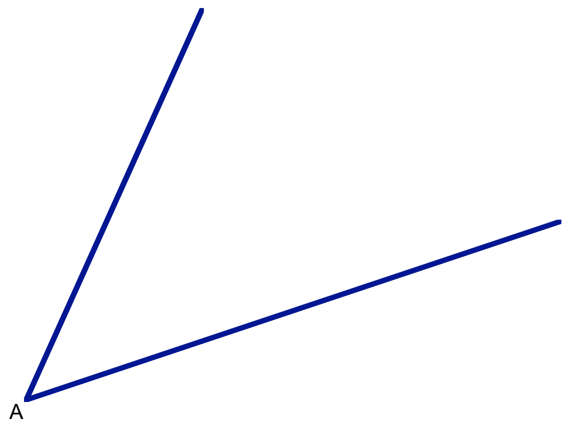
1- Using a Mira

2- Folding

Label the image A’B’C’D’.

8. **Assignment:** Use reflections to solve these problems.

#1 Bisect $\angle A$.

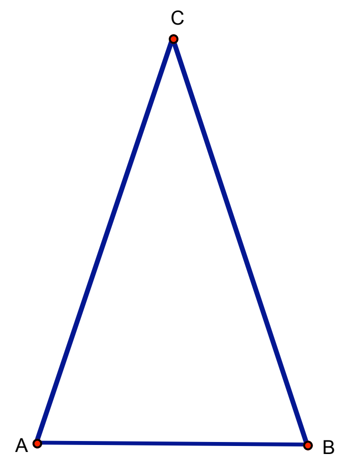


#2 Bisect $\overline{CD}$.

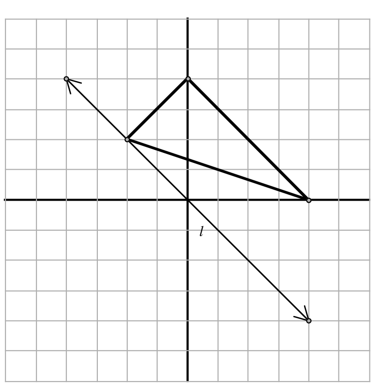
#3

If $\overline{AC} \cong \overline{BC}$ show

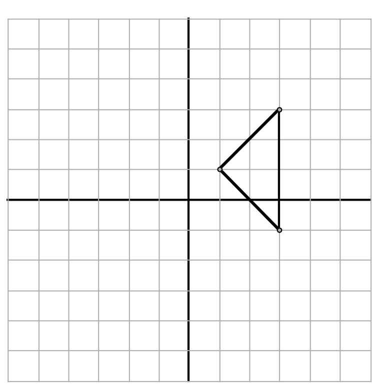
that $\angle A \cong \angle B$ by looking at a reflection.



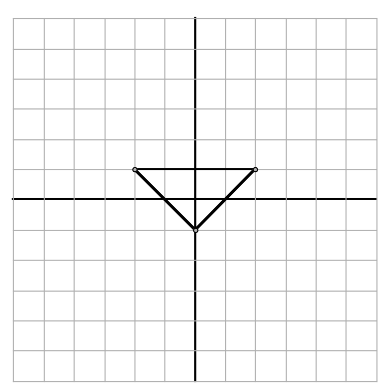
9. Assignment: Find the reflection of each triangle about the given line.



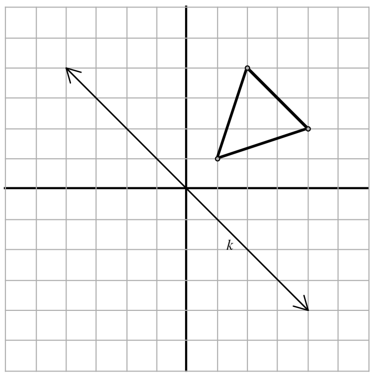
#1 reflect about line l



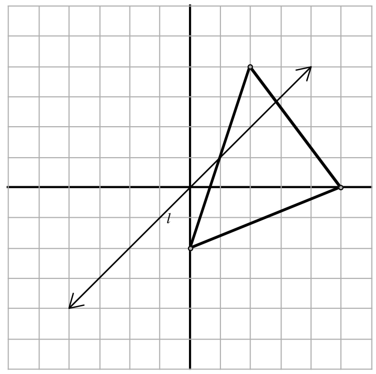
#2 reflect about the y-axis



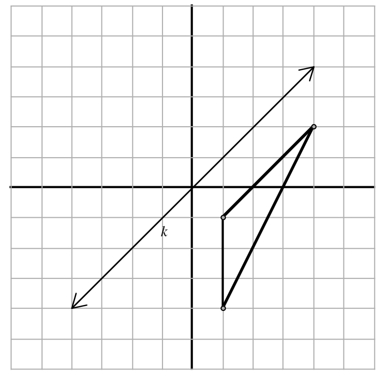
#3 reflect about the x-axis



#4 Reflect about the line k



#5 Reflect about the line l

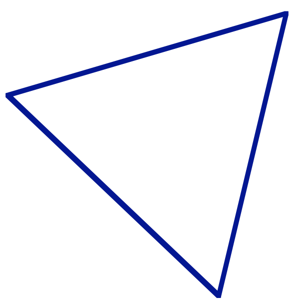


#6 Reflect about the line k

Line Symmetry A figure has *line symmetry* if it reflects back onto itself. *The line of symmetry* is the line of reflection. A figure may have many lines of symmetry or none at all.

10. **Assignment:** Find all of the lines of symmetry.

#1



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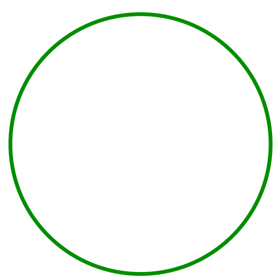

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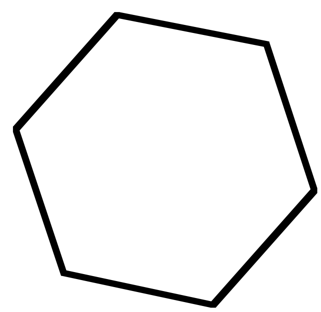

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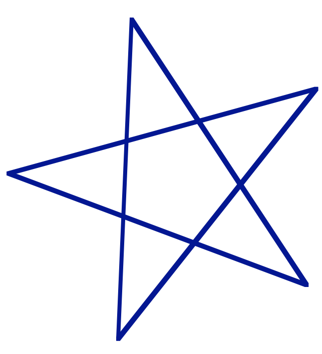

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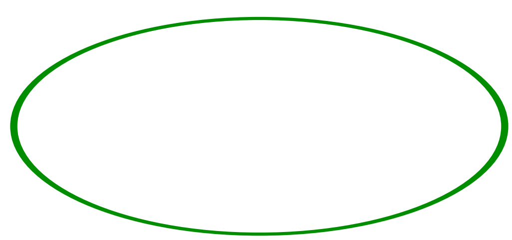


#6

A

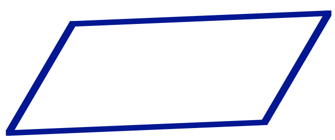
#7

S

#8

X

#9



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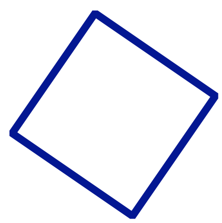
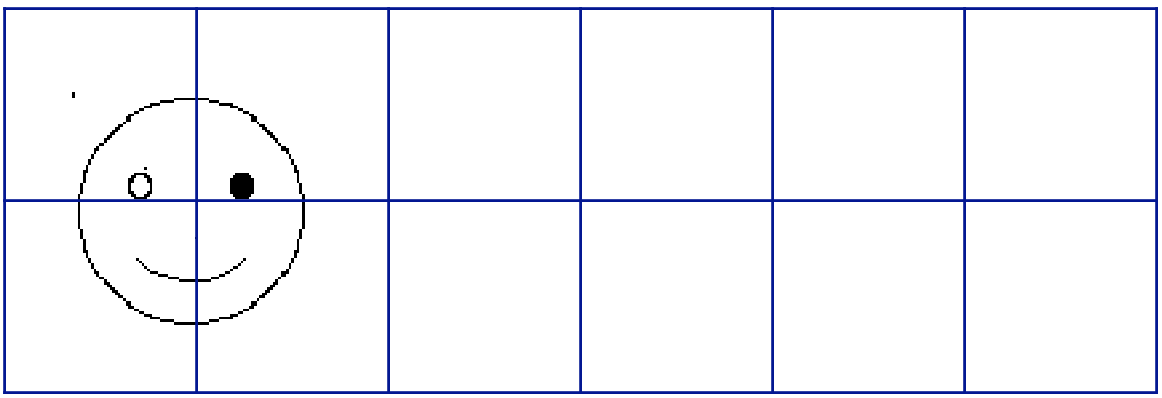


#11

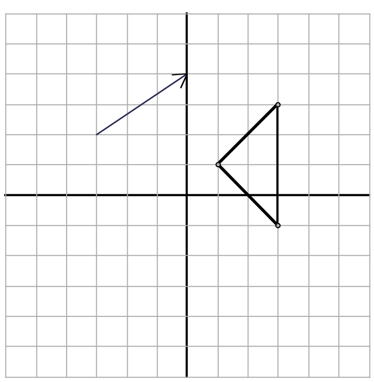
Translation
In a direction for a certain distance.

Translation is in some ways the simplest of the three simple isometries but in other ways, the hardest. To realize a translation we must move a figure without reflecting it (easy enough) and without rotating it (harder). One way to do this is to use a *guiding arrow* which tells what direction to move the object and how far to move it.

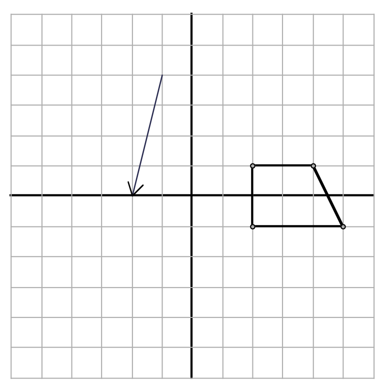
Example: Translate the smiley *three inches* to the *right*. Notice you are given a distance (three inches) and a direction (to the right). Draw a guiding arrow



Example: It's easy to construct a translation on grid paper.



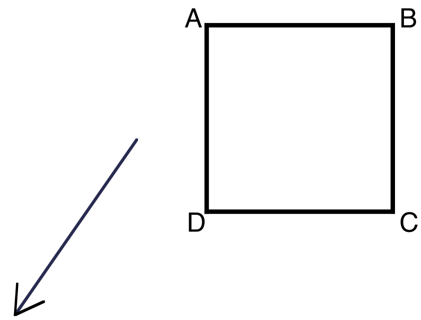
Translate the triangle to the right 3 units and up 2 units.



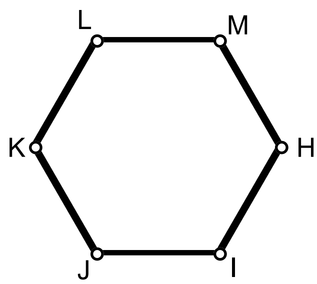
Translate the trapezoid to the left one unit and down four units.

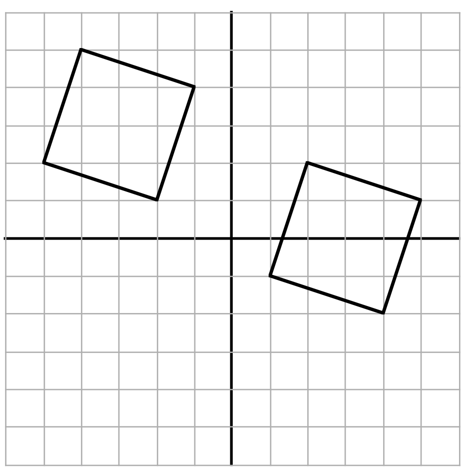
Another way to construct the image of a figure under translation is to use tracing paper.

Example: Trace the entire figure, including the arrow. Move the tracing paper so that the arrow always points in the same direction until the end of the arrow coincides with the tip of the arrow:

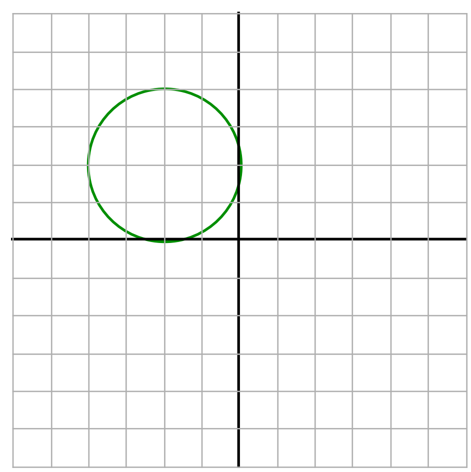


11. **Assignment:** Translate the hexagon so that the image of the point L coincides with the point H.





Draw and describe an arrow that Defines a translation that moves the left square onto the right square.

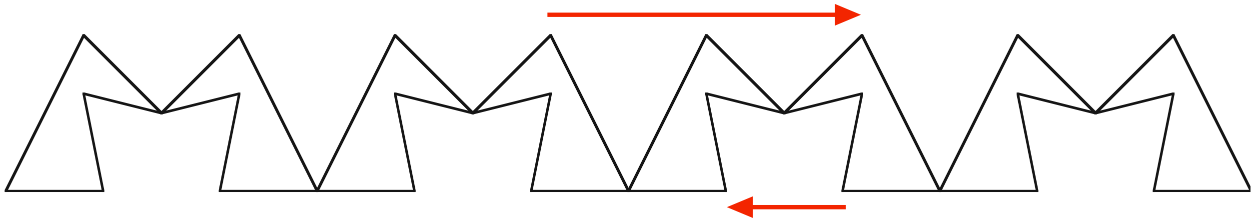


Translate the circle five units to the right and three units down. Draw and and describe the arrow that gives the same results.

For an object to have translation symmetry that object would have to be infinite in extent. Instead of saying that the object has translation symmetry, we say it *is invariant under a translation*.

Examples: This pattern is an example of a *frieze pattern.* You must imagine that the design continues to the left and to the right indefinitely. It is translation invariant in only two direction – but for many distances in each direction.

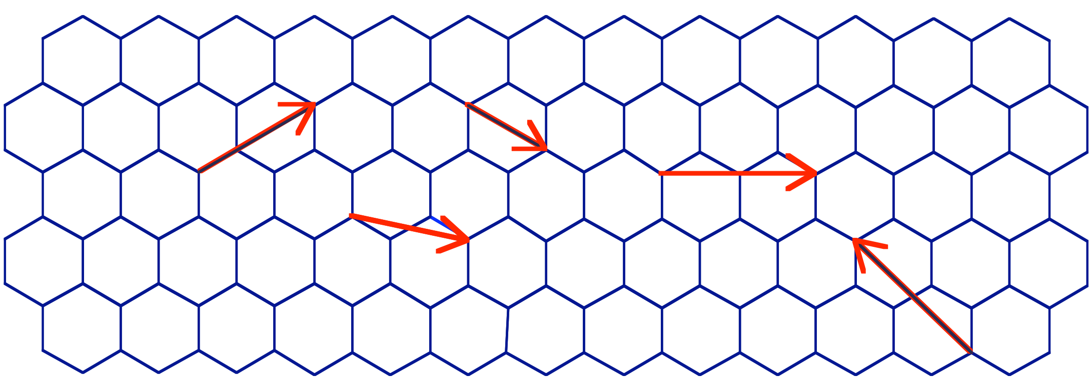
12. **Assignment:** Is this pattern translation invariant under each of these arrows? Draw in three arrows under which the pattern is translation invariant.



13. **Assignment:** Draw a frieze pattern of your own creation here. Make sure it is translation invariant.

The second example of a design that is translation invariant is a tessellation of the plane. This tessellation of hexagons is translation invariant in many directions. Imagine the pattern goes on and on in every direction and that we are only showing a small part of it.

14. **Assignment:** Circle each arrow that indicates a translation that transforms the pattern onto itself.



15. Go back to your tessellations from Lesson 2 and draw in arrows that indicate translations that preserves the tessellation.

Congruencies can be used to describe symmetries of polygons. For example, consider this rectangle. The rectangle has two lines of reflection. It also has rotational symmetry.

16. Draw the two lines of symmetry in the figure.

17. What is an accurate method to find the center of the rotational symmetry? What is the smallest angle of rotational symmetry?

The two reflective symmetries can be described by the congruencies

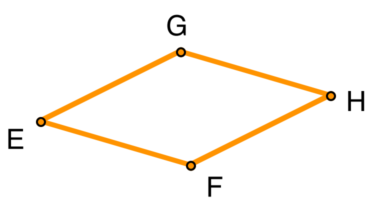
$ABCD \cong BADC$ -- a vertical line symmetry
$ABCD \cong DCBA$ -- a horizontal line symmetry

The rotational symmetry can be described by this congruency:

$ABCD \cong CDAB$ -- 180° rotational symmetry or point symmetry

Unless the rectangle is a square, the congruency $ABCD \cong CBAD$ (switching A and C) does NOT describe a symmetry because a rectangle does not have reflective symmetry through either diagonal.

18. Fold a piece of paper on one of the diagonals to show that the diagonal is NOT a line of symmetry. Sketch the result here:



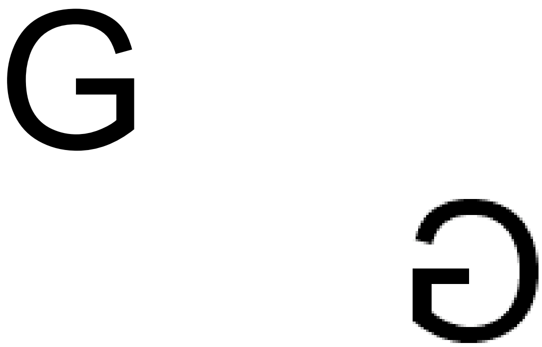
19. Describe a symmetry that shows the congruency $EGHF \cong HFEG$ for the rhombus, $EGHF$.

20. Complete the congruency that is realized by a reflective symmetry. $EGHF \cong$ _

Indicate the line of symmetry in the drawing.

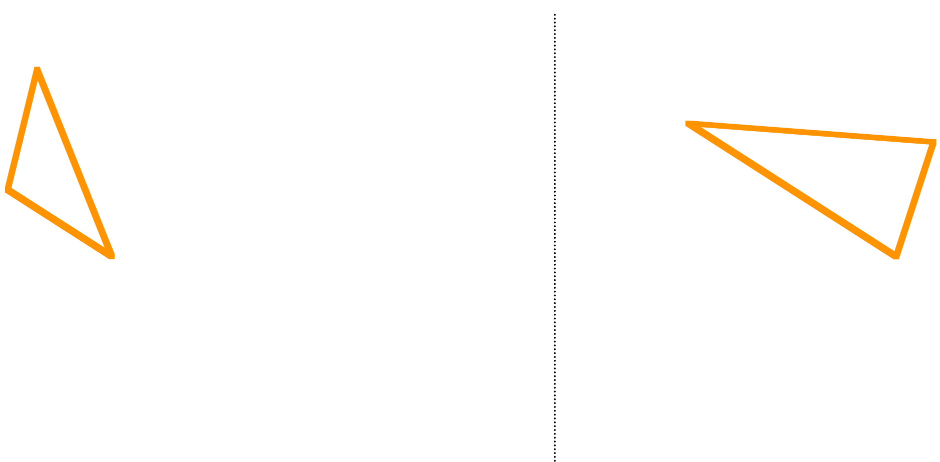
Lesson 3 Problems

21. Describe a translation followed by a reflection that takes one “G” to the other. **Show an arrow** to describe the translation and **draw a line** to describe the reflection.

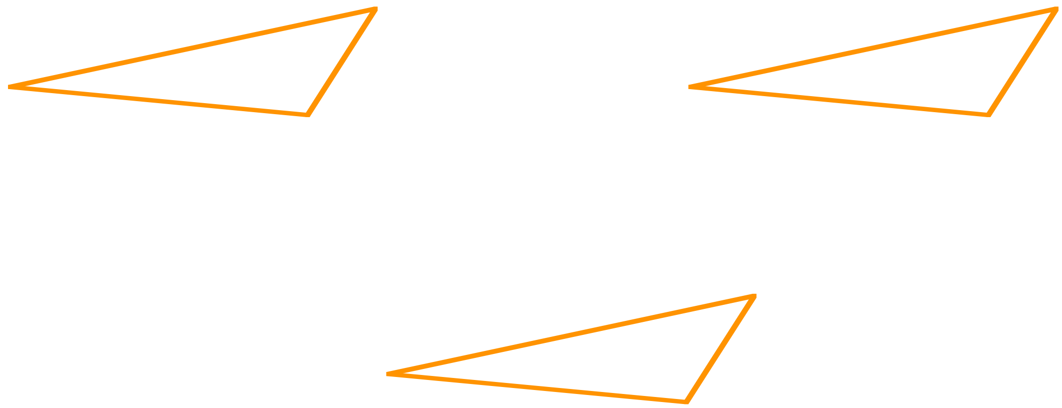


22. Describe a series of two isometries that take the one “S” to the other.

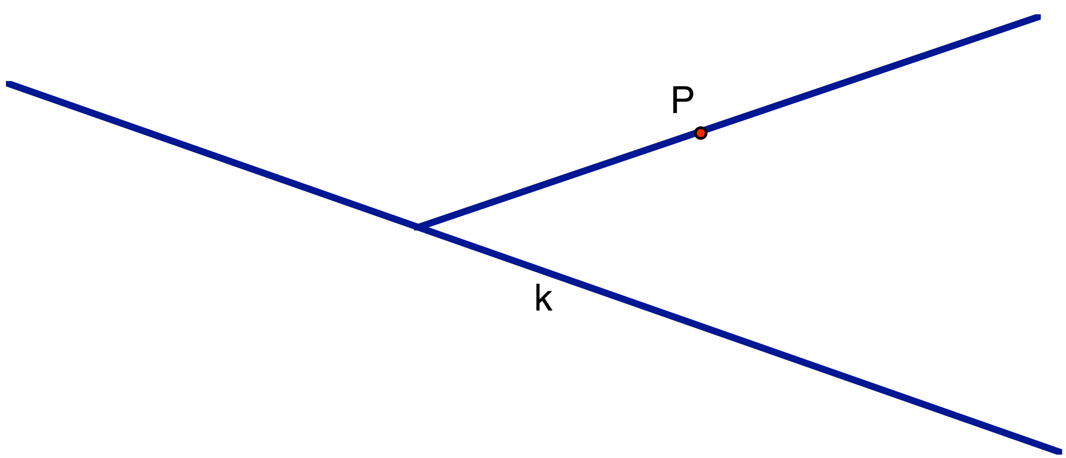
23. Reflect each isosceles triangle through one of its sides to make a rhombus.



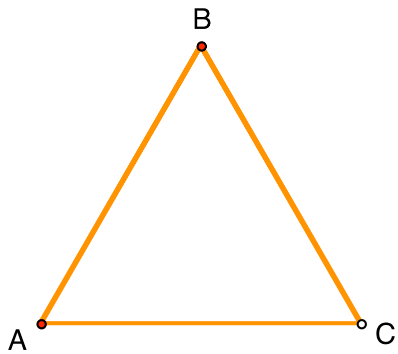
24. Construct three different kites by reflecting a scalene triangle through each of its sides.



25. Use a translation to construct a line that is parallel to k and goes through the point P.



26. There are 6 different symmetries of an equilateral triangle. Describe the transformation suggested by each congruency written below. Two are already given:



$ABC \cong ABC$, the "do nothing" isometry. Every polygon is always congruent to itself.

$ABC \cong BCA$, 120° rotational symmetry

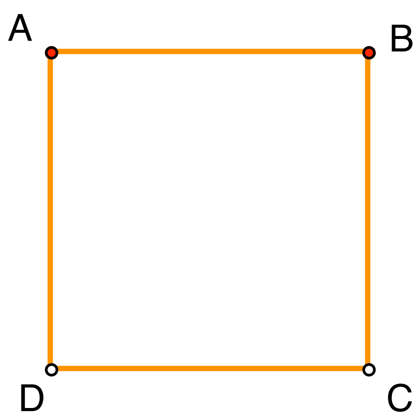
$ABC \cong CAB$

$ABC \cong ACB$

$ABC \cong CBA$

$ABC \cong BAC$

27. There are 8 different symmetries for a square. Write a congruency for each symmetry and describe the associated transformation. Two are given to get you started.



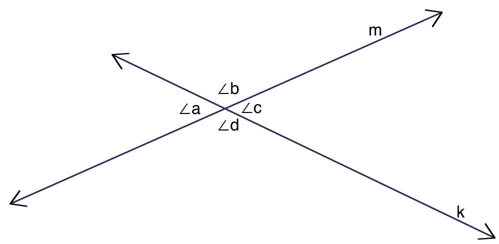
$ABCD \cong ABCD$

$ABCD \cong BCDA$

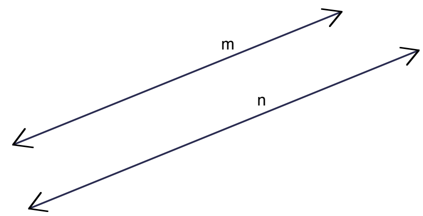
Lesson 4 – Postulates and Parallel lines and more about angles

There are two different ways to configure two lines in a plane:

1) The two lines intersect.



2) The two lines are parallel.
They do not intersect.



In the first case, four angles are formed. Two adjacent angles are *supplementary angles*, that is, they sum to 180˚. Angles opposite each other are called *vertical angles*. For example , $\angle a$ and $\angle c$ are vertical angles.

In the second case, there are no angles.

Fact: Vertical angles are congruent.

1. Explain why $\angle a \cong \angle c$.

A note on notation: $m(\angle 1) = m(\angle 2)$ and $\angle 1 \cong \angle 2$ imply the same thing. In the first case, we say "the measure of $\angle 1$ is equal to the measure of $\angle 2$". In the other case, we say "$\angle 1$ is congruent to $\angle 2$". Of course, congruent angles have the same measure and angles that have the same measure are congruent. Sometimes you will see something like $\angle 1 = \angle 2$ essentially equating an angle with the number that is it's measure. This is not correct use of language and may be misleading.

Group Project: Intersecting Lines

How many different ways can you configure three lines in a plane? Picture each way here. Use the words "intersect" and "parallel" to describe each configuration.

2. **Challenge:** Why are the words "In a plane" included? How many different ways can you configure two lines in space?

Euclid is probably the best-known mathematician of all time. He lived in the third century B.C.E. His treatise on geometry, known simple as *Euclid's Elements*, was the first comprehensive treatment of geometry. It is still the foremost used book on the fundamentals of geometry.

3. Research: Do some research on Euclid. Record here something you would tell a child about him. Write something else that you might tell one of your classmates.

Euclid's Elements begins with some definitions and some common notions and, most famous, five postulates. The postulates are the facts about geometry that Euclid assumes to be true before he begins to prove other theorems. The book proceeds with theorem after theorem, each one being proved using the five postulates and the previously proven theorems.

The five postulates were not given to Euclid. His first job in writing the treatise was to decide what – of all those things he knew to be true – would be the facts he would need to assume to prove the rest. While he decided on five postulates, mathematicians spent centuries debating his wisdom. Are all five necessary? What more is needed? Can any of them actually be proved using the other ones?

People were most concerned about the **fifth postulate**. Mathematicians worked for centuries to truly understand why Euclid included this particular fact in his list of postulates. Many of them worked desperately to find a proof of the fifth postulate using the other four postulates. But no one was successful for the very good reason that (as was finally shown in the 19th century) it cannot be proven from the first four postulates only. Eventually, Hilbert came up with a complete set of axioms and mathematicians were able to show that 1) All of the Hilbert axioms were needed. That is, not one of them could be proved using the other ones. And 2) no more were needed to completely cover Euclidean geometry.

EUCILD'S FIVE POSTULATES

I To draw a straight line from any point to any point.

II To produce a finite straight line continuously in a straight line.

III To describe a circle with any center and distance.

IV That all right angles are equal to one another.

V That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angle, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.

The first four postulates are straightforward. These postulates are just making clear what we already intuitively know about points and lines and angles. Each one can be easily rephrased:

POSTULATE I can be thought of with the familiar saying: "Two points determine a line."

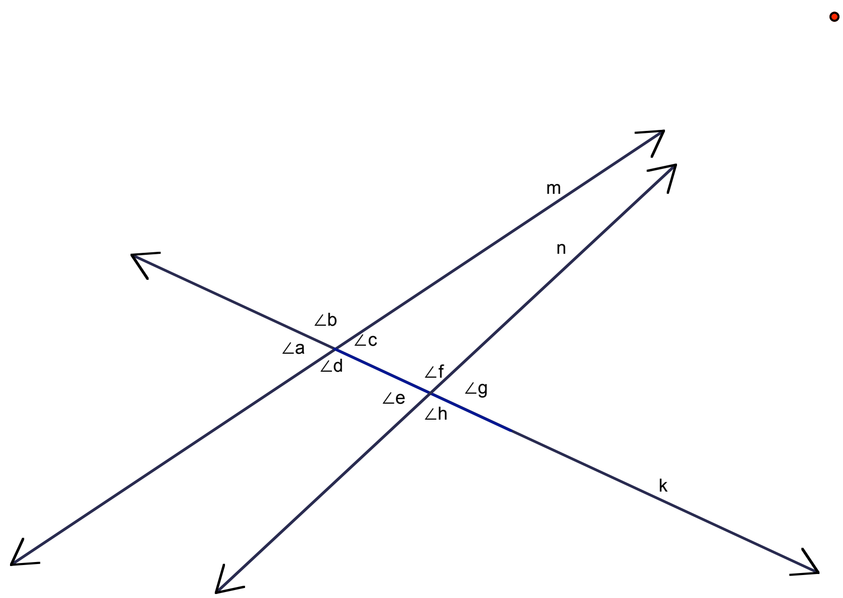
POSTULATE II means that any line segment can be extended indefinitely. We will see how useful it is to extend lines for doing some of the problems in later in this lesson.

POSTULATE III tells us that we can make circles of any size. We've already seen how circles can be useful in showing SSS.

POSTULATE IV guarantees that we can rotate, reflect and translate a right angle without altering the size of the angle.

POSTULATE V is more complicated than the other four. It's hard for us to understand what it says and it was hard for Euclid to know whether or not he should include it in his axioms.

But we can interpret the postulate with insight gained from this picture:



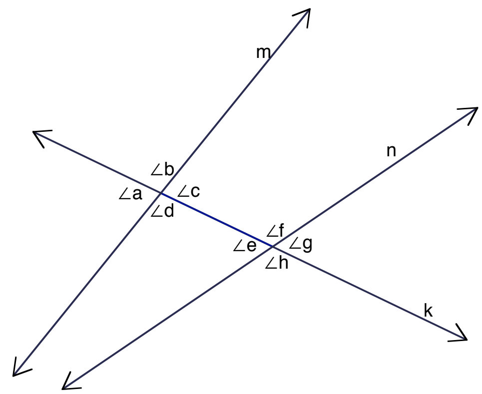
Fifth Postulate interpreted: If a straight line (line k in the picture) falling on two straight lines (lines m and n) make the interior angles on the same side($\angle c$ and $\angle f$) less than two right angle ($\angle c + \angle f < 90°$), the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles (lines m and n will intersect if we extend them to the dot).

The fifth postulate is describing the two cases that may arise when one line transverses two other lines:

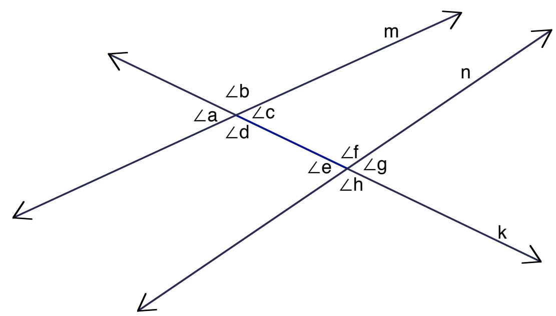
1) On one of the two sides of the transverse line, the two inside angles sum to less than 180˚. It is on this side that the two lines will eventually meet. The lines are NOT parallel.

2) The two inside angles sum to exactly 180˚ . The two lines are parallel.

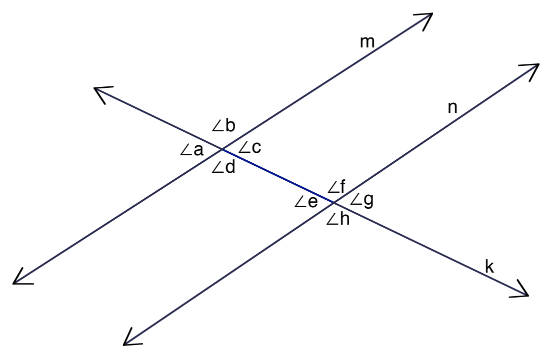
Case 1)
$\angle d + \angle e < 180°$
$\angle c + \angle f > 180°$



Case 1) alternative
$\angle d + \angle e > 180°$
$\angle c + \angle f < 180°$

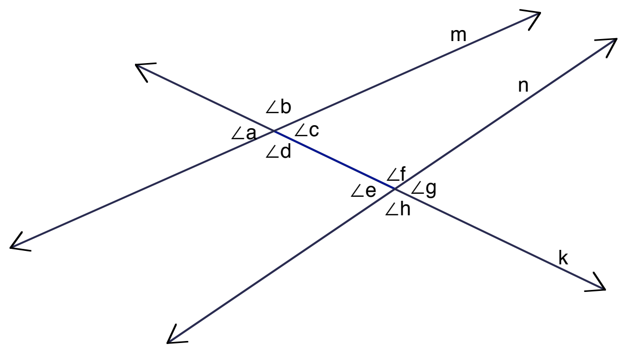


Case 2)
$\angle d + \angle e = 180°$ and
$\angle c + \angle f = 180°$



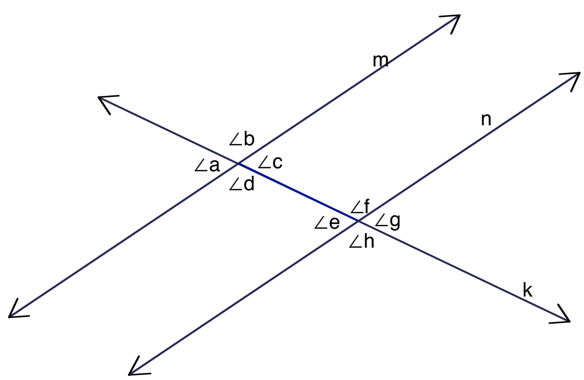
4. Assignment: In any case it is true that the four angles sum to 360˚, that is $\angle c + \angle f + \angle d + \angle e = 360°$. Explain why.

Today, Euclid's fifth postulate is usually replaced with a postulate about corresponding angles. In order to state this postulate, we need this definition:



Definition: In this configuration, there are four pairs of *corresponding angles*:

$\angle b$ and $\angle f$ $\quad \angle c$ and $\angle g$ $\quad \angle a$ and $\angle e$ $\quad \angle d$ and $\angle h$



Corresponding Angles are Congruent Postulate:
In the given configuration, if m and n are parallel lines, then corresponding angles are congruent. In particular, $\angle a \cong \angle e$, $\angle b \cong \angle f$, $\angle c \cong \angle g$, and $\angle d \cong \angle h$

5. Assignment: What other pairs of angles are congruent? Be sure to get all possibilities.

One way to think about this postulate is to think about translating the configuration of angles *a, b, c, d* along the line k. *If the lines* m *and* n *are parallel,* those angles will eventually coincide exactly with the configuration of angles *e, f, g, h.*

If the two lines m and n are NOT parallel, the configuration *a, b, c, d*, would never coincide with *e, f, g, h* and so corresponding angles would not be congruent. This is a different statement than the statement of the Corresponding Angle Postulate. It is the *converse* of the postulate and is not necessarily true just because the postulate is true. It is usually stated as another postulate:

> **Parallel postulate:**
> In the given configuration, if corresponding angles are congruent, then the lines m and n are parallel.

6. **Assignment:** In your own words, explain the difference between the Parallel Postulate and the Corresponding Angles Postulate.

Together, these two postulates can be stated as an "if and only if" statement:

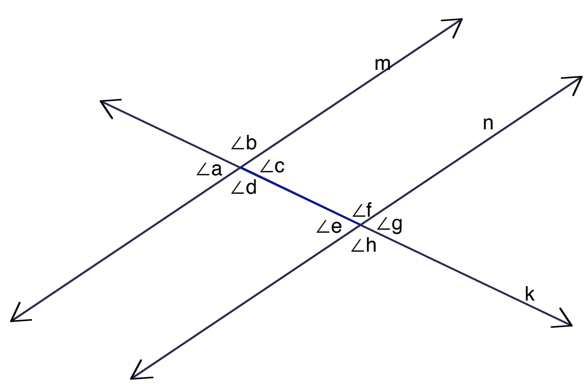
Corresponding angles are congruent	IF and ONLY IF	the lines m and n are parallel.

The "IF" part is the Corresponding Angles Postulate and the "ONLY IF" part is the Parallel postulate. These two facts are of fundamental importance in understanding geometry and are used as a starting place for understanding parallel lines and angles in polygons.

The first thing we will prove using these two postulates is two facts about alternate interior angles.

> **Definition:** In the configuration shown in the statement of the Z-Principle, there are two sets of *alternate interior angles,* $\angle c$ -- $\angle e$ and $\angle d$ -- $\angle f$.

The Z-Principle: In the given configuration, *alternate interior angles* are congruent if and only if the lines, m and n, are parallel.



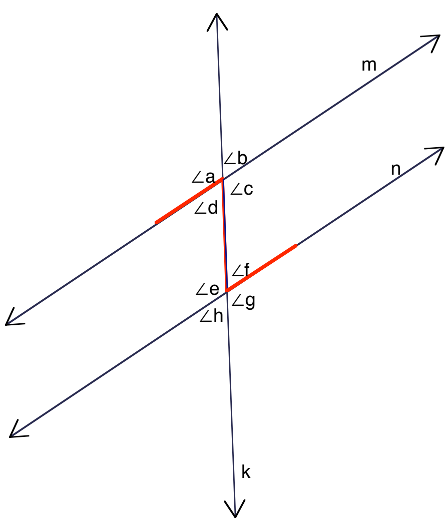
Proof of the Z-Principle: The "if and only if" make the Z-principle two facts, so we need a proof for each one. In the now familiar diagram where one line (k) transverses two lines (m and n) there are two sets of alternate interior angles: $\angle c$ *and* $\angle e$ or $\angle d$ and $\angle f$.

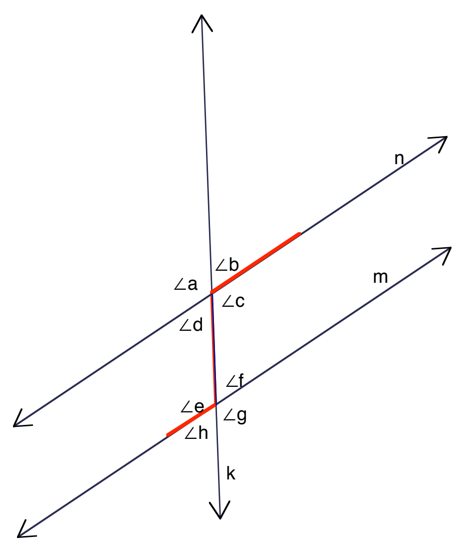
If either pair are congruent, say $\angle c \cong \angle e$, then it is also true that $\angle a \cong \angle e$ because $\angle c$ and $\angle a$ are vertical angles and hence congruent. Since $\angle a$ and $\angle e$ are corresponding angles, the two lines are parallel by the **Parallel Postulate**. This proves the "only if" fact.

Conversely, if the two lines are parallel, then $\angle a \cong \angle e$ by the **Congruent Angles Postulate**. Because $\angle c \cong \angle a$it follows that $\angle c \cong \angle e$ or "alternate interior angles are congruent. This proves the "if" fact.

7. Assignment: Write the Z-Principle as two different statements – one for the "if" and one for the "only if" part of the statement.

Together these facts are called the Z-principle because alternate interior angles sometimes appear to form the corners of a Z or something that resembles a Z.





The next thing we can prove is a theorem that is might be familiar to you.

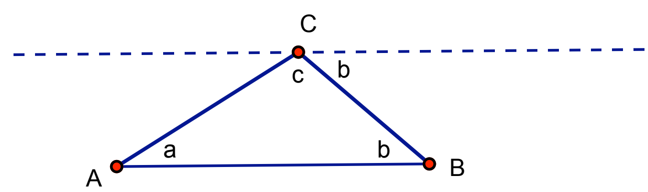
Theorem: The sum of the interior angles of a triangle is 180 degrees.

The proof is an easy application of both the "if" and the "only if" parts of the Z-principle. We proceed in two steps.

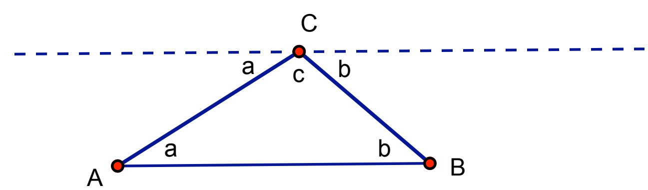
8. Assignment: Add lines and angles to the figures below that show the two lines and the transverse lines necessary to make the conclusions.

Proof: Start with any triangle ABC with angles that measure a, b and c.

Step 1: Construct an angle with measure b on the outer side of CB at point C, as shown. Extending the line constructs a line parallel to AB because alternate interior angles (the two angles that measure b) are congruent and the lines are parallel.
(Here's where we use the ONLY IF statement: lines are parallel only if alternate interior angles are congruent)



Step 2: The remaining angle at C must be equal to a because, together with the angle at A, these two angles are alternate interior angles and we know the two lines are parallel.
(This step uses the IF statement: If the lines are parallel then alternate interior angles are congruent.)



The 180˚ determined by the dotted line at the point C is the sums of the three angles a, b and c, as we wanted to prove.

This completes the proof.

A method often used in elementary school to see this fact is as follows: construct and cut out a triangle. Tear off two of the "corners" (say at A and B) and tape them back down at angle C. The resulting configuration looks just like the top part of the figure from our proof, with angles a, c, and b forming a straight line: 180˚.

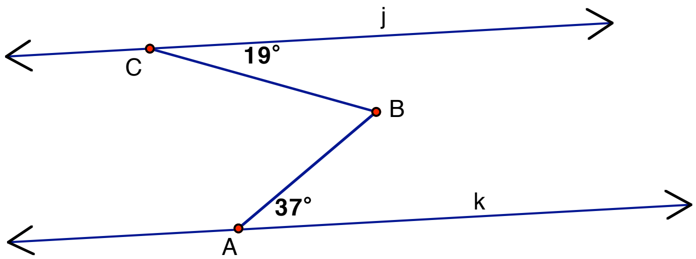
9. Assignment: Carry out these instructions and tape your results here. Which is more convincing for you: this demonstration or the proof given above?

Class Activity: Three problems

Sometimes it is useful to add auxiliary lines or extend line segments to help solve a problem. Add or extend lines to sole these three problems. Be prepared to justify your conclusions and to indicate whether you are using the "if" or the "only if" part of corresponding angles postulate or the alternate interior angles theorem.

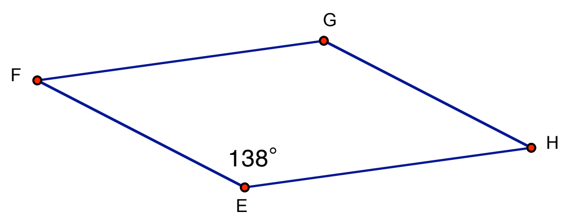
PROBLEM #1

Lines j and k are parallel. Find the measure of the angle at point B.



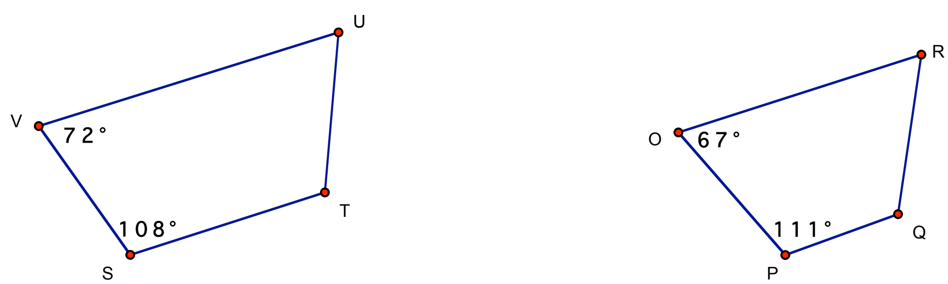
PROBLEM #2

In quadrilateral EFGH, the FG is parallel to EH and EF is parallel to HG. The interior angle at E measures 138˚. Find all the other interior angles.



PROBLEM #3

Are both of these quadrilaterals trapezoids? Explain.



10. Assignment: Which of the drawings on this page are accurate scale drawings and which are not? Make accurate scale drawings and measure to check your answers.

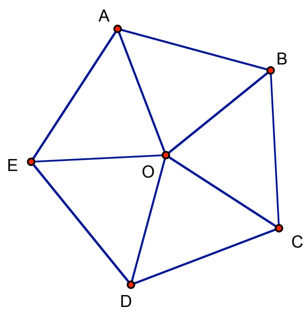
Once we know the theorem about the sum of the angles in a triangle we can ask about the sum of the angles in other polygons.

Class Activity: Sum of Angles in an n-gon

With your group make a drawing of a polygon with the number of sides indicated. In some way, figure out the sum of the angles in your polygon. Will your reasoning work for any polygon with that many sides? Be ready to justify your answers.

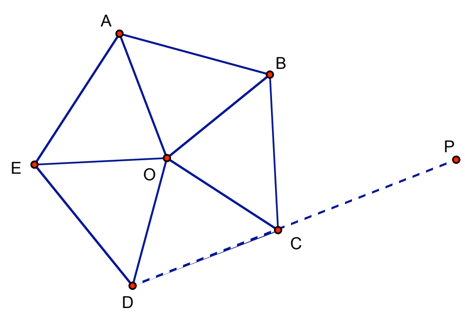
Number of sides of the polygon	Sum of interior angles	Sum of exterior angles
3	180˚	360˚
4		
5		
6		
n		

Class Activity: Formulas for Angles in Regular n-gons.



$\angle AOB$, $\angle BOC$, $\angle COD$, $\angle DOE$, and $\angle EOA$ are all ***center angles*** of the regular pentagon ABCDE.

$\angle ABC$, $\angle BCD$, $\angle CDE$, $\angle DEA$, and $\angle EAB$ are all ***interior angles*** of the regular pentagon ABCDE.



$\angle BCP$ is one of five ***exterior angles***.

Three formulas

Theorem: Any center angle in a regular n-sided polygon measures __________________.

Reasoning:

Theorem: Any exterior angle in a regular n-sided polygon measures _________________.

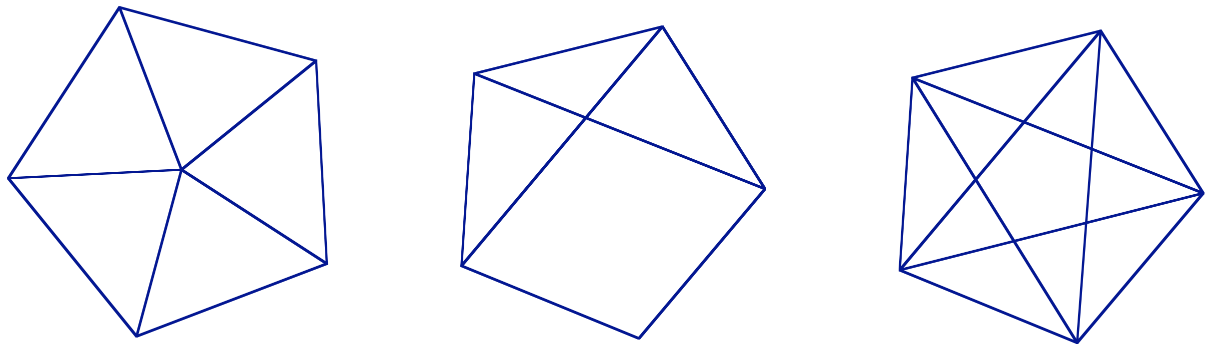
Reasoning:

Theorem: Any interior angle in a regular n-sided polygon measures __________________.

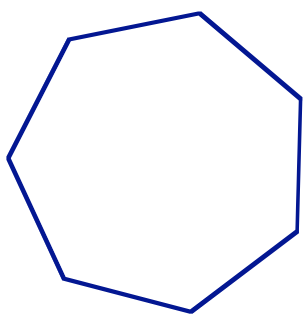
Reasoning:

Lesson 4 Problems:

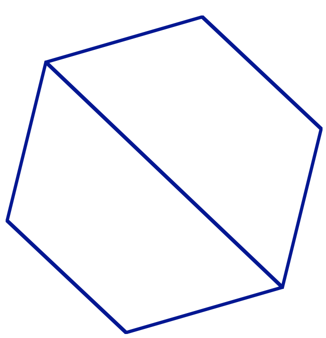
1. Here are three regular pentagons. Find the measure of all the angles.



2. Find an exact expression for the interior angle of a regular heptagon. What is the value accurate to 3 decimal places?



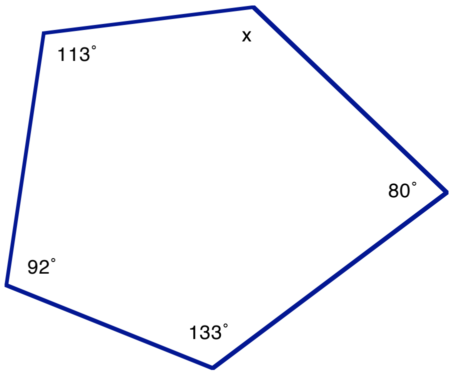
3. Use the Z-principle to show that the diagonal of a regular hexagon (as shown) is parallel to two sides.

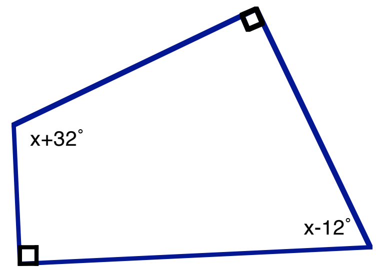


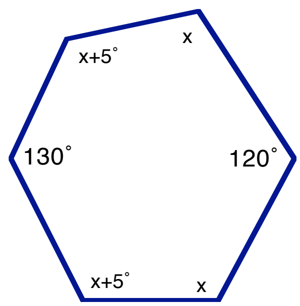
4. Use the Z-principle to show that opposite sides of a parallelogram are indeed parallel. Remember that our definition of parallelograms only says that opposite sides are congruent. [Hint: First, draw a diagonal of the parallelogram and show that the two triangles formed must be congruent]

5. Use the Z-principle to show that if the opposite sides of a quadrilateral are parallel then the quadrilateral is a parallelogram.

6. Find the missing angle measures in each of the following polygons.







Lesson 5 –Construction with Straightedge and Compass

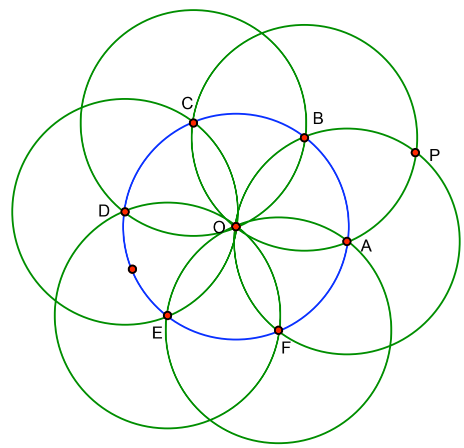
The compass is a tool for making circles; the straightedge, which is like a ruler but has no marks on it, is a tool for making straight lines. It is amazing what a wide array of geometric objects can be constructed with just these two simple tools. In fact, mathematicians have made a study of it: what geometric figures can be constructed and what cannot be constructed with straightedge and compass and no other tools.

In the following activity you will construct a regular dodecagon using only a compass and straightedge. Besides being quite a remarkable construction, there is a lot to learn by looking at the construction. If you have no experience with a compass you might want to start off by practicing making circles on another piece of paper.

Class Activity: Construct a Regular Dodecagon

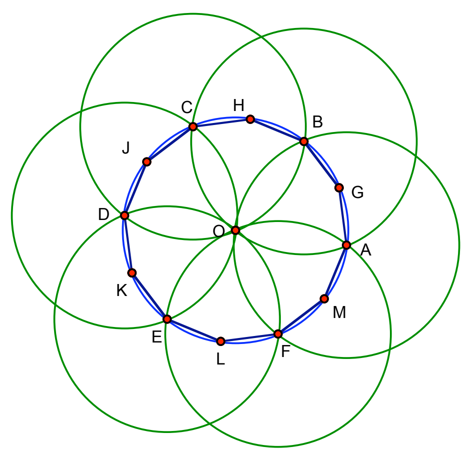
Make your construction on a separate piece of paper.

Step 1 Start out by making seven circles as shown in this picture: Each circle has the same radius and the centers of the outlying six circles are evenly spaced around the center circle. To construct it, first construct the circle with center, O. Then pick any point on the circle to be point A and construct the circle centered there with the same radius as the first circle. Continue constructing the circles and labeling points as shown in the diagram.



Connect A to B to C to D to E to F to A to form a regular hexagon.

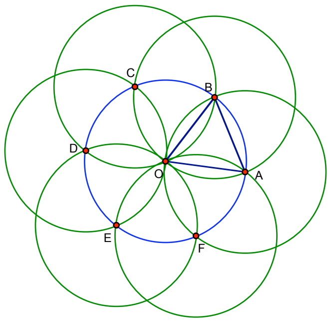
Step 2 Find the midpoint of segment $\overline{AB}$ by drawing segment $\overline{OP}$. Continue finding midpoints for each of the hexagon sides and labeling the point in order: G H J K L M.



Connect A to G to B to H to C to J to D to K to E to L to F to M to A.
You have constructed a regular dodecagon!

Analysis of Step 1: How do we know that the figure closes back on itself? Have we really constructed a hexagon?

Sometimes the construction of this figure doesn't work out properly. The compass may slip and accidentally change size or the compass not be located exactly on the center you intend. However, we do know that theoretically the figure is correct. That is, as we construct the circles centered at A, B, C, D, E and F, we know that the final circle centered at F does intersect the original circle at point A. This is easy to see when we draw in the equilateral triangles that are created by our construction:



FACT ONE: ΔOAB is an equilateral triangle.

Proof:
We know this is an equilateral triangle because all of the circles have the same radius – after all, you never (intentionally!) changed the size of the compass during the construction!

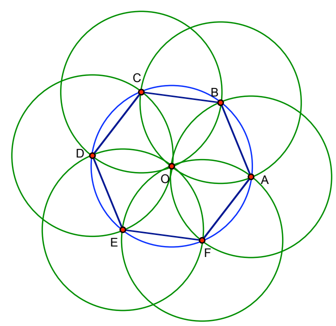
1. **Assignment:** Sketch all the congruent equilateral triangles created as you constructed each circle. Explain why they are all congruent to each other.

We are ready to see why the figure closes back on itself. Each angle in each equilateral triangle has measure 60˚, so by the time you've constructed that last circle you have six triangles joined around the center for a total of 6 X 60˚ or 360˚. 360˚ is the full circle, so the last side of the last triangle coincides with $\overline{OA}$. So in fact the figure does closes back on itself. Now we are ready to prove that our constructed figure ABCDEF is a regular hexagon:

FACT TWO: ABCDEF is a regular hexagon

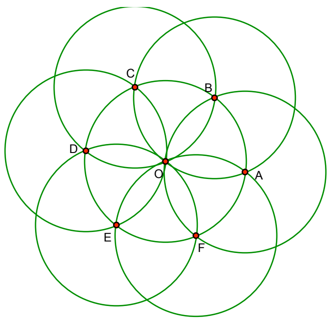
Proof:

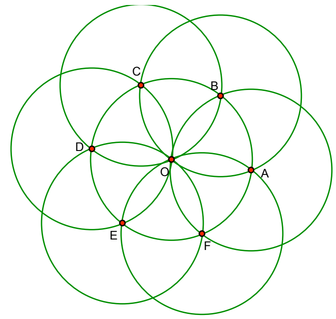
- All the sides are congruent because each is a radius of one of the circles and the circles all have the same radius.
- All the angles are congruent because . . . Well, imagine all the equilateral triangles that have a vertex at O that you found in Assignment 1. Combining them makes the hexagon. Two adjoining triangles make an angle of 120˚ (60˚ + 60˚) at each vertex. This shows that each angle in the hexagon has the same measure. (Just like the pattern blocks in lesson 2.)



These two parts show that ABCDEF has congruent sides and congruent angles so it is a regular hexagon.

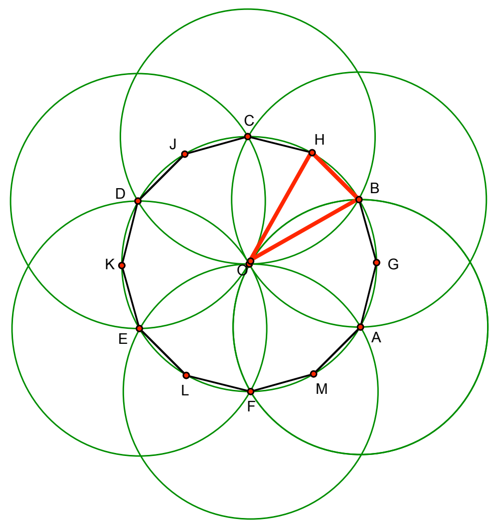
2. **Assignment:** Find all the rotational symmetries of the rosette that you constructed. Find all of the lines of symmetry





In fact the symmetries are the same as the symmetries of the regular hexagon, ABCDEF,

Analysis of Step 2: Is AGBHCJDKELFM really a regular dodecagon?



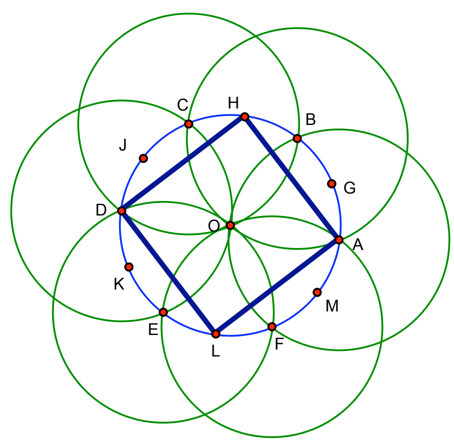
FACT THREE: Dodecagon AGBHCJDKELFM is a regular dodecagon.

Proof: Consider the isosceles triangle ΔOBH and all the other triangles with a vertex at O and a side that is also a side of the dodecagon.

- All the <u>sides</u> are congruent because of the symmetries you found in Assignment 2. Repeated reflections will take a side to any other side.
- All the <u>angles</u> are congruent because . . . same reason, repeated reflections will take any angle to any other angle.

These two parts show that AGBHCJDKELFM has congruent sides and congruent angles so it is a regular dodecagon.

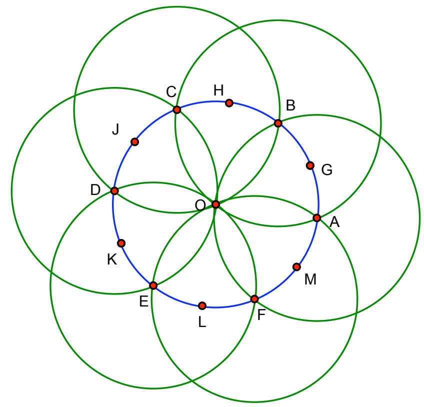
Many more polygons can be found in our construction. Example**:** One polygon not seen before is a square:

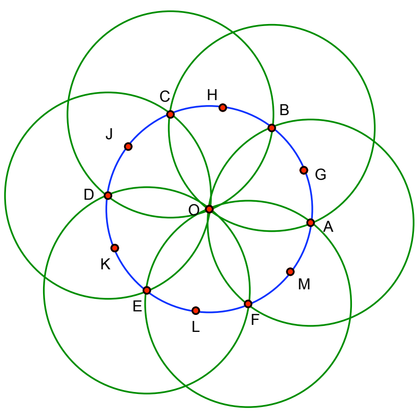


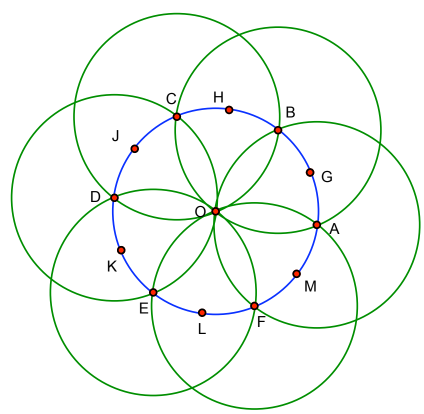
FACT FOUR: AHDL is a square.

Proof: I know the quadrilateral AHDL is a square because the entire figure has rotational symmetry of 90˚ which will also rotate AHDL to itself. So the sides are all congruent and the angles are all congruent – so each must be 90˚.

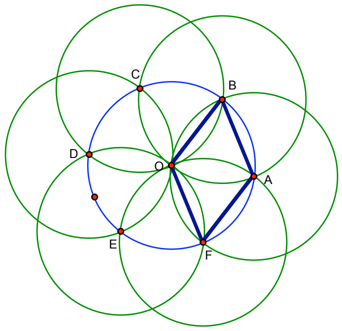
3. **Assignment:** Find more polygons, both regular or not regular, in the rosette. Find all the angles in each of your figures. Include justifications for each one.



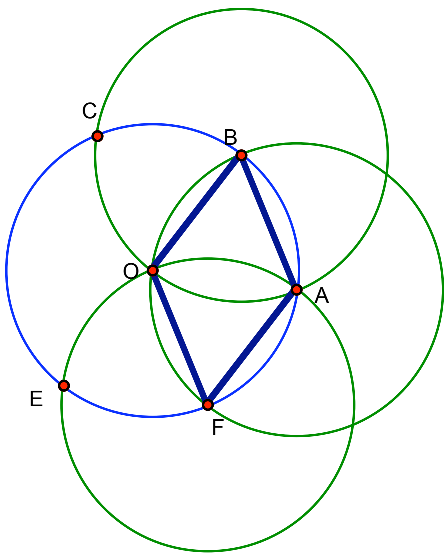




Learning more about constructions. One polygon you may have noticed is a rhombus with angles of 60˚ and 120˚. Many of these appear in the figure, but here is one that showed up in the hexagonal figure:



Let's look at this rhombus more closely. What are the symmetries of this figure? `Draw in the two lines of symmetry.`



Notice that they coincide with the diagonals of the rhombus. Because of these symmetries we can say two important things about the diagonals of a rhombus.

- **The diagonals bisect the angles of the rhombus.**

Proof: If we were to fold the figure on the line of symmetry BF the angles $\angle OBF$ and $\angle ABF$ would coincide. Similarly, folding on the line OA, shows that the other diagonal bisects the other angles.

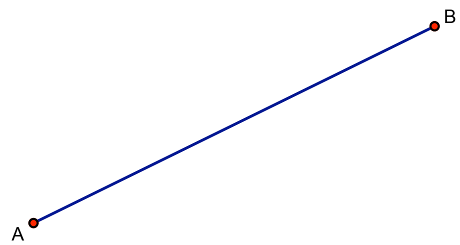
- **The diagonals are perpendicular to each other.**

Proof: If we fold on one diagonal and then the other the sides will all coincide and the angles formed by $\overline{BF}$ and $\overline{OA}$ will all be congruent and therefore all must be right angles. This also shows that

- **The diagonals bisect each other.**

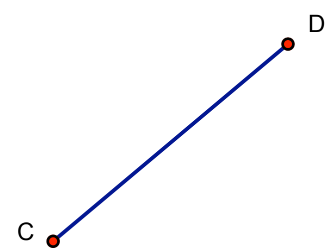
Constructing equilateral triangles and rhombuses.

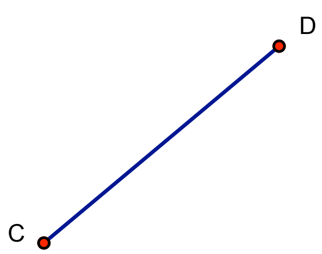
Example: Use your compass and straightedge to construct an equilateral triangle that has sides congruent to the line segment shown here:



Label the angles with the correct measurements.

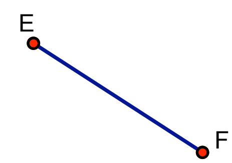
Example: Use your compass and straightedge to construct a rhombus that has sides congruent to this line segment:





4. **Assignment:** Use your compass and straightedge to construct a different rhombus having sides congruent to CD.

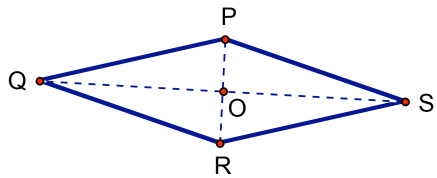
5. **Assignment:** Construct two different rhombuses so that $\overline{EF}$ is a diagonal.



Notice the following about all of the rhombuses you constructed:

Theorem: In any rhombus:

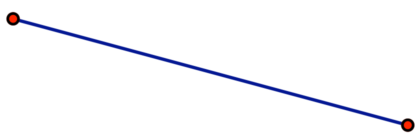
The diagonals are perpendicular.
The diagonals bisect the angles.
The diagonals bisect each other.



Proof: This rhombus has the same symmetries as the rhombus we considered earlier that had angles measuring 60˚ and 120˚ but it is not so easy to prove this because we do not know that the opposite angles are congruent. However, we can see, by SSS that $\Delta QPS \cong \Delta QRS$. So when we fold the rhombus on the diagonal $\overline{PR}$ folds onto itself. The same is true for the other diagonal, so both are lines of symmetry for the rhombus. Now the same argument we used for the special rhombus works to convince us of these three facts for any rhombus.

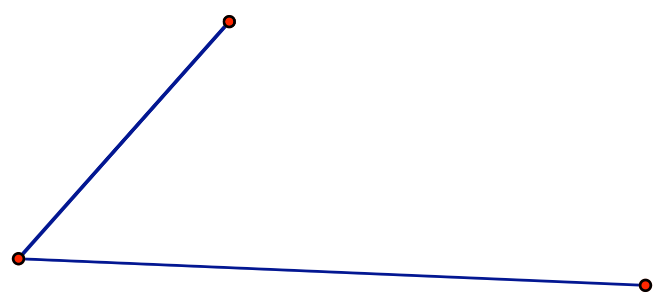
Using rhombuses to understand constructions

We can use these facts about rhombuses to see how to do many constructions.



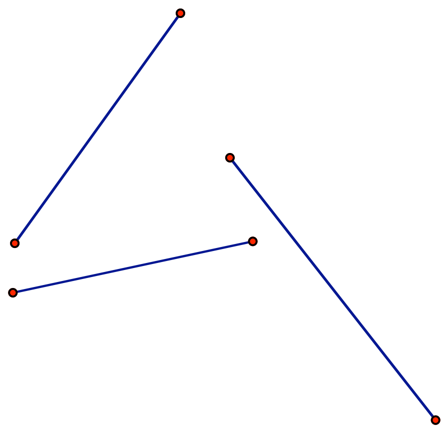
6. **Assignment:** Use your compass and straightedge to construct the perpendicular bisector of this line segment.

7. **Assignment:** Use your compass and straightedge to construct the angle bisector of this angle.

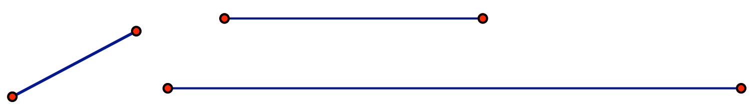


Recall from Lesson 1 how to construct a triangle given three side lengths.

8. **Assignment:** Use your compass and straightedge to construct a triangle that has sides congruent to the three line segments shown here.



9. **Assignment:** What happens when you try this construction with these three line segments?



Finally, a word from Antiquity

You have constructed many polygons and angles with a compass and straightedge, but not all figures and angles can be constructed with those tools. There are 3 famous construction problems from about 400 B.C .that can not be solved using compass and straightedge only.

1. the quadriture of any given circle
2. the duplication of any given cube
3. the trisection of any given angle

Problem #1 asks you to construct a square with the same area as a circle. You will study area in depth during the next unit, but you may remember these two formulas:

Area of a circle = πr^2 and Area of a square = s^2.

If the square you construct has the same area as the circle, s^2 must have the same value as πr^2. If you take the square root of both of these values, s = $r\sqrt{\pi}$. While it may seem obvious to you that it is impossible to construct a length of $\sqrt{\pi}$, mathematicians worked until 1882 to prove its impossibility.

Problem #2 asks you to construct a cube with twice the volume as a given cube. Volume is one of the topics you will study in Unit 3, but you might remember the formula Volume of a cube = s^3. The volume of a cube with twice the volume must be $2s^3$. If you solve V = $2s^3$ for s, s = $\sqrt[3]{\frac{V}{2}}$. Again, mathematicians needed algebra (unknown in 400 B.C.) to prove that it is impossible to construct a length equal to the cube root of a known length.

Problem #3 asks you to trisect (cut into 3 congruent angles) any given angle. Of course, some angles can but trisected with only a compass and straightedge.

However, not all angles can be constructed in that manner. The first rigorous proof that this problem is sometimes impossible was completed in 1837.

Some problems in mathematics take a long time to solve!

10. **Assignment:** What angles would you be able to trisect with only a compass and straightedge?

Lesson 5 Problems:

Recall how, in earlier lessons you did some of these same constructions by folding. One could also do them by measuring with a ruler or protractor. Not all of following problems can all be completed with compass and straightedge and no other instruments. So you will need to consider other tools. For each construction, first try with toolbox I. Use toolbox II or III if that seems easier. You may want to use a combination of these three toolboxes. Use toolbox IV only if you see no other way to complete the construction.

Toolboxes:

I	**II**	**III**	**IV**
Compass Straightedge Pencil	Tracing paper Pencil (for folding and copying)	3X5 card (a right angle and a straight edge) Pencil	Ruler Protractor Pencil

11. Discuss the relative advantages and disadvantages of each toolbox

For problems in which part of the construction is already shown, you may make a copy of the figure on a different piece of paper or on tracing paper before completing the construction.

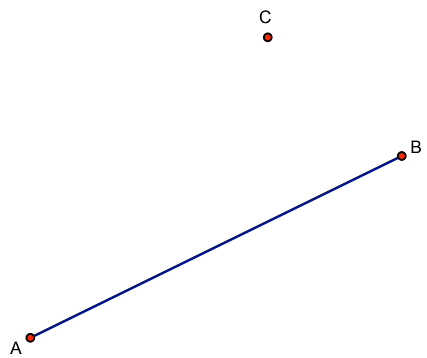
12. Construct an angle that measures 60˚ without using a protractor.

13. Construct an angle that measures 30˚ without using a protractor.

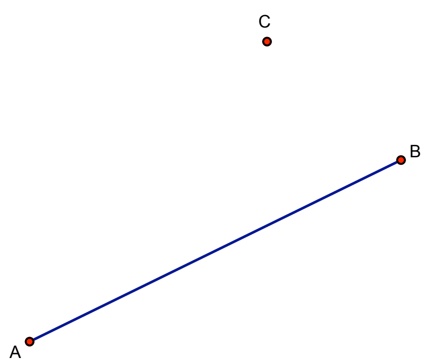
14. Construct an angle that measures 45˚ without using a protractor.

15. Explain how one could onstruct angles that measure 75˚, 90˚, 105˚, and 110˚ without using a protractor.

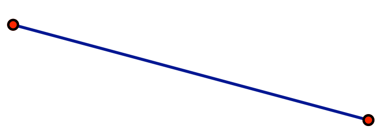
16. Construct a line perpendicular to a line $\overline{AB}$ through the point, C. Repeat the problem using each of the four toolboxes.



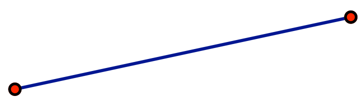
17. Construct a line parallel to a line $\overline{AB}$ through the point, C.



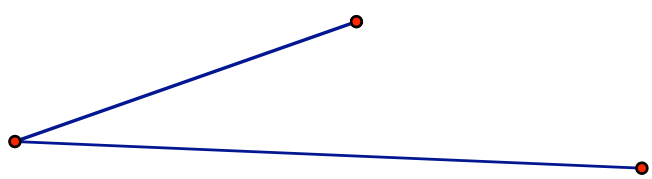
18. Construct a square that has this line segment as one of the sides.



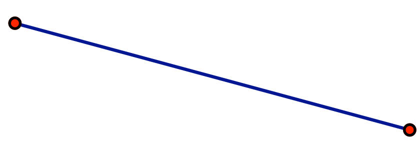
19. Construct a square so that this line segment is a diagonal of the square.



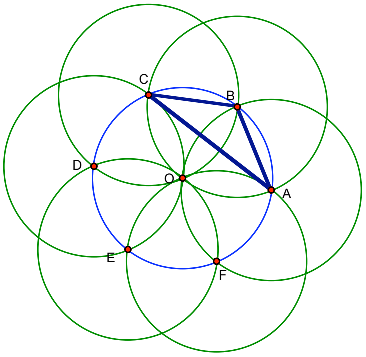
20. Complete the parallelogram given these two sides.



21. Construct a kite that has this line segment as one of the diagonals.



More polygons Here are some more polygons that can be found in our rosette:

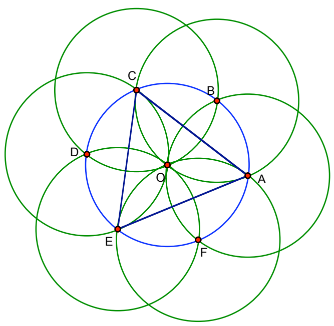


The triangle ΔABC is an isosceles triangle.

Proof: As before, the line segments $\overline{AB}$ and $\overline{BC}$ are congruent Because they are both radii of congruent circles.

22. How many triangles congruent to ΔABC can you find in the rosette? Write them by given the letter of each vertex:

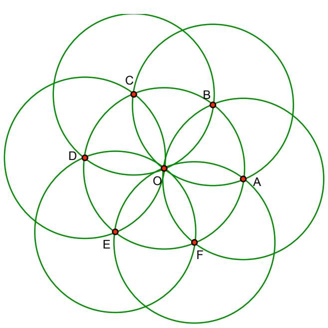
Now, you try one:

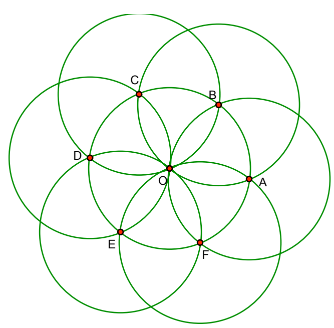


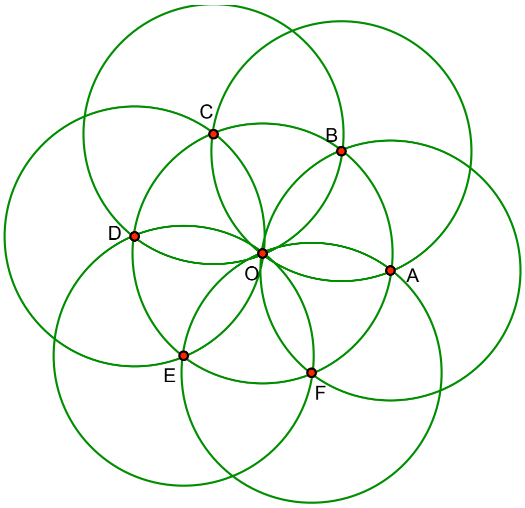
ΔACE is an equilateral triangle.

Proof:

23. Draw more examples on these pictures, name the polygon and supply a proof. Find polygons that are NOT congruent to the ones already shown.






C
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D
O
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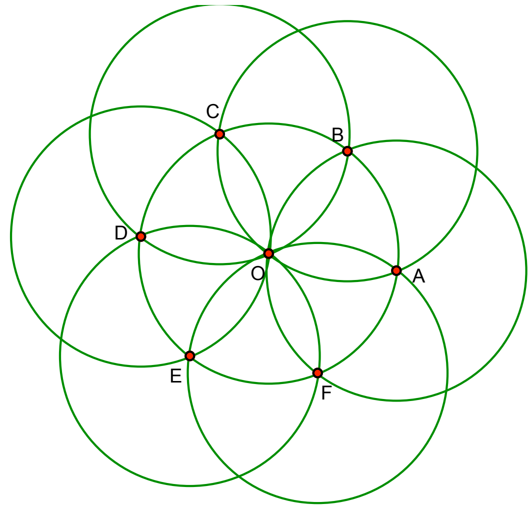
F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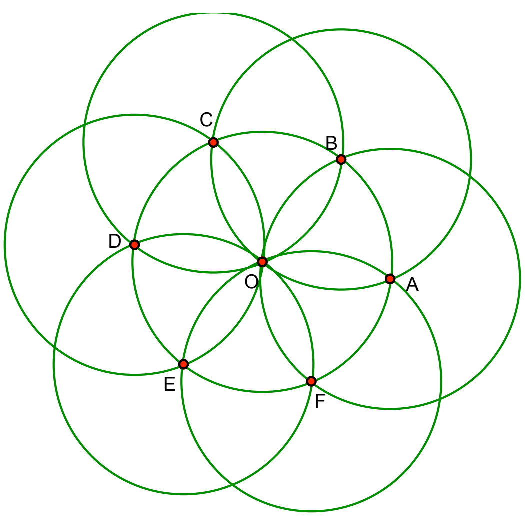
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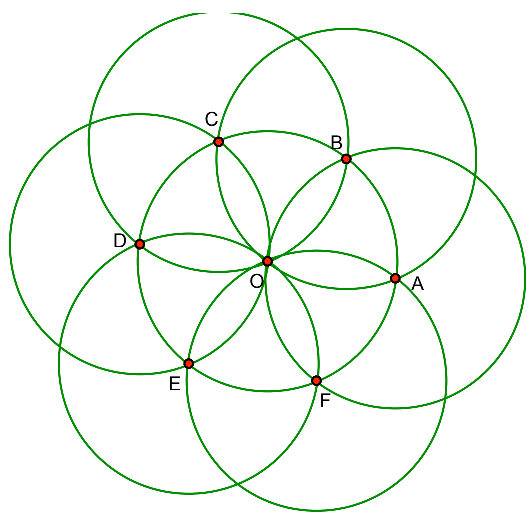
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Lesson 6 – Quadrilaterals, Conjecture and Proof

Class Activity: Quadrilateral Chart

Complete the charts on this page and the next. Here are some things you may consider doing while investigating quadrilaterals:

- construct several different examples,
- use geostrips,
- try folding
- try tracing
- use your compass and straight edge,
- measure with ruler or protractor.

In the end you will want to be able to provide a good justification for your answers.

Side Properties

	Parallelogram	Rhombus	Rectangle	Square	Kite	Trapezoid
At least 1 pair of parallel sides						
2 pairs of parallel sides						
All sides congruent						
At least 1 pair of opposite sides congruent						
2 pairs of opposite sides congruent						

Angle Properties

	Parallelogram	Rhombus	Rectangle	Square	Kite	Trapezoid
4 right angles						
Opposite angles congruent						
Adjacent angles supplementary						

Diagonal Properties

	Parallelogram	Rhombus	Rectangle	Square	Kite	Trapezoid
At least 1 diagonal forms 2 congruent triangles.						
Each diagonal forms 2 congruent triangles.						
Diagonals bisect each other.						
Diagonals are congruent.						
Diagonals bisect the vertex angles.						
Diagonals are perpendicular						
Diagonals form 4 congruent triangles						

Symmetric Properties

	Parallelogram	Rhombus	Rectangle	Square	Kite	Trapezoid
Number of lines of symmetry						
Smallest positive angle of rotational symmetry						

Example: Filling in the column for the square

1. Draw a line from each X on left to the statement on the right that justifies the statement.

Side Properties

	Square
At least 1 pair of parallel sides	X
2 pairs of 1parallel sides	X
All sides congruent	Defn.
At least 1 pair of opposite sides congruent	X
2 pairs of opposite sides congruent	X

Angle Properties

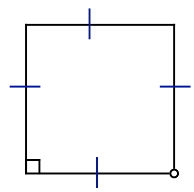
4 right angles.	X
Opposite angles congruent.	X
Adjacent angles	

Diagonal Properties

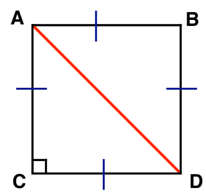
At least 1 diagonal forms 2 congruent triangles	X
Both diagonals form each form 2 congruent triangles.	X
Diagonals bisect each other.	X
Diagonals are congruent.	X
Diagonals bisect the vertex angles.	X
Diagonals are perpendicular	X
Diagonals form 4 congruent triangles	X

From the definition of a square, se start knowing that all sides are congruent and that <u>one</u> angle is a right angle. First, we'll show that the other angles are also right angles:

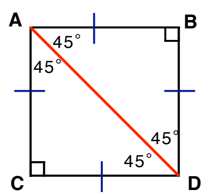
- By transformation: Rotate transparency over the square to confirm that the sides are congruent and that all angles are congruent. Since one is a right angle, all of them must be right angles.
- By construction: Make a rhombus from geostrips and show how fixing one angle forces the others.
- By rigorous proof:



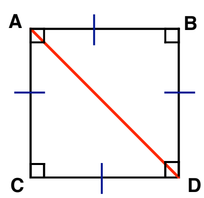
1. Start with a square. All we know for sure is labelled at the left: All sides are congruent and ONE angle is 90° The rest we need to prove.



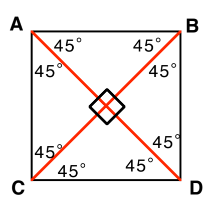
2. **ΔABC** and **ΔACD** are congruent because the three sides are congruent, that is **AB≅AC, BD≅CD** and **AD≅DA**.



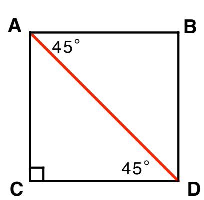
3. By the sum of the angles formula the other two angles in both **ΔABC and ΔACD** measure 45°



4. This also shows that the other three angles in the square are 90°



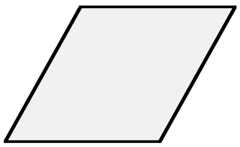
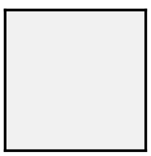
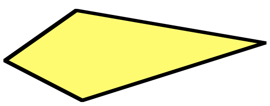
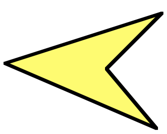
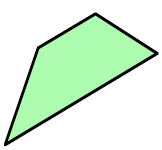
5. Adding the other diagonal constructs four congruent right isoseles triangles. The 45° angles ensure that the center angles are all right angles.



6. The above diagram shows that the opposite sides of a square are parallel. Lines $\overline{AB}$ and $\overline{CD}$ are cut by the tranversal $\overline{AD}$. The alternate interior angles marked are congruent so the lines must be parallel.

Group Project

Complete this chart. Be prepared to justify any of your conclusions.

Quadrilateral	Definition	One Diagonal	Other diagonal
	Parallelogram opposite sides are congruent	congruent obtuse (possibly right) triangles	congruent acute (possibly right) triangles
	Rhombus		
	Rectangle		
	Square		
	Kite (convex)		
	Kite (concave)		
	Trapezoid		
	Isosceles Trapezoid		

Always, Sometimes, Never

Each of the following is a conditional statement. Mark whether it is true always, sometimes or never. If you mark always, give a convincing argument that the statement is always true. If you mark sometimes give two examples of quadrilaterals: one for which both parts of the statement are true and one where the first is true but the second is not.

Always, Sometimes, Never If the diagonals of a quadrilateral are congruent, then they are perpendicular.

Always, Sometimes, Never If a quadrilateral has 180˚ rotational symmetry (about the point of intersection of the diagonals) , then the diagonals bisect each other.

Always, Sometimes, Never If a quadrilateral has 180˚ rotational symmetry, then the diagonals of a quadrilateral are congruent.

Always, Sometimes, Never If a quadrilateral has 90˚ rotational symmetry, then the diagonals of a quadrilateral are perpendicular.

Always, Sometimes, Never If a quadrilateral has 90˚ rotational symmetry, then the diagonals of a quadrilateral are congruent.

Always, Sometimes, Never If one of the diagonals is a line of reflective symmetry for the quadrilateral, the quadrilateral is a rhombus.

Always, Sometimes, Never If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Always, Sometimes, Never If a quadrilateral is a parallelogram, then the diagonals bisect each other.

Always, Sometimes, Never If the diagonals of a quadrilateral are congruent, then the quadrilateral is a rectangle.

Always, Sometimes, Never If a quadrilateral is a rectangle, then the diagonals are congruent.

Lesson 6 Problems:

1. For each part below, describe all the quadrilaterals you can make by adjoining the given two triangles along congruent sides.

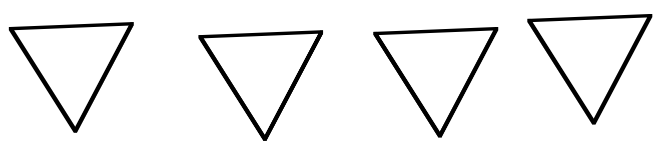
   a) two congruent scalene triangles.

   b) two congruent isosceles triangles.

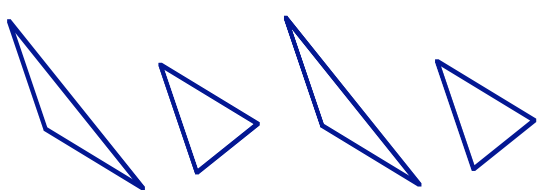
   c) two congruent right triangles.

2. Which of these three sets of four triangles can be puzzled together to construct a quadrilateral? What different kinds of quadrilaterals can you make from each set? Explain.

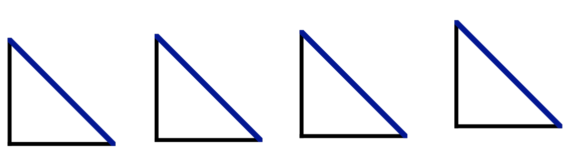
SET 1

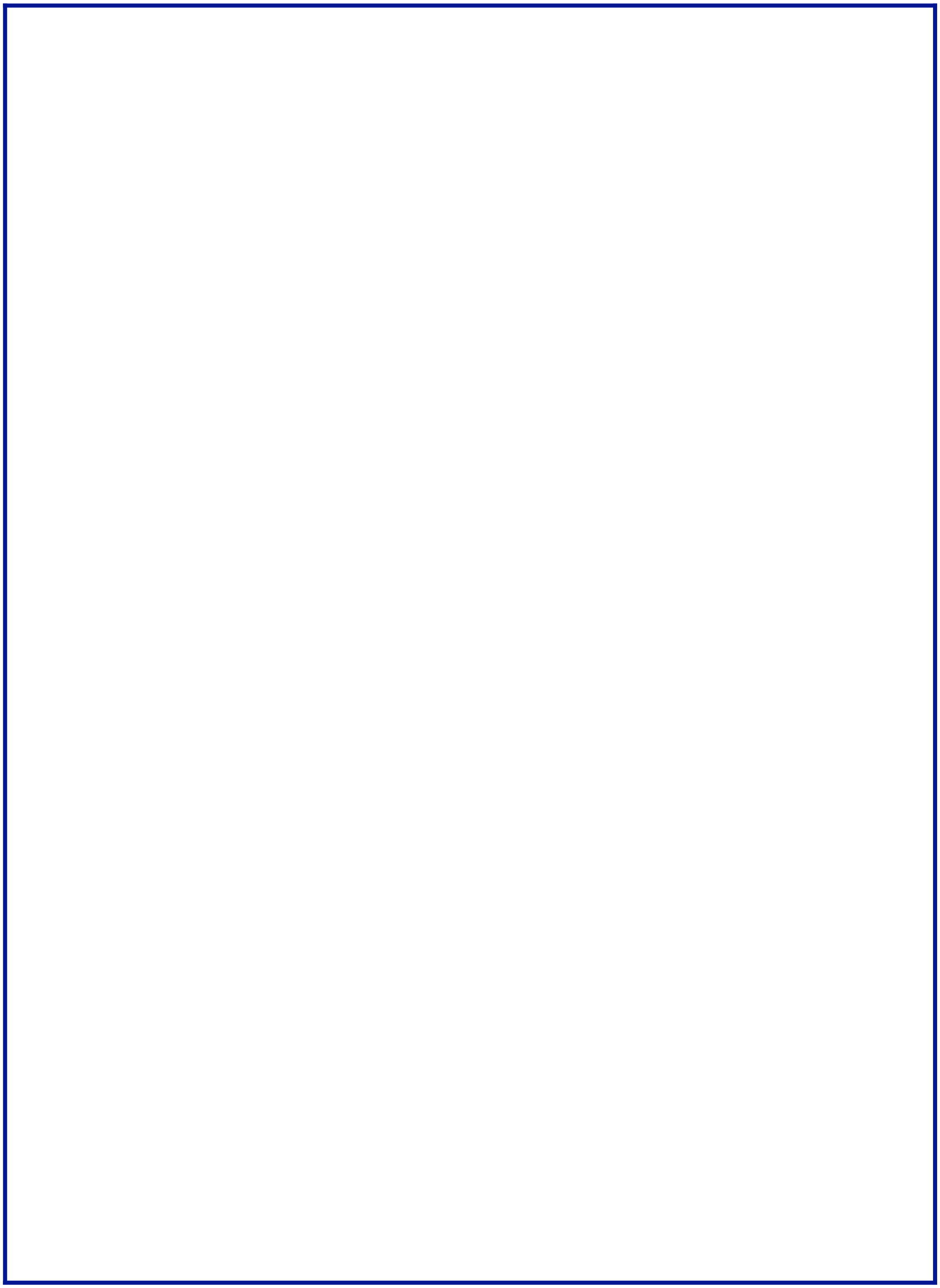


SET 2



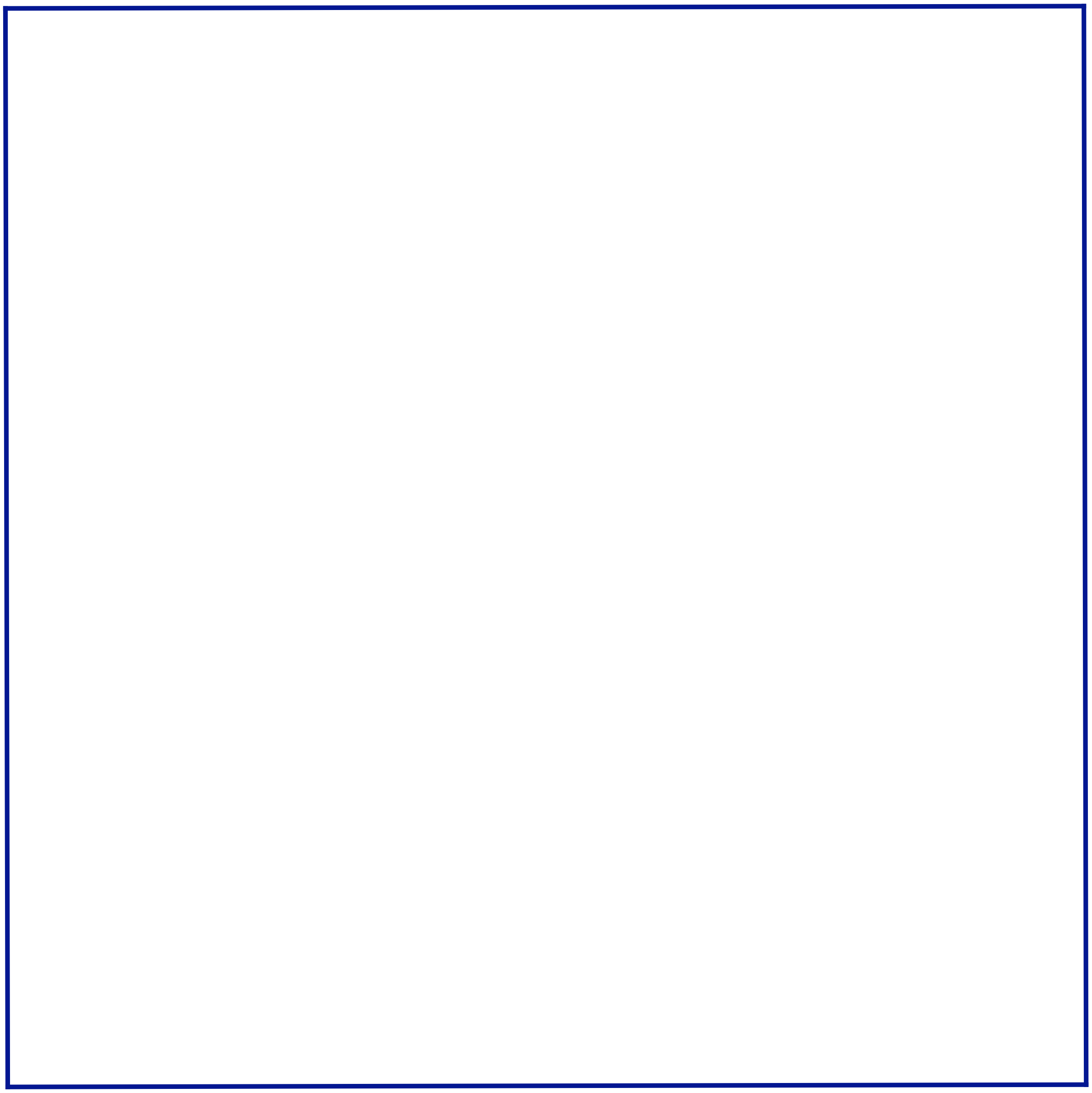
SET 3





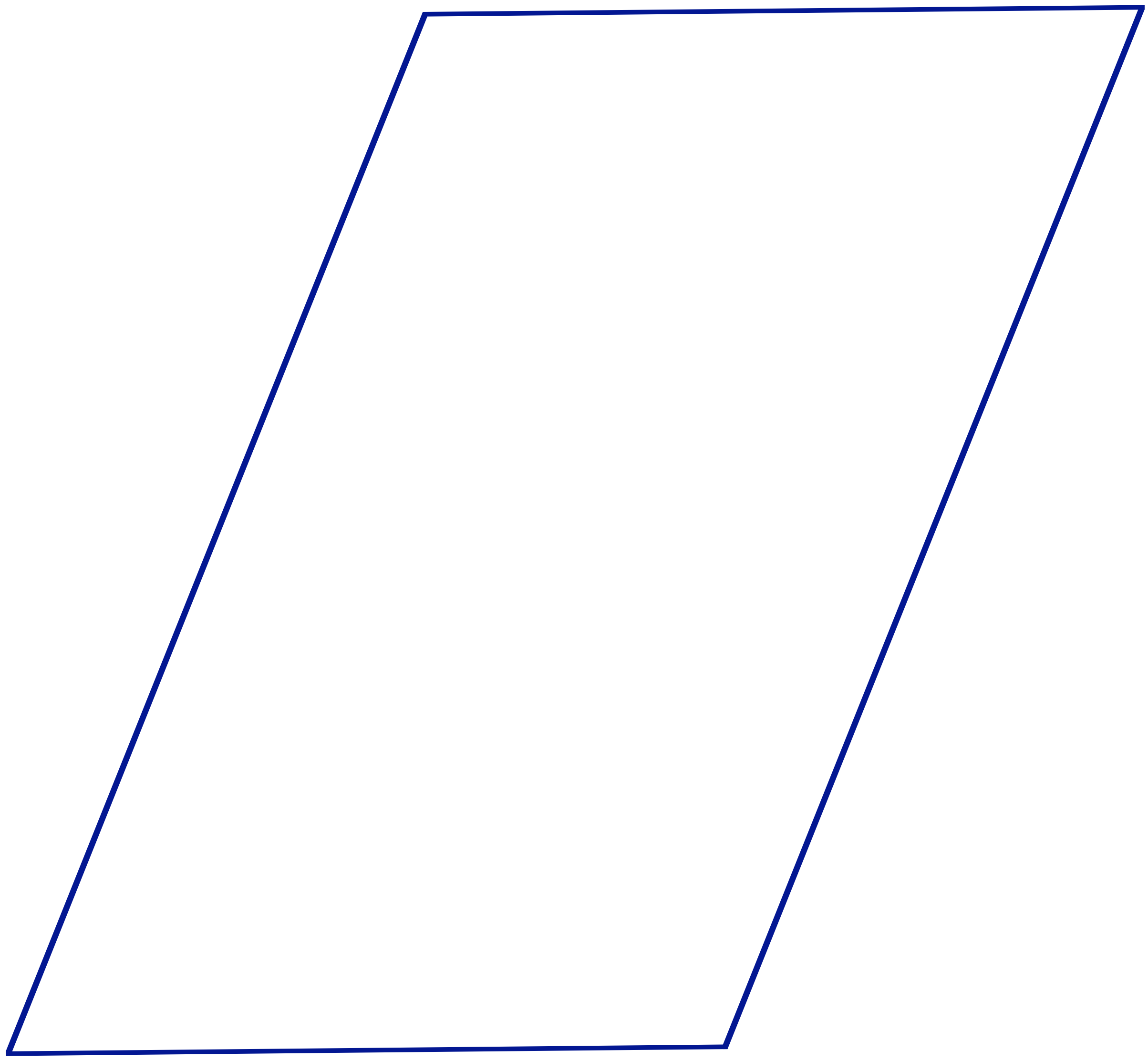
A rectangle that is not a square.

But a plain piece of paper may be a better example

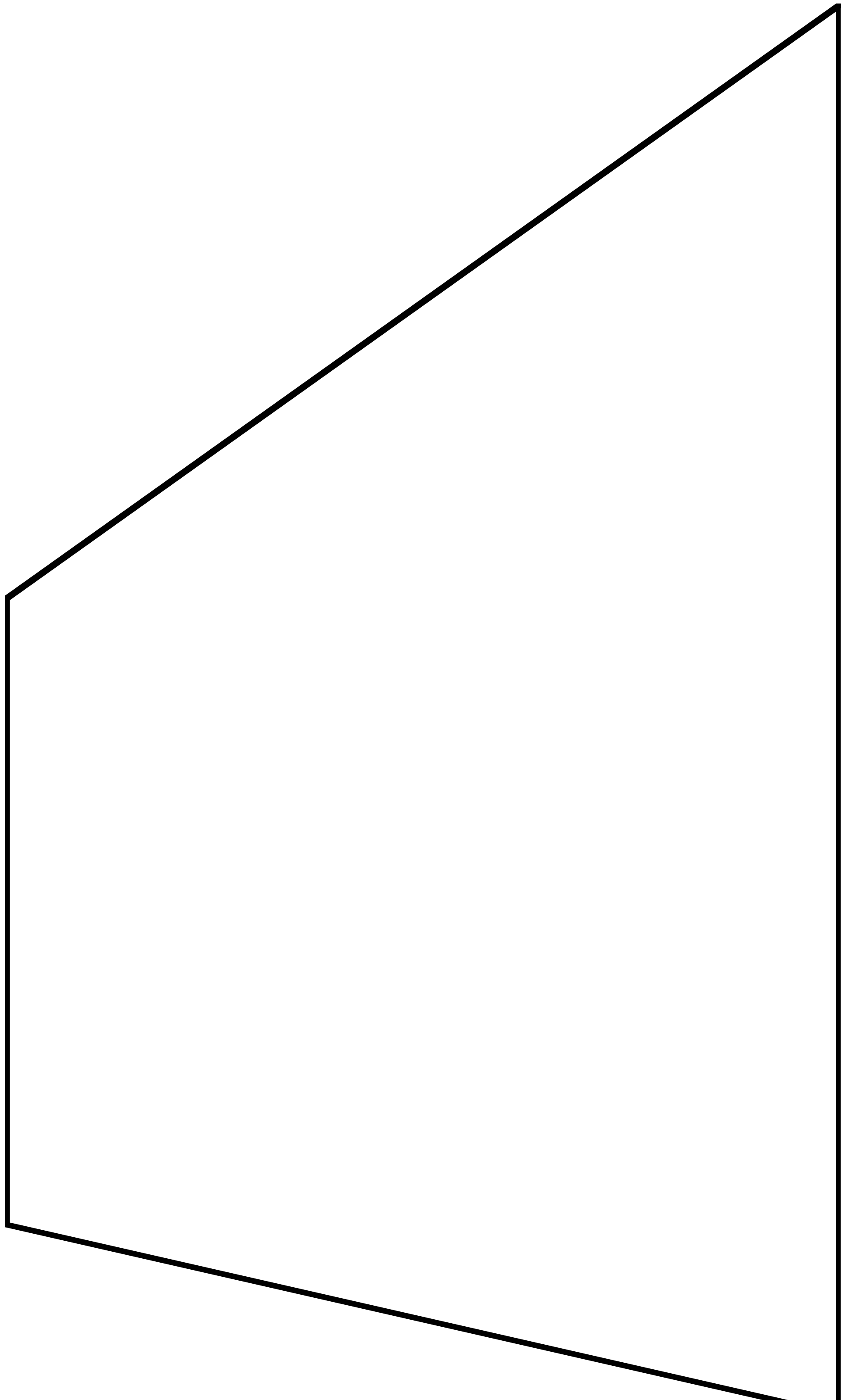


A square.

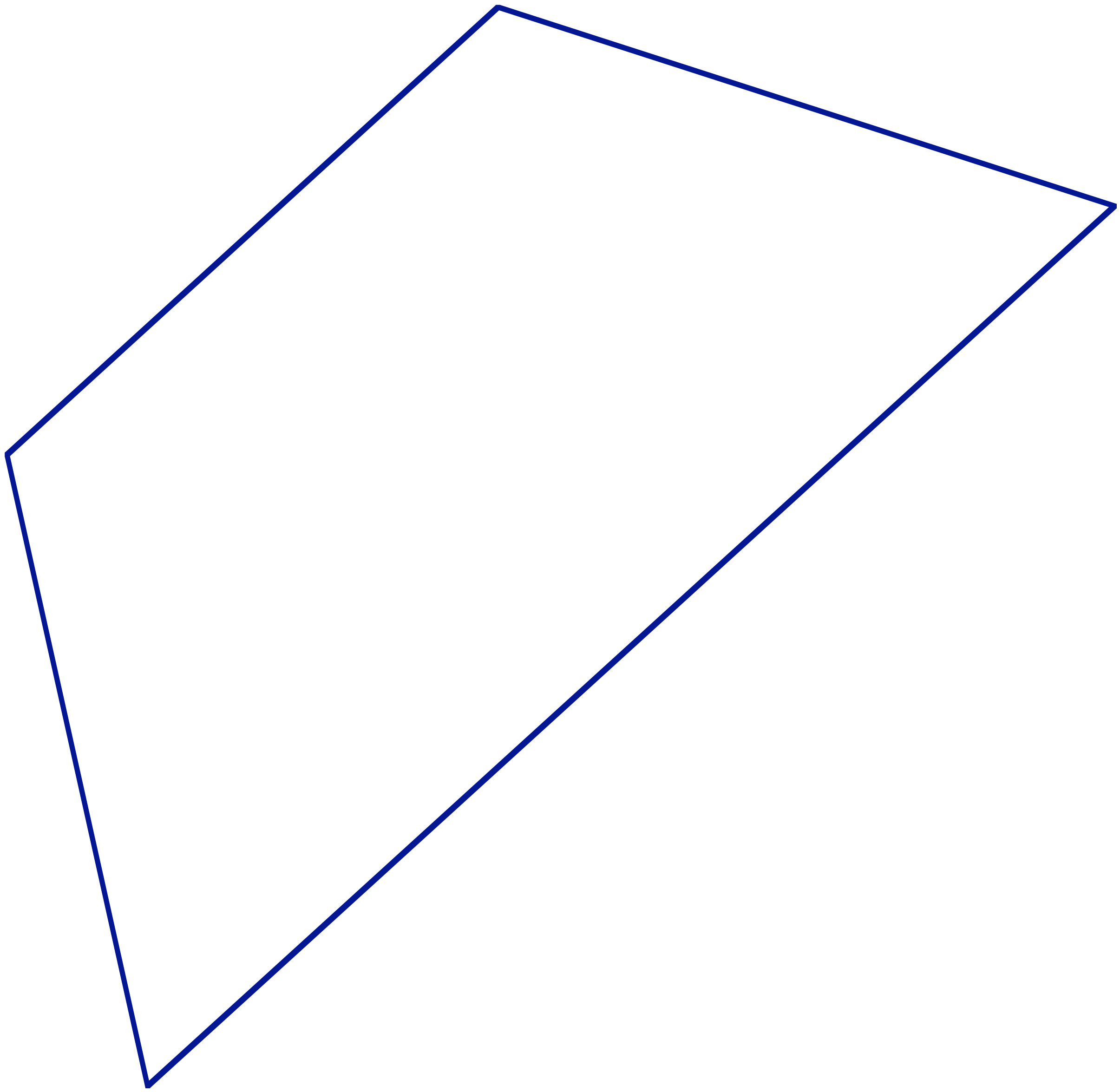
Do you know how to make a square from a rectangular piece of paper?



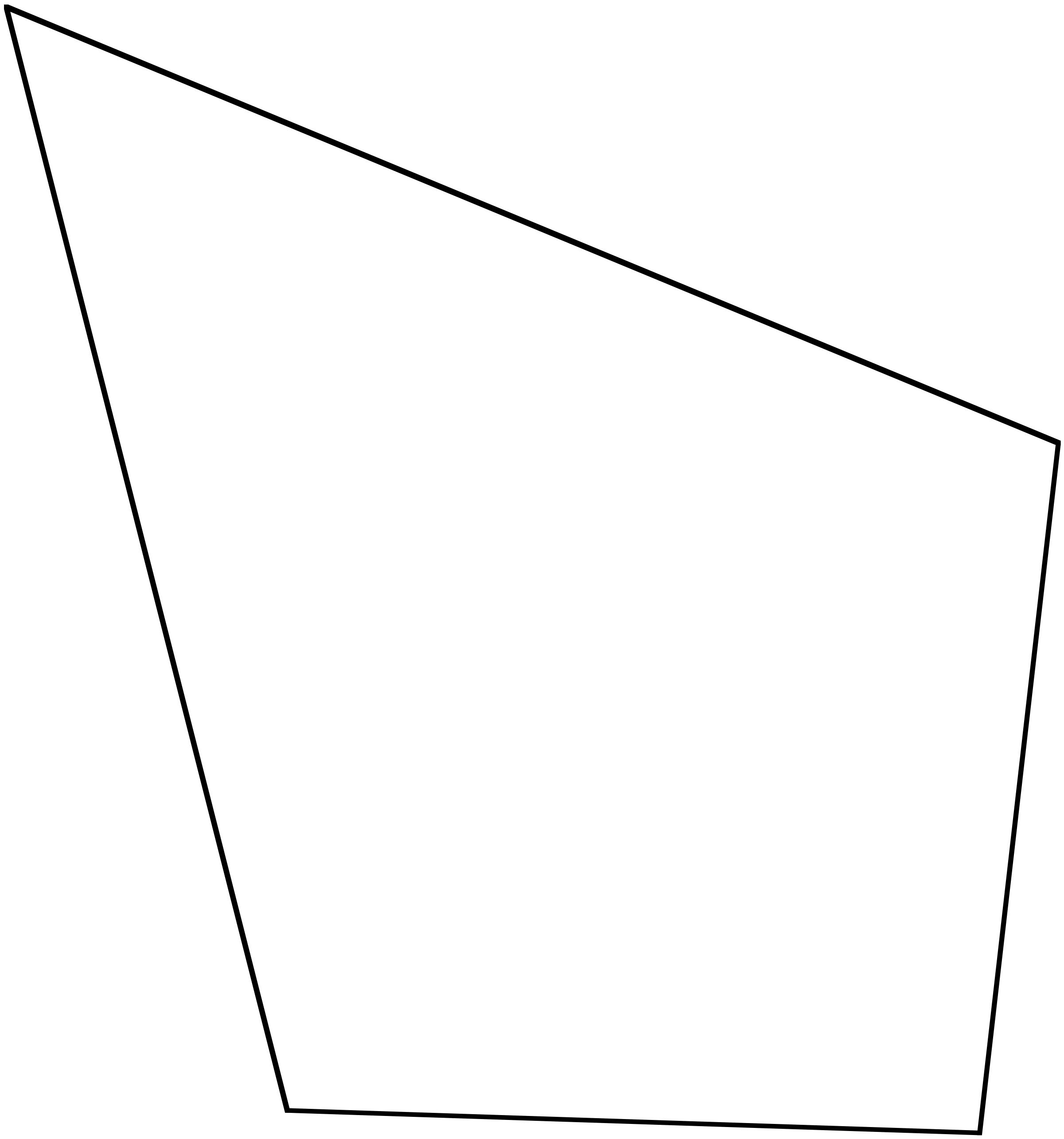
A parallelogram that is not a rectangle or a rhombus



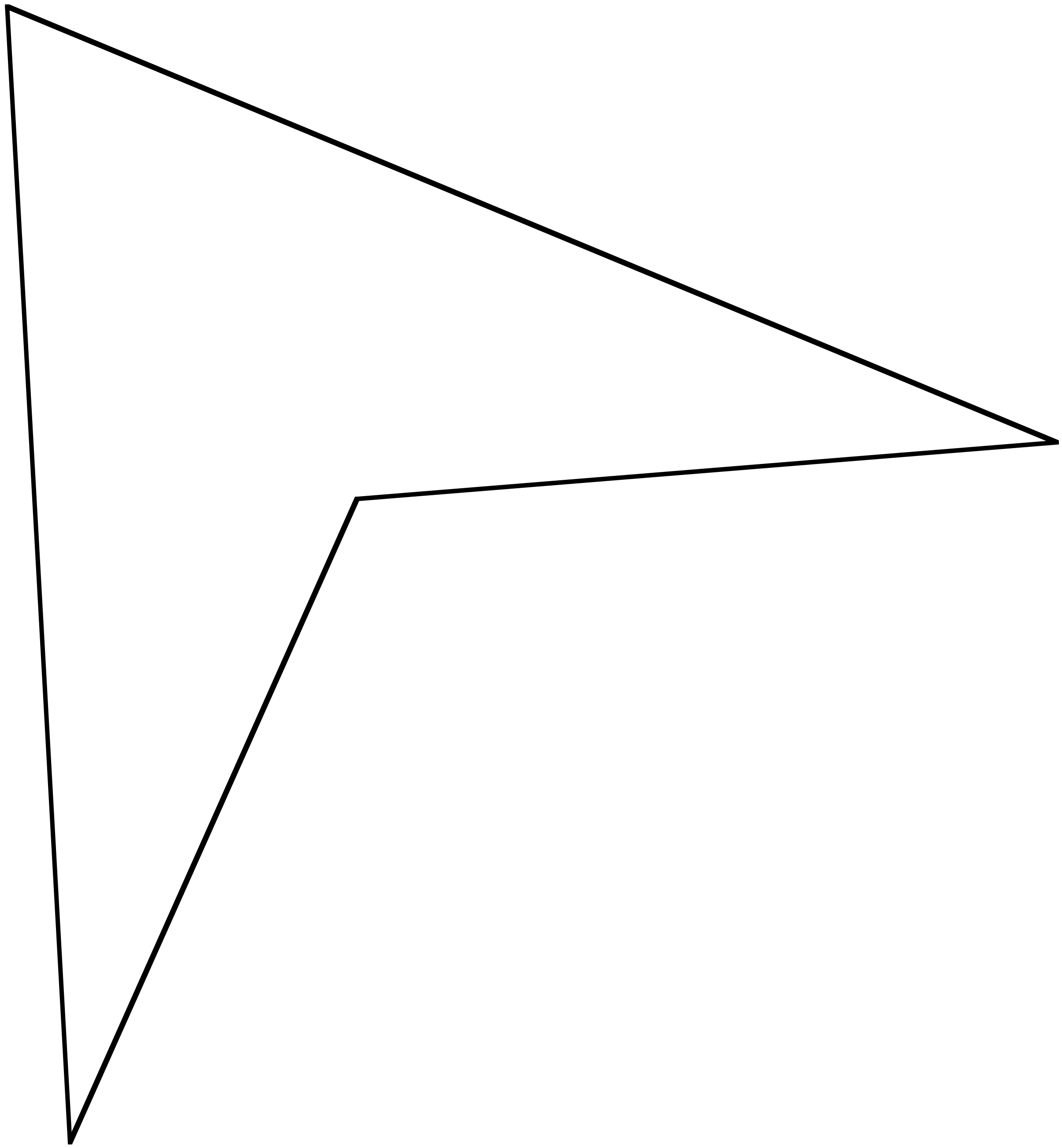
A trapezoid that is not a parallelogram or an isosceles trapezoid.



An isosceles trapezoid.



A convex kite. that is not a rhombus



A concave kite

Tessellation Project 1: Tessellating with a scalene triangle.

Cut out a scalene triangle that fits on a 3X5 card. Label the angles a, b, c and color each angle a different color. Tessellate this entire page with your triangle in such a way that every side of a copy of the triangle coincides exactly with a side of the adjoining copy of the triangle. As you copy each triangle label the angles with the correct letter and color.

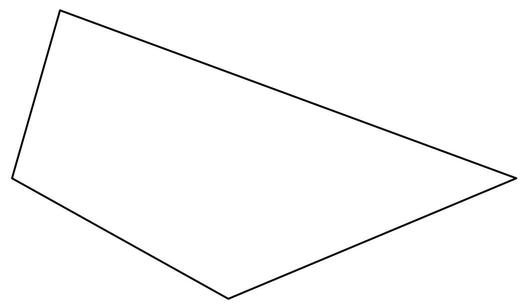
One can learn many things from this tessellation. For these two facts, trace the parts of your tessellation that demonstrate it:

The sum of the measures of the interior angles of a triangle is equal to 180˚.

Given two lines cut by a transversal, corresponding angles are congruent if, and only if, the lines are parallel.

Tessellation Project 2: Tessellating with a quadrilateral.

The same technique you used to make the tessellation of the plane with the scalene triangle will work with a quadrilateral. Make a tessellation of the plane using copies of this quadrilateral. Label and color the angles as you did with the triangle tessellation.

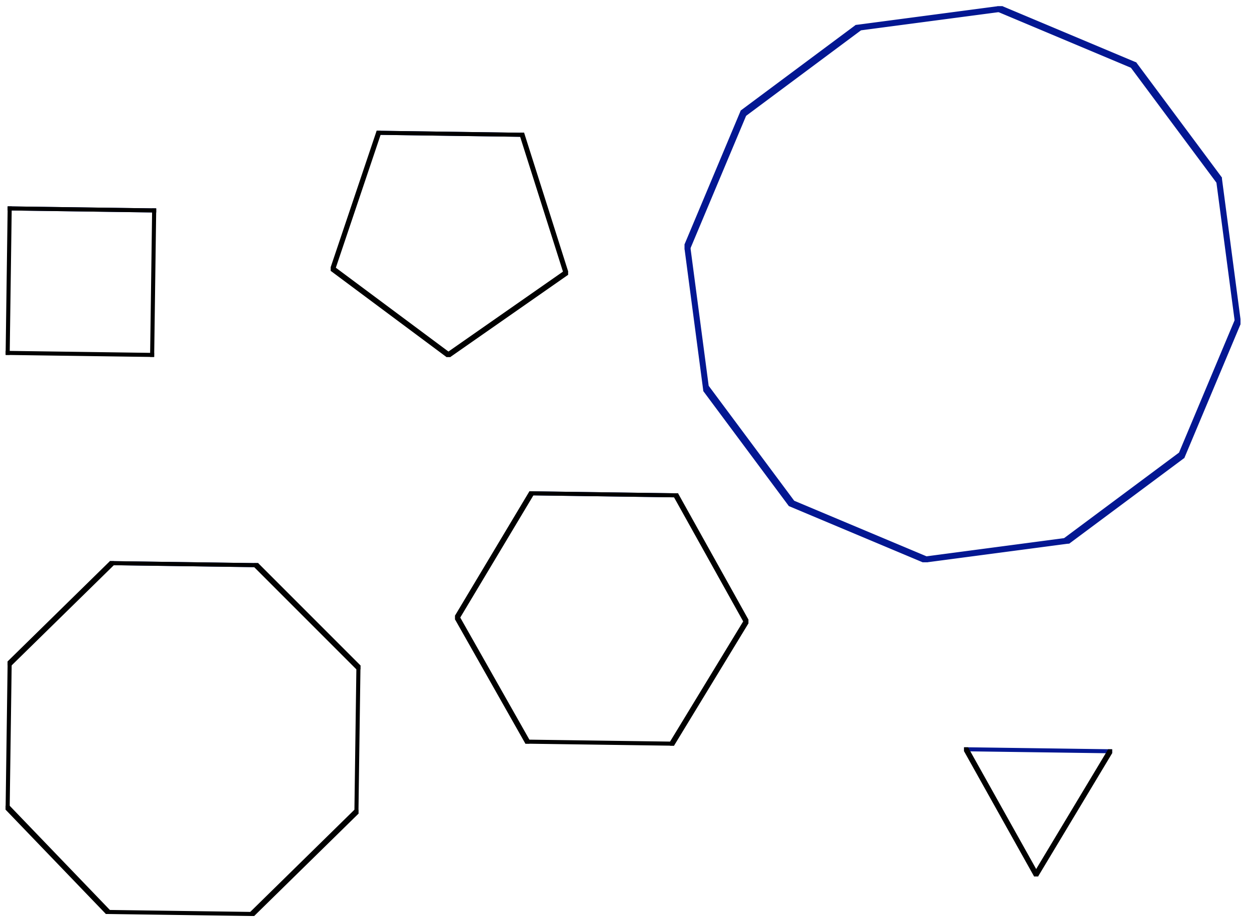


Research question: Is it possible to make a tessellation with copies of an arbitrary (not necessarily regular) pentagon?

Tessellation Project 3: Tessellating with regular polygons.

Which of these regular polygons will tessellate the plane? For each one, explain why or why not in terms of the measurement of the interior angle.

For each that does tessellate make a half-page tessellation on a separate piece of paper. You may use tracing paper.



Are there any other regular polygons that will tessellate the plane? Explain your reasoning.

Tessellation Project 4: Tessellating with two or more regular polygons: Semi-regular tessellations

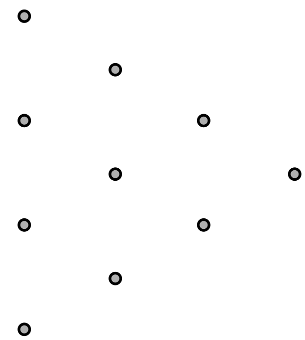
Review the definition of semi-regular tessellation from Lesson 2. Since every vertex in a semi-regular tessellation must be congruent to every other vertex, we can name each semi-regular tessellation by listing each regular polygon, in order, around a vertex. For example, the two different semi-regular tessellations with squares and triangles are named **3-3-3-4-4** or **3-4-3-4-3**. Notice that, although order matters in this naming convention, it does not matter where your start.

Number of sides of each polygon	3	4	6	8	12	Sum of angles at a vertex
Interior angle ->	60˚	90˚				
3-3-3-4-4	3	2				
3-4-3-4-3	3	2				

There are only eight possible semi-regular tessellations. Complete the table to list them all and make a half-page tessellation for each one.

Unit One Problems:

1.



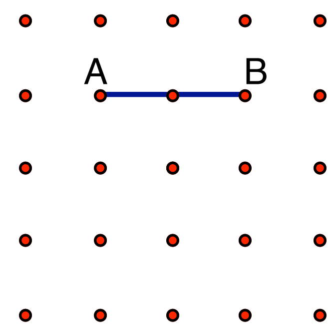
a) How many equilateral triangles can you construct by joining the points in this isometric grid?

b) How many parallelograms?

c) How many trapezoids?

In each case, how do you know you have found all possibilities?

2. Given the square grid shown, draw quadrilaterals having segment AB as one of the sides. All four vertices of the quadrilateral must be grid points.



a) How many parallelograms are possible?

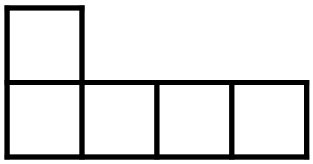
b) How many rectangles are possible?

c) How many rhombuses are possible?

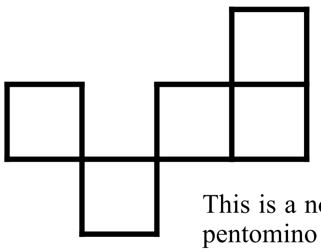
d) How many squares are possible?

In each case, be prepared to make an argument that there are no more.

3. A pentomino is made by five connected squares that touch only on a complete side. There are 12 non-congruent pentominos.



This is a pentomino



This is a not pentomino

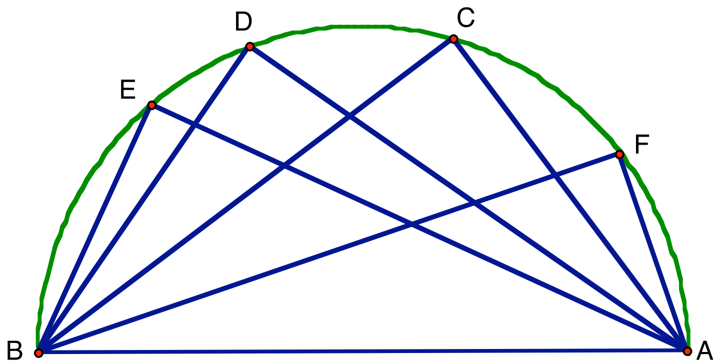
Draw all 12 pentominos on a sheet of centimeter graph paper.

Label the pentominos that have reflection symmetry and draw all lines of symmetry.

Label the pentominos that have rotation symmetry and give the smallest angle of rotational symmetry.

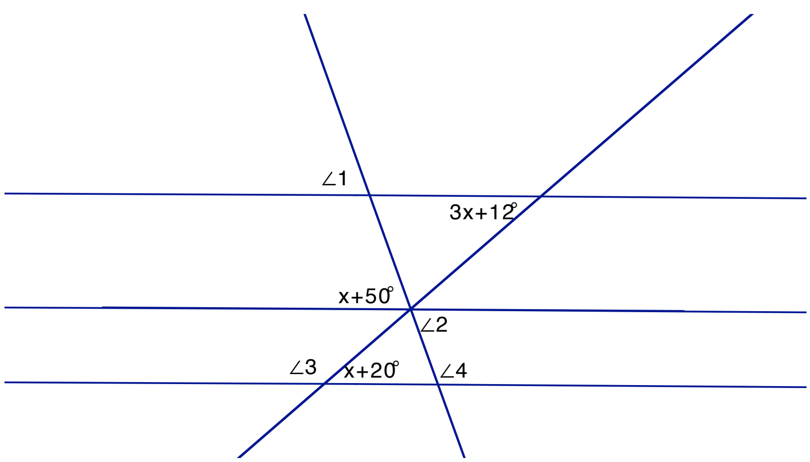
4. Use 3 pentomino shapes to form a 3 x 5 rectangle. Use 5 pentomino shapes to form a 5 x 5 square. Use all of the pentominos to make a 6 x 10 rectangle.

5. Use your protractor to measure angles E, D, C, and F with dots on their vertices in the following semicircle. Make a conjecture based on your findings.

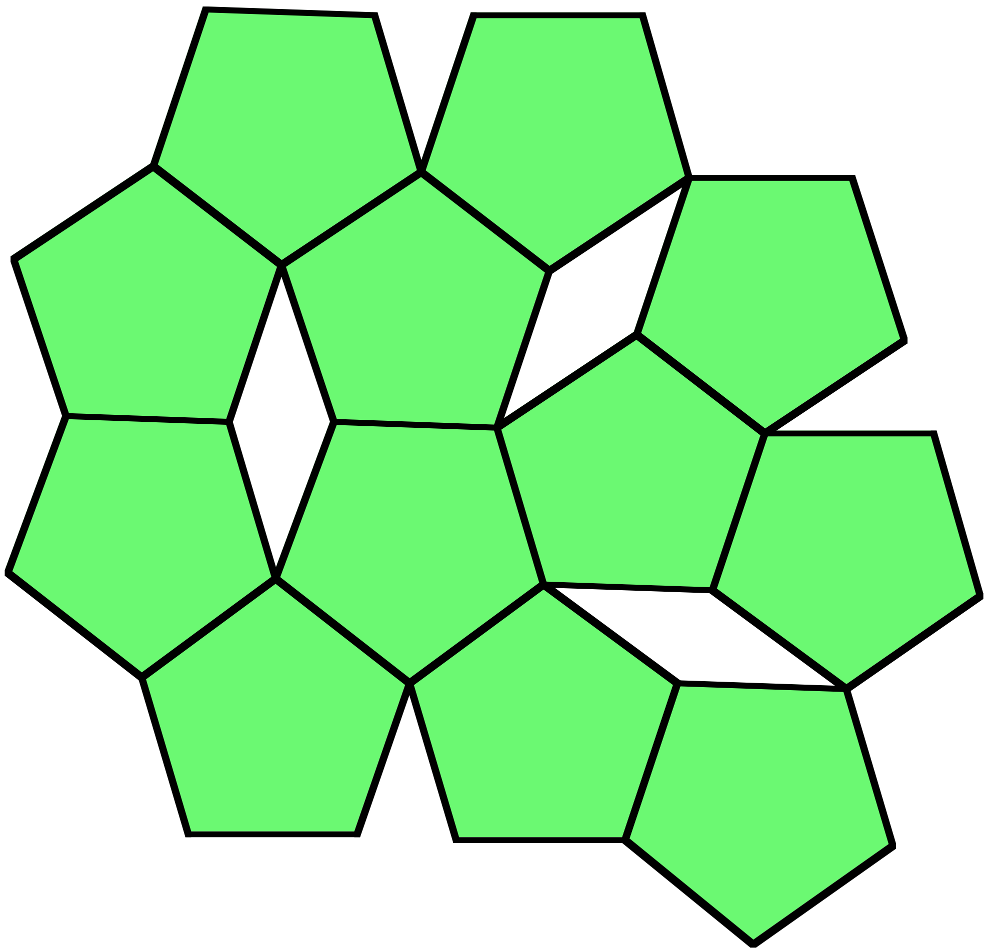


What are some differences between a conjecture and a proof?

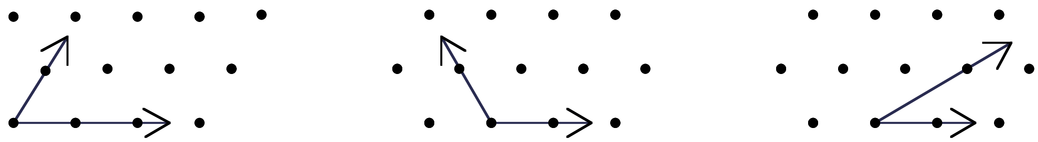
6. Find the measures of $\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$ if lines l, m, and n are parallel. Is this picture drawn to scale?



7. Saima is attempting a tessellation with regular pentagons. She ends up with gaps as shown in this figure. What kind of quadrilateral is made by a gap? What are the angle measurements of that quadrilateral? Be prepared to justify your answer without using a protractor.



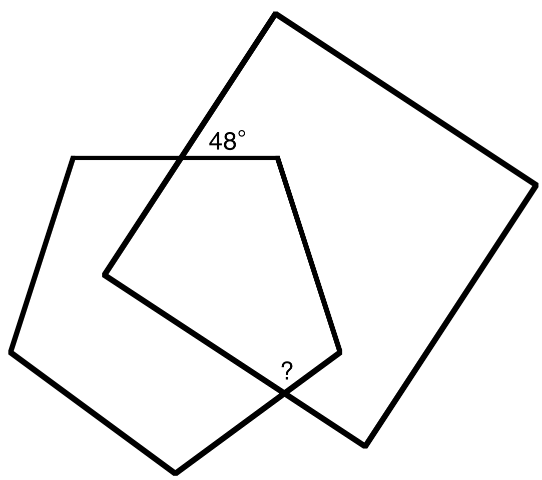
8. Find the measure of the following angles drawn on isometric grids without using your protractor. What are the properties of the isometric grid that allow you to make your conclusions?



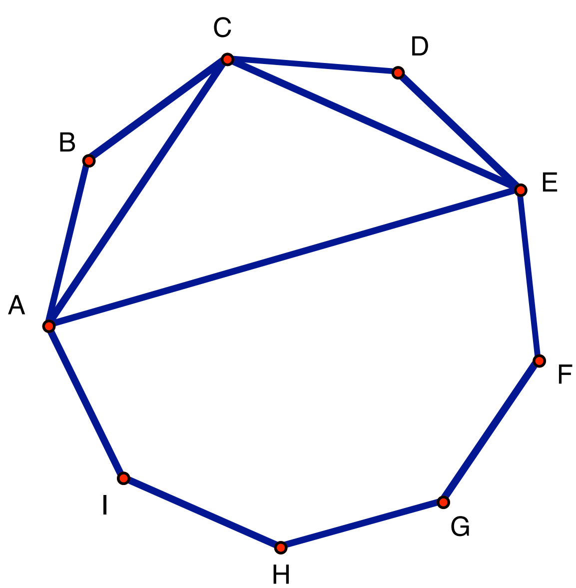
9. Sometimes one must deal with very small angles. What angle does the hour hand sweep out in one second of time.

10. When it is exactly twelve o'clock, the angle between the hour hand and the minute hand is 0 degrees? Because the minute hand passes the hour hand once every hour there will be eleven more times when the hands coincide. When? Please note that the hands are NOT coincident at 1:05 because at that time the hour hand has already passed the 1:00 mark.

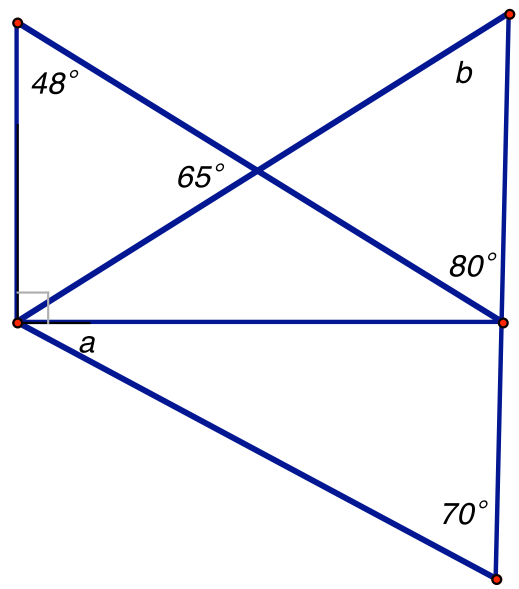
11. A square and a regular pentagon are shown. Find the measure of the indicated angle. Is the sketch drawn to scale? That is, can you find the answer by measuring?



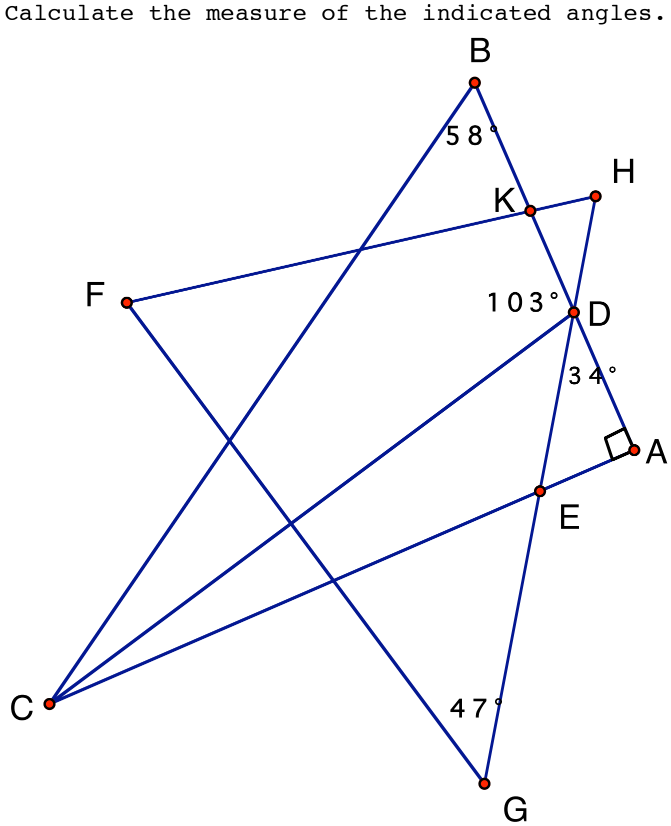
12. A regular 9-gon is shown. Is the triangle ACE obtuse, acute, or right? Is the sketch drawn to scale? Can you find the answer by measuring?



13. Find the measure of angles a and b. Show your work. Is the drawing a scale drawing?



14. Calculate the measure of the indicated angles. Is the diagram drawn to scale?



Given that
$\angle F \cong \angle H$

Find the measure of these angles.

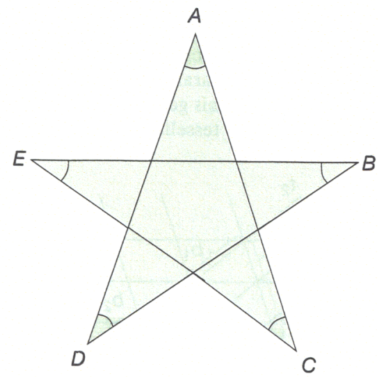
$\angle DCA$

$\angle ECD$

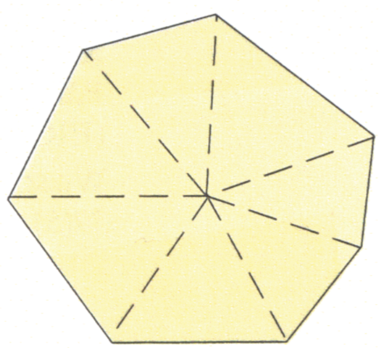
$\angle F$

$\angle BKF$

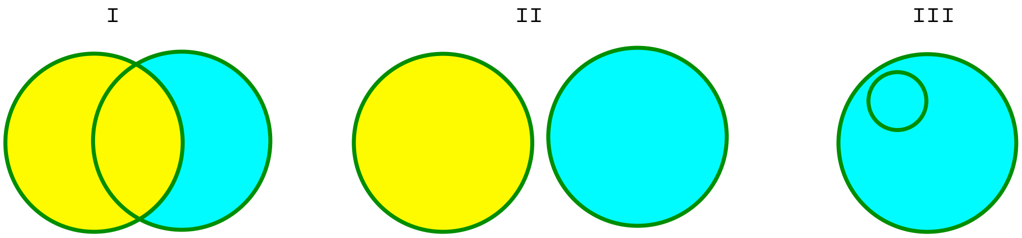
15. In the five-pointed star, what are the measures at the angles at points A, B, C, D, and E? Assume that the pentagon in the center is a regular pentagon.



16. A ninth grade student drew this figure to investigate the sum of the angles in a convex 7-gon. He said that, because there are seven triangles in the picture, the sum of its vertex angles equal to 7 times 180° = 1260°. Where has this student gone wrong? *Working from his picture* show how to find the correct answer.



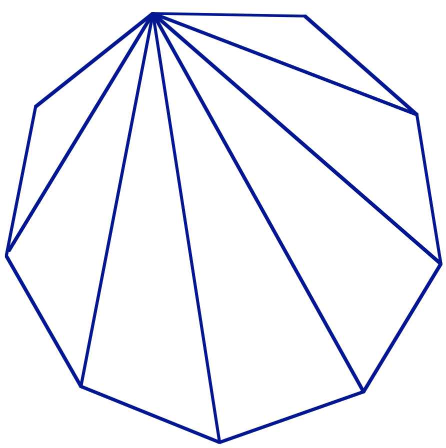
17. If one circle represents the set of rectangles and the other circle represents the set of parallelograms, which of the diagrams best represents the relationship between rectangles and parallelograms?



a) If one circle represents the set of rhombuses and the other circle represents the set of rectangles, which of the diagrams best represents the relationship between rhombuses and rectangles?

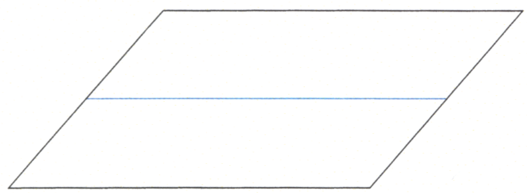
b) Find a different relationship to illustrate the remaining diagram.

18. Find the measures of all angles of this regular nonagon when the diagonals from one vertex are included in the diagram.

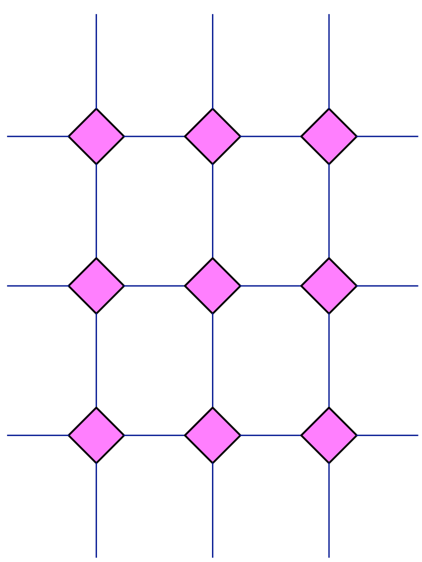


19. These conversations were overheard in the classroom, in stores, and in other places where people discuss mathematics. Comment on the reasoning expressed or implied.

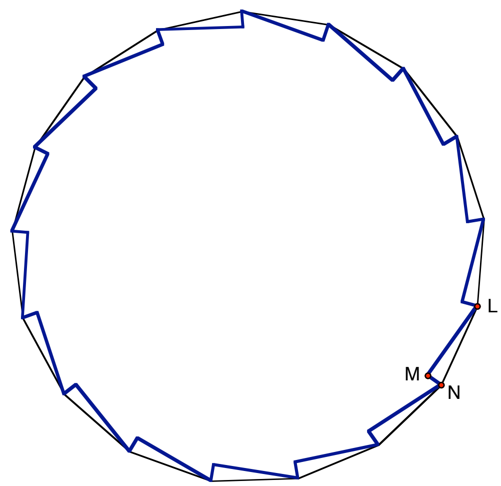
a) Gail draws a horizontal line through a parallelogram and says, "If I cut along this line, the two pieces fit on top of each other. So my line must be a line of symmetry!"



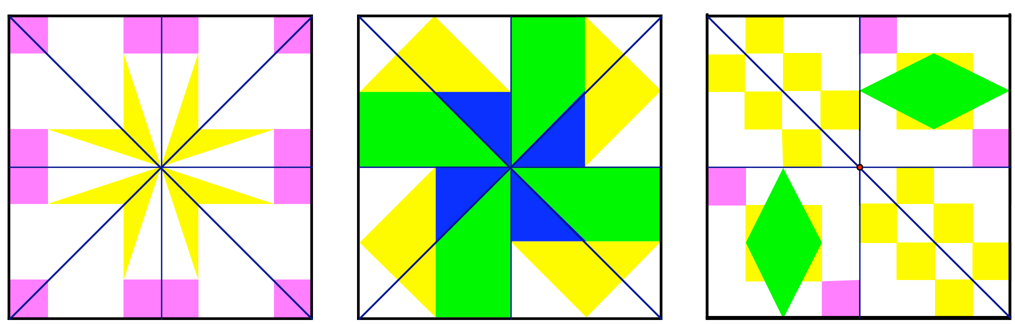
b) Fred chose the floor tiling pattern shown below using squares and octagons. After studying it for a few minutes, he decided that each angle of the octagon measures 135°, even it is not a regular octagon.



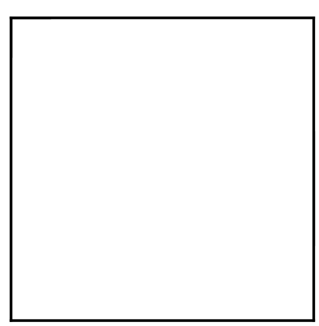
20. A saw blade is made by cutting 17 congruent right triangles out of a regular 17-gon as shown below. If the angle M is the right angle, what is the measure of the angle made by the sharp points of the blade? The line segment MN is coincident with a radius of the 17-gon.



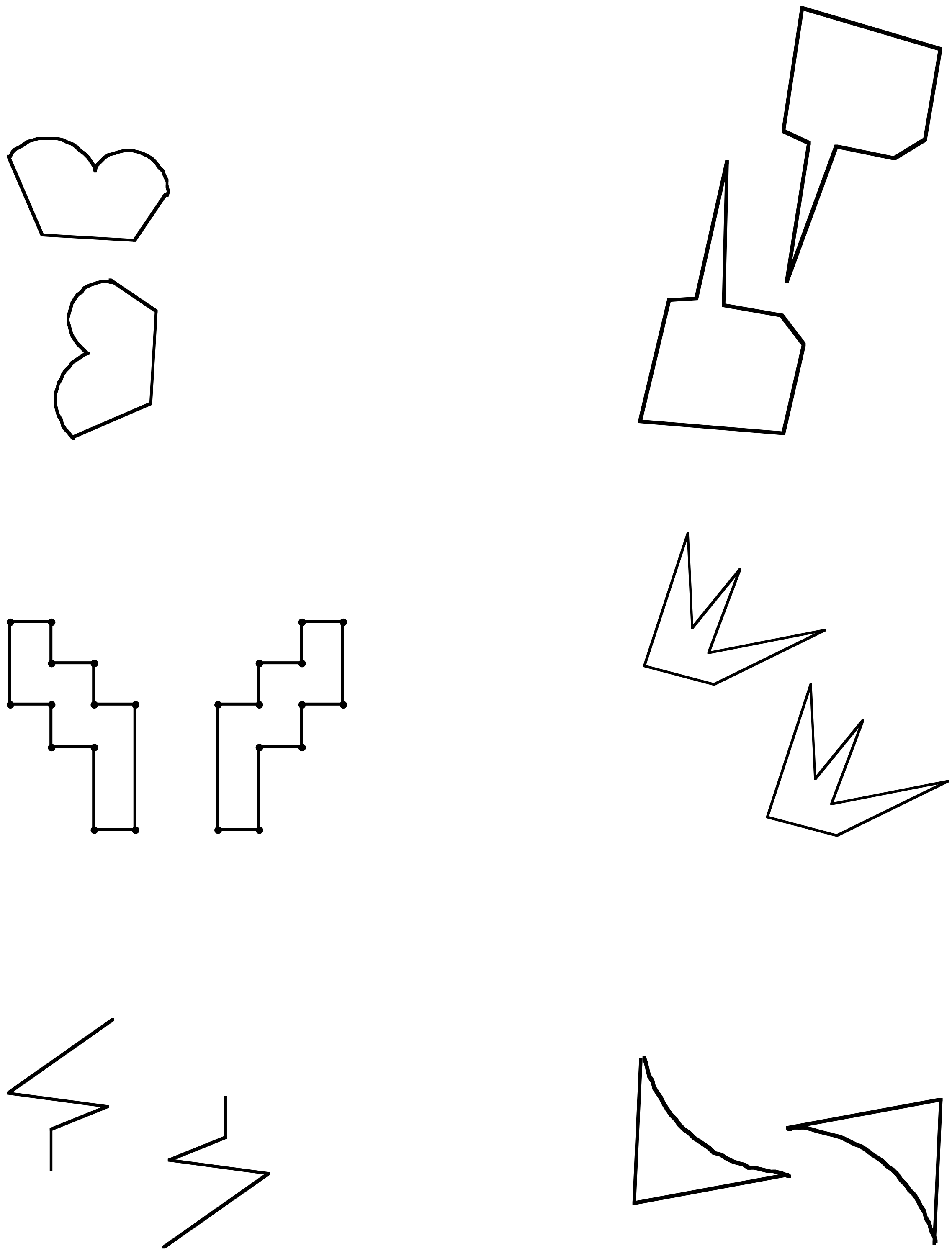
21. Identify the types of symmetry present in the following patterns.



Create a pattern in this square that has symmetry through a vertical line but no other symmetries.



22. Determine the isometry that maps the shape on the left onto the shape on the right. All but one can be done with one simple isometry (rotation, translation or reflection). Which one? In each case be ready to demonstrate the transformation using tracing paper. Include the center of rotation, line of reflection, arrow of translation as needed.



Scale Drawings,

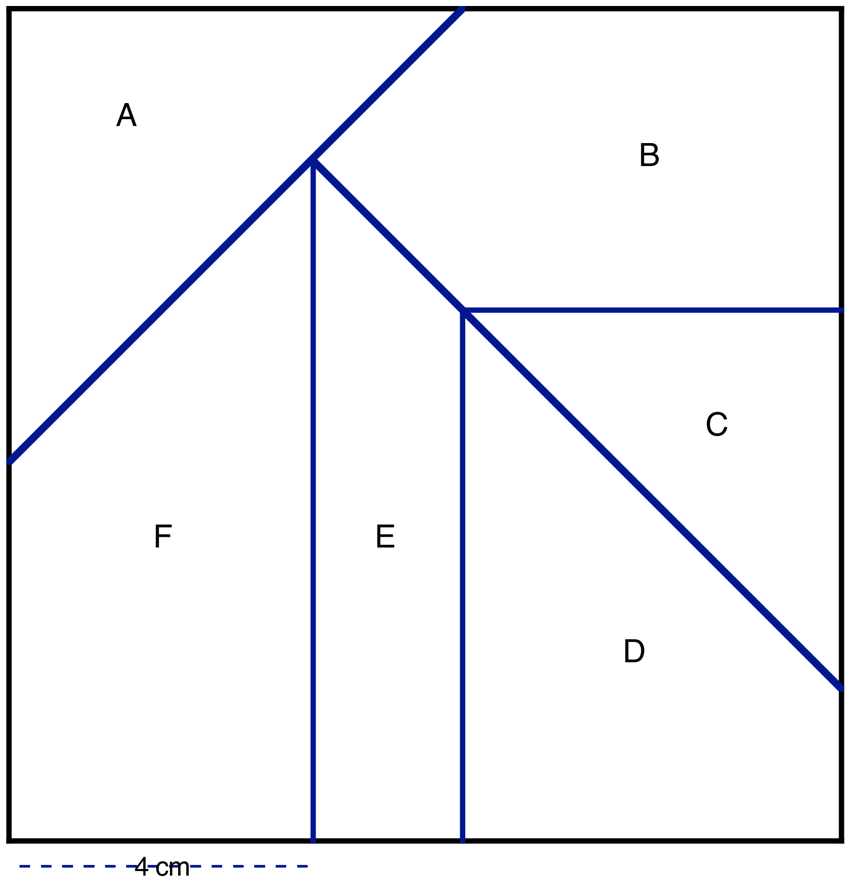
Proportional Reasoning

And

Similar Figures

Group Project: A Puzzle from Didactique

1. **Assignment:** Make an enlargement of this puzzle such that the length, that now measures 4 cm, will be 7 cm long.



Each person in your group should be assigned at least one piece to enlarge. Do not work on this together, but bring your piece, or pieces, back to class tomorrow. Use any tools. When your group has all the pieces together, assemble the larger puzzle.

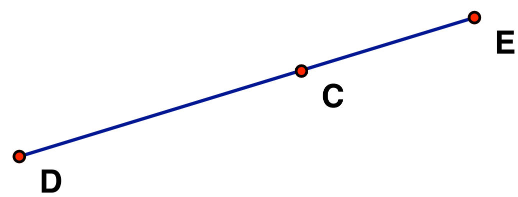
2. Write down your method here:

3. If the puzzle does not fit together, explain what went wrong.

Activity: Centimeters vs. inches

1. Measure the line segment $\overline{AB}$ using inches. Write your answer to the nearest $\frac{1}{16}$ of an inch and to the nearest tenth of an inch.

2. Measure the line segments $\overline{DE}$ and $\overline{DC}$ in centimeters then compute the length of $\overline{CE}$. Measure to the nearest millimeter. Confirm your answer by measuring the length of $\overline{CE}$.



3. Complete the second column of the table by using your ruler to draw line segments using inches and then measuring the segments in centimeters.

Find out the exact measurement in centimeters of 1 inch, and then fill in the third column

inches	centimeters measured	centimeters exact
1 in.		
1.5 in.		
2 in.		
2.5 in.		
3 in.		

4. What patterns do you see in the table?

5. Show how to use the meaning of multiplication to find the length of a 5-inch line segment in centimeters.

6. My height is 5’4” what is my height in centimeters?

7. A meter stick is 100 centimeters long. Use division to find its measure in inches?

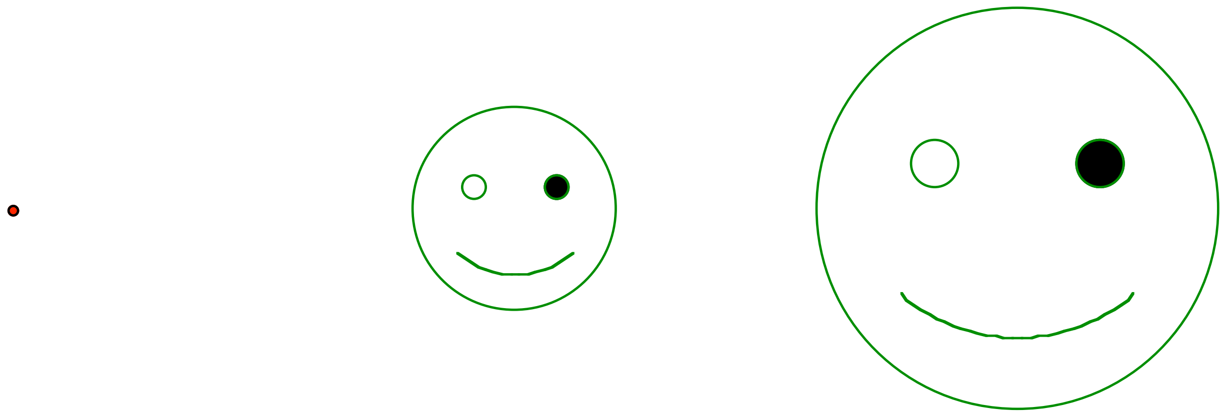
Use proportions to find the measurements in the next two problems:

9. Compute the measure in centimeters of the line segments in 1. Confirm by measuring.

10. Compute the measure in inches of the line segments in 2. Confirm by measuring.

Lesson 1 – Size Transformations

*



A scaling transformation changes the size of an object but preserves the shape of the object.. Notice that a line drawn through the point on the left through the small face ends up at the corresponding point on the large face. In fact for any such line the distance from the point to the large face is twice the distance from the point to the small face. Try this for several lines.

Also notice that the distance between any two points in the large face – say the distance between the center of the eyes – is twice that of the corresponding distance on the small face.

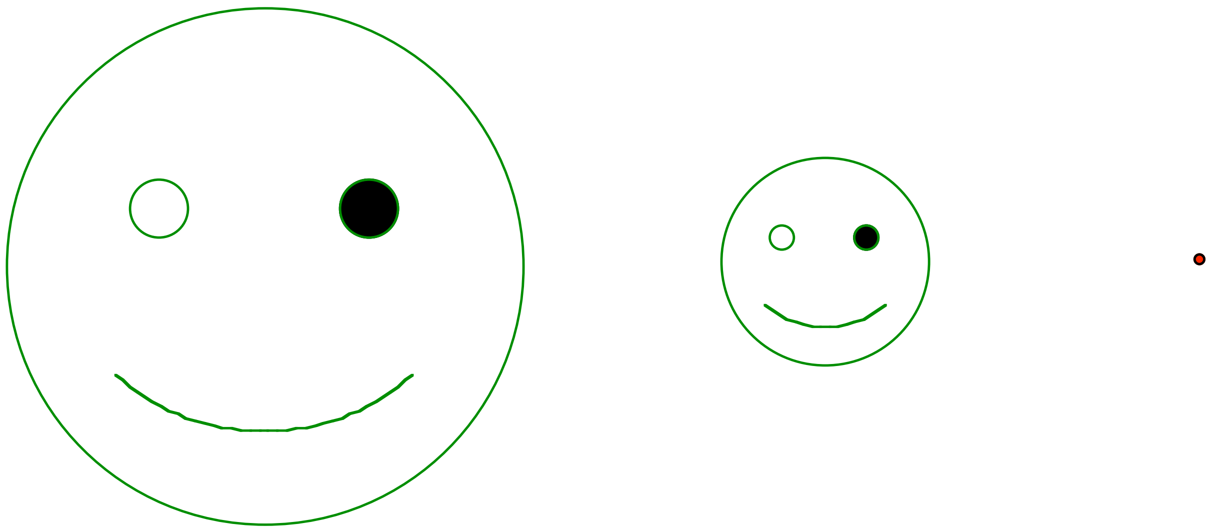
The small face has been scaled to twice its size.

One way to construct a figure that is twice the size is to use rubberbands. Link two rubberbands of equal size. Fasten one end of the double rubberband to the point on the left. Fix a pencil to the other end. Move the pencil to draw the enlarged smiley while tracing the original figure with the knot that links the two rubberbands.

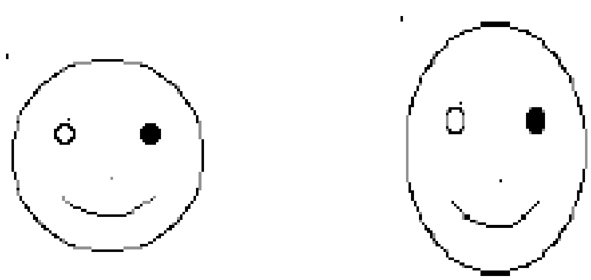
Iif you are left handed, turn the page upside down

A size (or scaling) transformation
enlarges (or shrinks) a figure by a factor of k and preserves all angle measurements.

Example 1: The large face on the left has been shrunk by a factor of 2.5 (so $k = \frac{1}{2.5} = 0.4$).



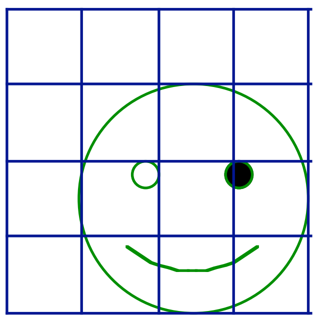
Example 2: This is <u>not</u> a size transformation:

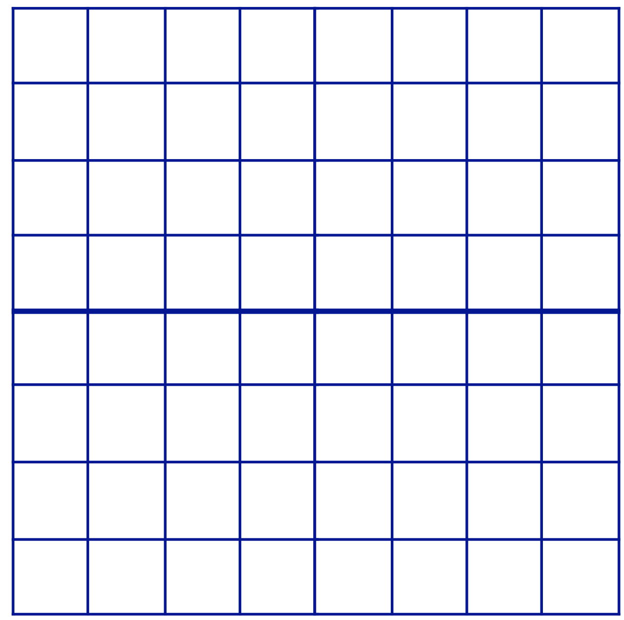


The face has been stretched vertically but not horizontally. It did not retain its shape. The circles are no longer circles and the smile is exaggerated.

Another way to think about size transformations is to think about the object on a grid.

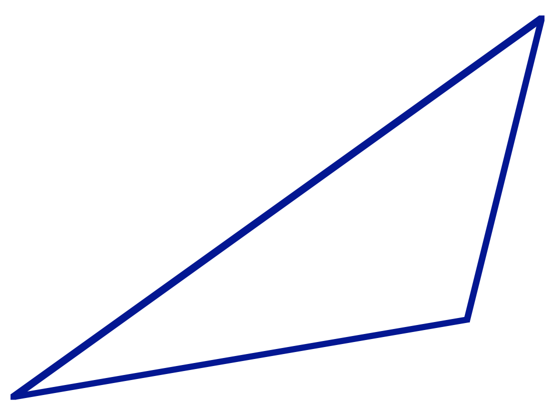
Enlarge the smiley face by drawing it into a grid that is twice as large



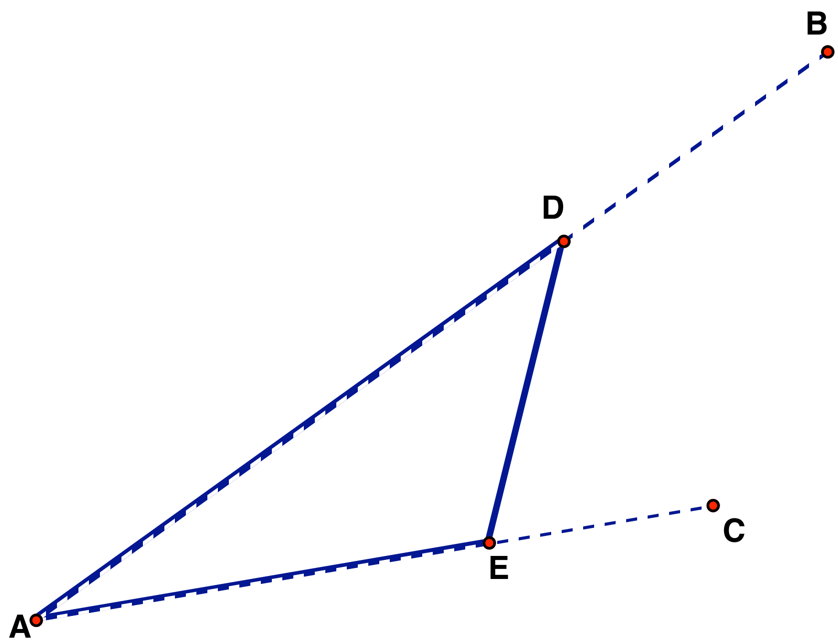


Activity: Practice with Size Transformations

Use both methods to make various sizes of this triangle.



Here is another way to enlarge a triangle. In this example we enlarge the triangle by a factor of 1.5. Pick one of the vertices and extend the two sides until the required length.



Verify by measuring that **AB** is 1.5 times **AD** and that AC is 1.5 times **AE**

Connect **B** and **C** and verify that **BC** is 1.5 times **DE**.

Try making several more enlargements of the triangle using this method. Compare with your other examples.

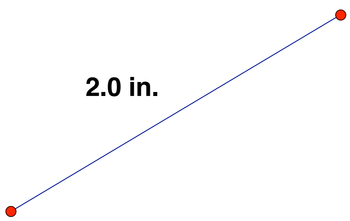
All of these triangles are said to be similar to each other: sides are in proportion and all the angles are the same.

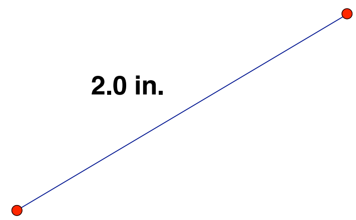
Making scale drawings

1. Anya has a garden and she wants to make a scale drawing of it. The garden is in the shape of a triangle. She measures each side. Her garden is 20 feet, 15 feet, 10 feet.

She draws a line segment 2 inches long to represent the side that is 20 feet long.
Finish the scale drawing for her.

Two triangles have the same shape when corresponding sides are in proportion.





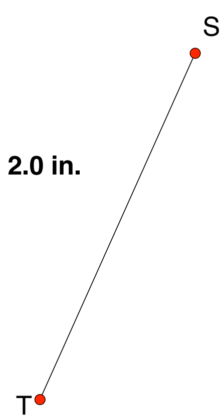
This drawing does not satisfy Anya it is too small.
Suppose she wants the 2.0 in. segment to represent the 10-foot side of the garden.
Finish this scale drawing for her.

What is the prorportion between the first and the second scale drawing?

2. Anya has another garden and she wants to make a scale drawing of it. The garden is in the shape of a triangle. She measures each angle. The three angles measure 40°, 60°, 80°.

She draws a line segment 2 inches long. The 40°-angle is at point S and the 60°-angle is at point T.
Finish the scale drawing for her.

Two triangles have the same shape when corresponding angles are equal.

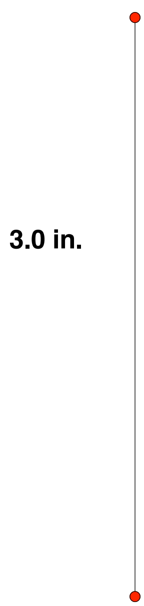


If the side of the garden represented by $\overline{ST}$ is actually 20 feet long, what are the lengths of the other sides of the garden?

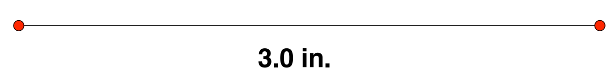
Rectangle problems

3. Anya has a garden and she wants to make a scale drawing of it. The garden is in the shape of a rectangle. She measures each side. Her garden is 20 feet x 30 feet.

She draws a line segment 3 inches long. Finish the drawing for her.

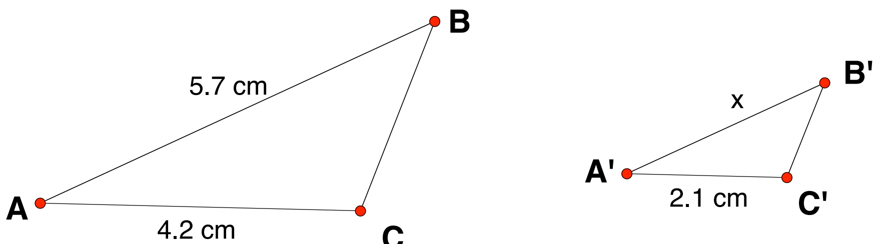


4. Suppose the 3.0 inch line is suppose to represent the short side of the garden. Finish the scale drawing for her.



Using similarity to solve problems

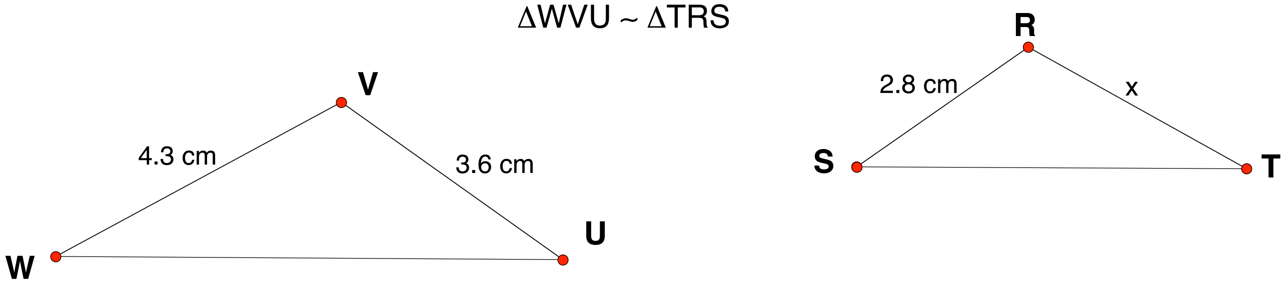
1. Find x. The drawing is exact to 0.1 cm so you may check your answer by measuring.



$\Delta ABC \sim \Delta A'B'C'$

2. Find x. The drawing is exact to 0.1 cm so you may check your answer by measuring.

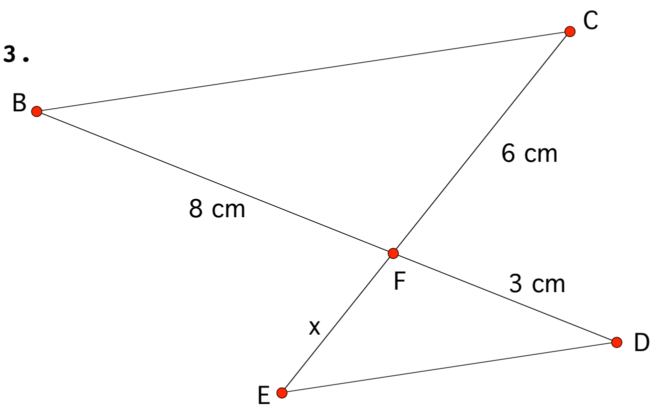
$\Delta WVU \sim \Delta TRS$



Using similarity to solve problems

In the next two problems you must find the missing length, x. Find x by first identifying similar triangles. Then write and solve proportions.

3.



$\overline{BC} \parallel \overline{ED}$

$m(\overline{CF}) = 6$ cm
$m(\overline{BF}) = 8$ cm
$m(\overline{DF}) = 3$ cm

What is the $m(\overline{EF})$?

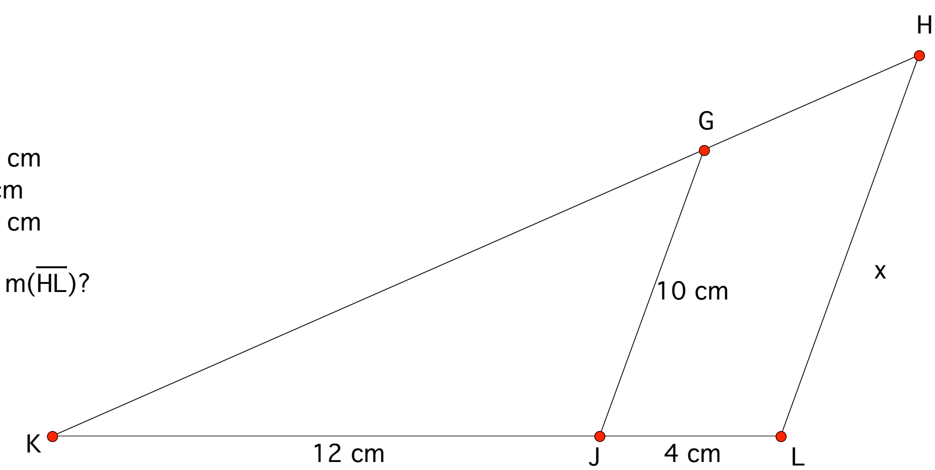
Complete this statement:

ΔBFC ~ Δ_______ because . . .

$\overline{GJ} \parallel \overline{HL}$

4. $m(\overline{GJ}) = 10$ cm
$m(\overline{JL}) = 4$ cm
$m(\overline{KJ}) = 12$ cm

What is the $m(\overline{HL})$?



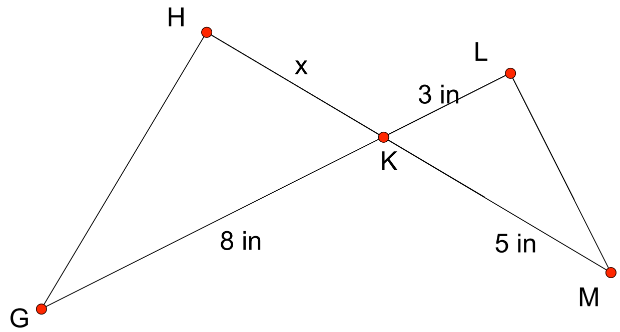
Complete this statement:

ΔKJG ~ Δ_______ because . . .

5.

m∠GHK = 90°

m∠MLK = 90°

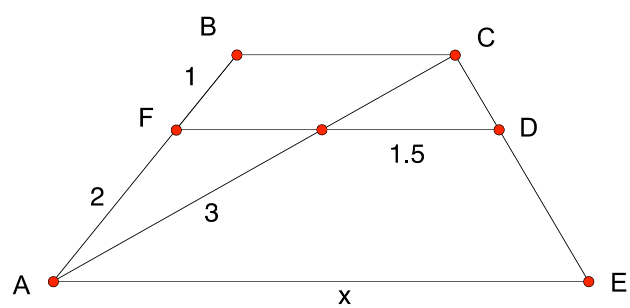


6. ABCE is a trapezoid.

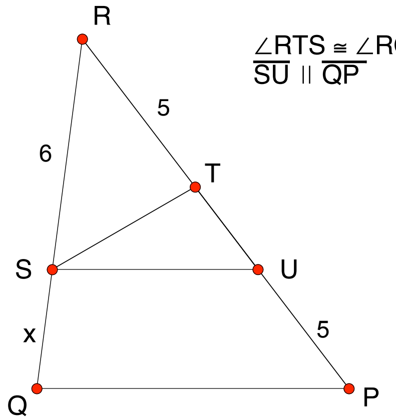
$\overline{FD} \parallel \overline{BC}$

$\overline{AC}$ is a diagonal of ABCE

Find x.



7. Identify three triangles that are similar to each other and write proportionality statements to find x.

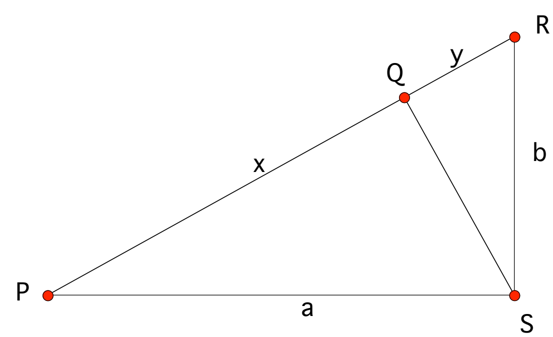


$\angle RTS \cong \angle RQP$
$\overline{SU} \parallel \overline{QP}$

8. Find three similar right triangles in this picture. Write down 2 different proportional statements involving the variables, a, b, c, x, y.

$\overline{QS} \perp \overline{PR}$
$\overline{PS} \perp \overline{RS}$

$m(\overline{PR}) = c$ (not shown in picture. It is the same as x+y)
$m(\overline{PQ}) = x$
$m(\overline{QR}) = y$
$m(\overline{PS}) = a$
$m(\overline{RS}) = b$



9. Answer the following questions **TRUE** or **FALSE**. If true give a reason why; if false give a counterexample.

TRUE or FALSE Any two equilateral traingles are similar.

TRUE or FALSE Any two equilateral traingles are congruent.

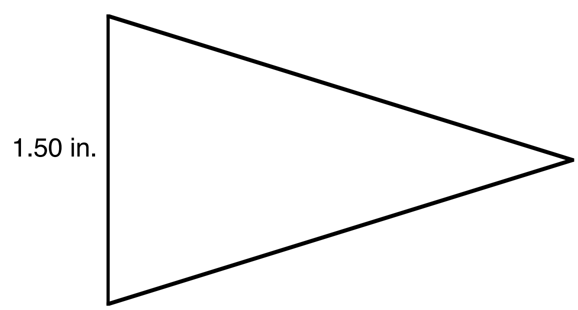
TRUE or FALSE Any two right traingles are similar.

TRUE or FALSE Any two rectangles are similar.

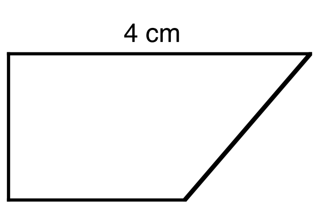
TRUE or FALSE Any two regular hexagons are similar.

TRUE or FALSE Any two hexagons are similar.

10. Construct a triangle that is similar to this isoceles triangle but the base is 1 inch long.



11. Construct a trapezoid that is similar to this trapezoid but the length that is 4 cm. is now 7 cm.

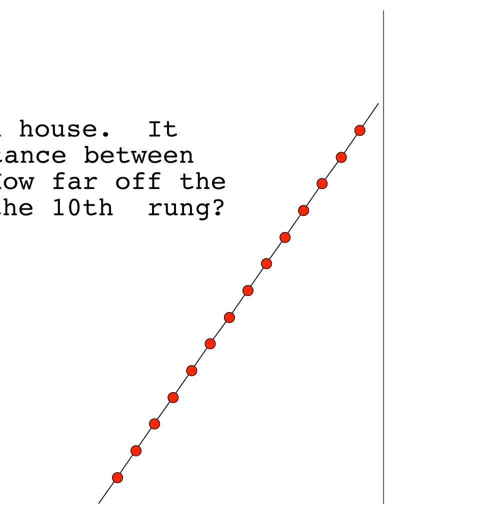


More problems with using similar triangles

Draw (or complete) a picture for each problem.

12. At 4:00 pm the shadow of a tree measures 5 feet 7 inches and the shadow of a 6'2" person measures 28 inches. How tall is the tree?

13. A 20 foot ladder is leaning against a house. It reaches a height of 17 feet. If the distance between each rung of the ladder is 16 inches. How far off the ground are you when you are standing on the 10th rung?



Dimensional Analysis

Cooks, carpenters, engineers , chemists, seamstresses and nurses, all need to be able to change from one unit of measurement to another in their work. In this section, we discuss a method of converting units called dimensional analysis and see examples of converting different units of length, area, volume, time, and speed.

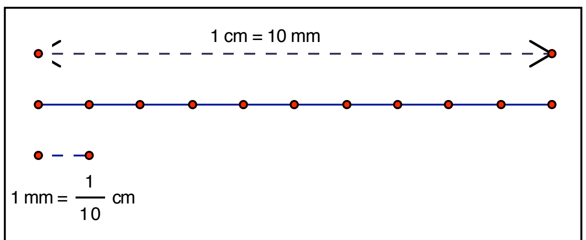
If you want to change the unit of measure, you can sometimes use the **meaning of multiplication**. For example, we know that 36 inches = 1 yard since 12 inches = 1 foot and 3 feet = 1 yard. *We have* 3 *groups (feet) with* 12 *objects (inches) in each group, so*

$$3 \text{ feet X } 12 \text{ inches in each foot} = 36 \text{ inches}$$

We often write $\frac{\text{inches}}{\text{foot}}$ instead of "inches per foot". With this notation, it looks like the feet and foot cancel each other as if they were like numbers in a fraction problem.

$$3 \text{ feet X } 12\frac{\text{inches}}{\text{foot}} = 36 \text{ inches}$$

However, you cannot always use the definition of multiplication. Sometimes it is necessary to use the **meaning of division**. For example, to express 240 mm as some number of cm, we figure out how many groups of 10 mm there are in 240 mm.



picture is enlarged from actual size

Here each group is 1 cm that consists of 10 mm. Writing this as a division problem looks like

$$240\,\text{mm} \div 10\frac{\text{mm}}{\text{cm}}.$$

This can look like a multiplication problem, when we remember that division by 10 is the same as multiplying by $\frac{1}{10}$, so

$$240\,\text{mm} \div 10\frac{\text{mm}}{\text{cm}} = 240\,\text{mm} \times \frac{1\text{ cm}}{10\,\text{mm}} = 24\,\text{cm}.$$

Normally, this kind of problem would just be done in your head. Here is another example. How many feet in 30 inches? Think of this as a "how many groups?" division problem, where each group is one foot – a group of 12 inches. How many groups of 12 inches are there in 30 inches? The answer is 30 divided by 12.

$$30 \text{ inches} \div 12\frac{\text{inches}}{\text{foot}} = 1\frac{1}{2} \text{ feet}$$

by writing this expression as multiplication by the reciprocal of 12, we again see how the units (this time the inches) appear to "cancel out"

One method for changing any unit measurement to another, which takes advantage of this apparent canceling of units, is called **dimensional analysis.** Dimensional analysis uses the idea of a *unit ratio*.

> **Definition:** A *unit ratio* is a fraction that has the value of 1 if both the numerator and denominator are expressed in the same units.

For example, $\frac{1\text{ foot}}{12\text{ inches}} = \frac{1\text{ foot}}{1\text{ foot}} = 1$, so $\frac{1\text{ foot}}{12\text{ inches}}$ is a unit ratio.

You know that a fraction has many names. For example, $\frac{1}{2} = \frac{2}{4} = \frac{20}{40}$. One way to find different names for a fraction is to multiply one form by another fraction representing the value of 1.

$$\frac{1}{2}\cdot\frac{2}{2} = \frac{2}{4} \qquad \text{or} \qquad \frac{1}{2}\cdot\frac{10}{10} = \frac{20}{40}$$

Dimensional analysis tells us to multiply any measurement by a unit ratio, using the given unit of measure to guide our choice of unit ratio, to express the measurement in an equivalent form.

For example, suppose that you want to change

$$180\text{ inches to } \_\_\_\_ \text{ feet.}$$

We know that $12\text{ inches} = 1\text{ foot}$. Both $\frac{12\text{ inches}}{1\text{ foot}}$ and $\frac{1\text{ foot}}{12\text{ inches}}$ are unit ratios, because numerator and denominator of each represent the same length. However, if we multiply: $180\text{ inches} \times \frac{12\text{ inches}}{\text{foot}}$, we get a more complicated, not meaningful expression. However, if, using the meaning of multiplication, we multiply by $\frac{1\text{ foot}}{12\text{ inches}}$, the unit "inches" cancel, and we have

$$180\text{ inches X } \frac{1\text{ foot}}{12\text{ inches}} = 15\text{ feet}$$

Since the commutative property applies to the operation of multiplication, dimensional analysis can be done in different orders and even in several steps. In each of these examples, notice how the units appear to "cancel."

Example 1: $156.92\text{ cm} = \_\_\_\_\text{ m}$

$$156.92\text{ cm} \times \frac{1\text{ m}}{100\text{ cm}} = 1.5692\text{ m}$$

Example 2: $3467\text{ inches} = \_\_\_\_\_\_\text{ yds}$

$$3762 \text{ inches} \times \frac{1 \text{ foot}}{12 \text{ inches}} = 313.5 \text{ feet}$$ and then

$$313.5 \text{ feet} \times \frac{1 \text{ yard}}{3 \text{ feet}} = 104.5 \text{ yards}.$$

Another way to do this is in one long string

$$3762 \text{ inches} \times \frac{1 \text{ foot}}{12 \text{ inches}} \times \frac{1 \text{ yard}}{3 \text{ feet}} = 3462 \div 12 \div 3 \text{ yards} = 104.5 \text{ yards}$$

Example 3: 54 ft^2 = ____ yd^2

We do not need new facts when using dimensional analysis to convert area measurements. One ft^2 is the area of a square that is 1 foot by 1 foot. As a unit of area we short cut and think of ft^2 as ft(ft) just as we can think of x^2 as x(x). Here are three methods to convert ft^2 to yd^2.

Method 1: $54 \text{ ft}^2 \times \left(\frac{1 \text{ yd}}{3 \text{ ft}}\right) \times \left(\frac{1 \text{ yd}}{3 \text{ ft}}\right) = 6 \text{ yd}^2$

Method 2: $54 \text{ ft}^2 \times \left(\frac{1 \text{ yd}}{3 \text{ ft}}\right)^2 = 6 \text{ yd}^2$

A third method, uses the fact that 1 yd^2 is the same area as 9 ft^2

Method 3: $54 \text{ ft}^2 \times \frac{1 \text{ yd}^2}{9 \text{ ft}^2} = 6 \text{ yd}^2$

1. Assignment: Draw a picture and make a convincing argument that shows that 1 yd^2 is 9 ft^2.

Example 4: $16 \text{ m} =$ ______ ft

In order to solve example 4, we must know another conversion fact. It is not enough to know some relationships among the metric measurements and to know some relationships among the English measurements. We must also know at least one relationship between a metric measurement and an English measurement. A convenient conversion to use is 1 inch = 2.54 cm.

Method 1:

$$16 \text{ m} \times \left(\frac{100 \text{ cm}}{1 \text{ m}}\right) \times \left(\frac{1 \text{ in}}{2.54 \text{ cm}}\right) \times \left(\frac{1 \text{ ft}}{12 \text{ in}}\right) \approx 52.4934 \text{ ft}$$

(to 4 decimal place accuracy)

Method 2: Some people remember that 1 foot = 30.48 cm.

$$16\ m \times \left(\frac{100\ cm}{1\ m}\right) \times \left(\frac{1\ ft}{30.48\ cm}\right) \approx 52.4934\ ft$$

The following example shows how to use dimensional analysis to convert speeds, by using unit fraction s for both distance and time units.

Example 4: 16 $\frac{\text{feet}}{\text{sec}}$ = ______ $\frac{\text{meters}}{\text{min}}$

$$16\frac{\text{feet}}{\text{sec}} \times \frac{12\,\text{inches}}{1\,\text{foot}} \times \frac{1\,\text{meter}}{39.37\,\text{inches}} \times \frac{60\,\text{sec}}{1\,\text{min}} = 292.61\frac{\text{meters}}{\text{min}}$$

2. Use dimensional analysis, to convert the following units. You may have to do some research to find appropriate unit ratios if you don't know ones that will do the problem.

17 yards = ______ cm

25 miles = ______ kilometers

12 ft^2 = ______ in^2

107 cm^3 = ______ ft^3

37 weeks = ______ hours

60 $\frac{\text{miles}}{\text{hour}}$ = ______ $\frac{\text{feet}}{\text{second}}$

24 pints = ______ gallons

17 tablespoons = _______ cups

1 year = _______ minutes

45 ft^2 = ______ yd^2

3. Make a list of convenient unit ratios that you might want to commit to memory.

The unit ratios we've used so far are given to us. If we didn't know how many centimeters in one inch, we could look it up in a book or on the internet. In the following problem the unit ratios are derived from the information given in the problem.

4. There are about 1 billion people in China. If they lined up four to a row and marched past you at the rate of 25 rows per minute, how long would it take the parade to pass you:

$$10^9 \text{ people} \times \frac{1 \text{ row}}{4 \text{ people}} \times \frac{1 \text{ minute}}{25 \text{ rows}} = 10^7 \text{ minutes}$$

5. Explain why $\frac{1 \text{ row}}{4 \text{ people}}$ is a unit ratio.

Because it will take so long for all the Chinese people to march past, minutes is not the best unit to use. We can use dimensional analysis to convert minutes to years:

$$10^7 \text{ minutes} \times \frac{1 \text{ hour}}{60 \text{ minutes}} \times \frac{1 \text{ day}}{24 \text{ hours}} \times \frac{1 \text{ year}}{365 \text{ days}} \approx 19 \text{ years}$$

6. Check this calculation with your own calculator and give the answer rounded to 3 decimal points. Do the calculation without retyping any numbers into your calculator.

7. Challenge: Give the answer to this example in years, days and hours instead of using a decimal fraction of a year.

More problems

8. Which is faster – a car traveling at 60 mph or a cheetah running at 25 meters per second?

9. Will Pete have enough paint to cover one wall of his room if he buys a can of paint that covers 30 sq ft and his wall has an area of 8 sq yd?

10. If gasoline in "A-land" costs 72.9 A¢ per liter, is it more expensive than gasoline in Chicago? Note that 1 A¢ = .79 US¢ and 1 gal = 3.7854 L

11. Prior to conversion to a decimal monetary system, the United Kingdom used the following coins.

1 pound = 20 shillings
1 shilling = 12 pence (NOTE: pence is the plural of penny)
1 penny = 2 half-pennies = 4 farthings

a) How many pence were there in a pound?

b) How many half-pennies were in a pound?

c) How many farthings were equal to a shilling?

12. A restaurant chain has sold over 80 billion hamburgers. A hamburger is about one-half inch thick. If the moon is 240,000 miles away, what percent of the distance to the moon is the height of a stack of 80 billion hamburgers?

13. A teacher and her students established the following system of measurements for the Land of Names.

1 jack = 24 jills
1 james = 8 jacks
1 jennifer = 60 james
1 jessica = 12 jennifers

Complete the following table.

	Jill(s)	Jack(s)	James(s)	Jennifer(s)	Jessica(s)
Jill =	1				
Jack =		1			
James =			1		
Jennifer=				1	
Jessica=					1

Unit #2 Measurement

INTRODUCTION

Measurements are numbers that we attach to objects. There are many reasons for doing this and many different systems for measuring. We have already looked at measurement of angles. In this unit we study measurements of length and area. In Unit #3 we will study volume. In other mathematics or science courses you may have studied speed, mass and weight.

For a first step, consider these words that John Quincy Adams, the 6th president of the United States, wrote in a report to congress in 1821, when he was Secretary of State.

> "Weights and measures may be ranked among the necessaries of life to every individual of human society. They enter into the economical arrangements and daily concerns of every family. They are necessary to every occupation of human industry; to the distribution and security of every species of property; to every transaction of trade and commerce; to the labors of the husbandman; to the ingenuity of the artificer; to the studies of the philosopher; to the researches of the antiquarian; to the navigation of the mariner, and the marches of the soldier; to all the exchanges of peace, and all the operations of war. The knowledge of them, as in established use, is among the first elements of education, and is often learned by those who learn nothing else, not even to read and write. This knowledge is riveted in the memory by the habitual application of it to the employments of men throughout life."
>
> JOHN QUINCY ADAMS - Report to the Congress, 1821

Assignment: Do you think that it is still true that some people learn about measurements but cannot read or write? Do you think that learning about measurements is more important than learning to read?

John Quincy Adams understood the importance of the study of measurements in 1821, and today the importance is recognized by the National Council of Teachers of Mathematics in the 2002 Standards for School Mathematics. In fact Measurement is one of the five Content Standards listed in that document. The standard is stated in this way:

> Instructional programs from prekindergarten through grade 12 should enable all students to--
>
> • understand measurable attributes of objects and the units, systems, and processes of measurement;
> • apply appropriate techniques, tools, and formulas to determine measurements.
>
> The study of measurement is crucial in the pre-K–12 mathematics curriculum because of its practicality and pervasiveness in so many aspects of everyday life. The study of measurement also provides an opportunity for learning about other areas of mathematics, such as number operations, geometric ideas, statistical concepts, and notions of function.
>
> Measurement is the assignment of a numerical value to an attribute of an object. In the earliest grades, students can compare and order objects using language such as longer and shorter. As they progress through the grades, students' collection of measurable attributes, their understanding of the relationships among attributes, and their understanding of precision in measurement should expand. By high school, for example, students should recognize the need to report an appropriate number of significant digits when computing with measurements.

Table of contents for Unit #2

Lesson 1 - Comparing Areas and Perimeters: Understanding the Importance of Units.

The study of geometry originated in problems of determining measurements of land. In fact the word, geometry, comes from the Greek roots "geo" which means earth or land and "metron" which means a measure. The American Heritage Dictionary of the English Language (Fourth Edition. 2000) gives the etymology

> Middle English geometrie, from Old French, from Latin gemetria, from Greek gemetri, from gemetrein, to measure land: ge-, geo- + metron, measure

Over 3000 years ago, the Egyptians were among the first to approach problems of land measurement. Their methods tended to be technical and computational rather than the theoretical. For example, they could compute the area of any given triangle with sufficient accuracy but it is not clear that they understood the general formula, *1/2 base times height,* for the area of a triangle. The following quote from Herodotus, a Greek historian and traveler living in the 5th century BCE, refers to Pharaoh Sesostris, who ruled in Egypt in the 12th Century BCE.

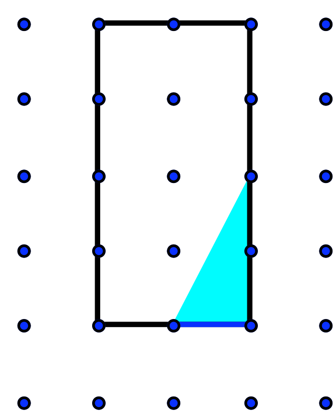
> "The king divided the land among all Egyptians so as to give each one a quadrangle of equal size and to draw from each his revenues, by imposing a tax to be levied yearly. But everyone from whose part the river tore anything away, had to go to him to notify what had happened; he then sent overseers who had to measure out how much the land had become smaller, in order that the owner might pay on what was left, in proportion to the entire tax imposed. In this way, it appears to me, geometry originated, which passed thence to Hellas."

An overseer's problem

1. ***Assignment:*** Look at the next to the last sentence in the quote from Herodotus. What is meant by the phrase: "in proportion to the entire tax imposed?"

Suppose that a quadrangle is a rectangle twice as long as it is wide. Suppose the river "tore away" a corner of someone's quadrangle, taking away a triangular region as shown on this grid.

If the tax on one quadrangle is $189, what is the tax after the river has washed away this region?

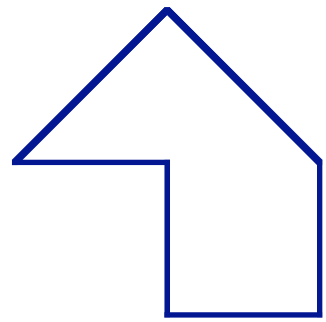


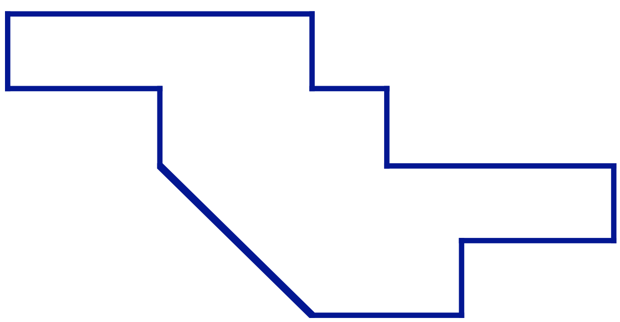
We begin our investigation of measurement with an activity that takes us back to the ancient Egyptians. In this lesson we want to develop an understanding of the important concepts of measurement without getting directly involved with formulas (that will come later) and other theoretical considerations.

Group Project: The Pharaohs Problem

Three Pharaohs have hired you to help them determine how to divide their land for their children. Each one has an accurate scale drawing of their land. In each case you must convince the Pharaoh and his children that your solution is fair.

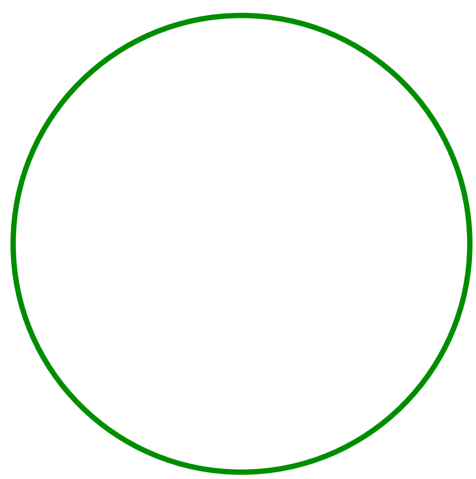
Pharaoh I must divide his land fairly between his **two** children.





Pharaoh II must divide his land fairly among his **three** children.

Pharaoh III must divide his land fairly among his **five** children.



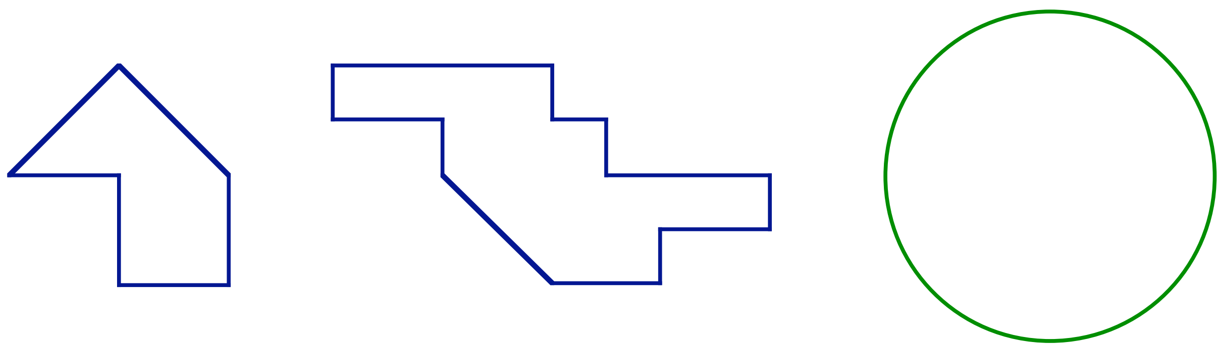
2. Challenge: Which of the children gets the most land? You may assume that each of the plots is drawn to the same scale.

Which Pharaoh has the most valuable estate may depend on many things. Measuring area seemed the best way to determine the value of the land in the previous problem. What if the situation were different? Let's consider this variation on the Pharaoh's problem:

Group Project: A Different Pharaohs Problem

Suppose that the Pharaohs' estates are in fact island-estates in the Mediterranean Sea. The amount of coastline might determine the value of the land more than the actual area of the land.

Which of the Pharaoh's have the longest coastline? NOTE: These drawings are not the same scale as the one on the previous page. That should not be important to solving this problem. Use the enlargements provided on the next three pages



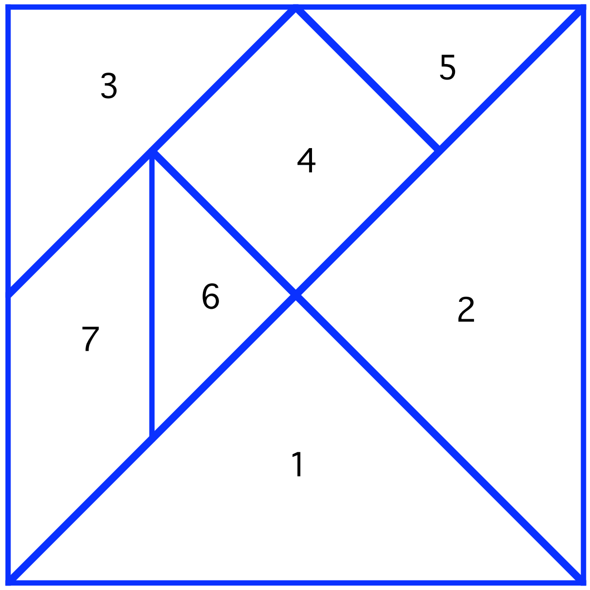
3. **Challenge problem:** Can you divide each of the Pharaoh's Island estates fairly for the right number of children in such a way that they each get the same amount of coastline as well as the same amount of land?

Area and perimeter are two concepts that are understood by their properties rather than by a definition. For example, we can measure the area of a shape if we know the area of a basic shape, like a square, and can divide that shape up into squares or fractions of squares. You may have noticed doing this when thinking about the Pharaohs' problems.

Perimeter also depends on our understanding of length. We can measure the length of a line segment by comparing the length to a ruler that has units of length, like inches or centimeters. Determining the amount of the coastline for each estate is measuring the **perimeter** of the shape. If the shape is a polygon, then the perimeter is the sum of the lengths of each side. If the shape has curved edges we might measure the perimeter by using a tape measure or a flexible ruler that can bend around the shape.

Practice with units: areas and perimeters

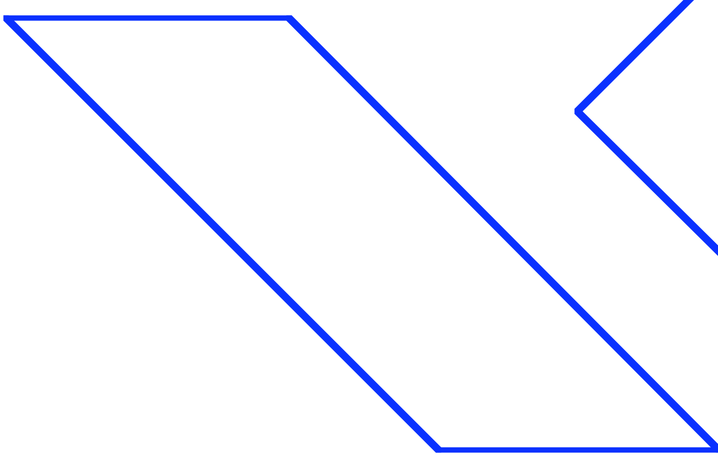
4. List the tangram pieces in increasing order by AREA. Indicate which ones have the same area. List them in increasing order by PERIMETER. Indicate which ones have the same perimeter. Do not use rulers or formulas. You may want to cut out the tangram pieces.

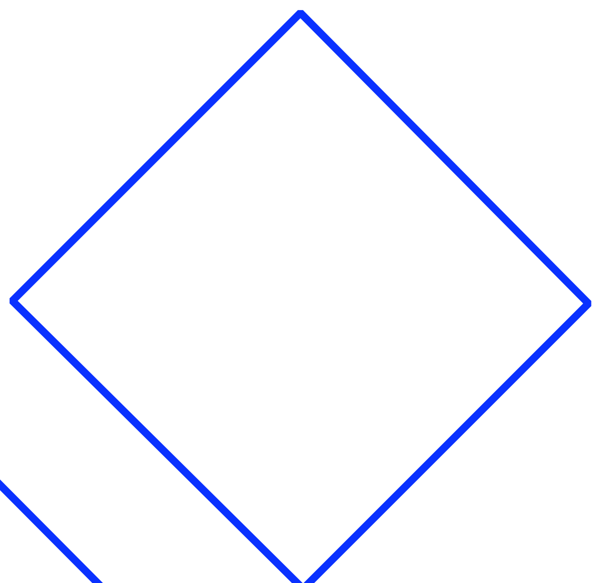


What are the differences in the two lists?

5. Suppose the small triangle (piece 5) has area 1 tangram_area_unit. What is the area of each of the other pieces in the set in tangram_area-units?

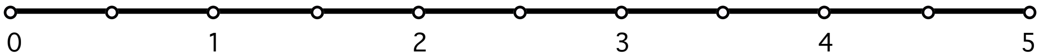
6. What is the area of each of these two shapes in tangram_area_units? Do not use rulers or formulas. Compare areas with the tangram pieces.





Area parallelogram = ____ tangram_area_units   Area square = ____ tangram_area_units

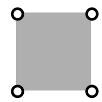
7. This is a tangram ruler. It measures in tangram_length_units. One tangram_length_unit is equal in length to the length of the side of the small square (piece 4) in the tangram set. Measure the perimeter of the parallelogram and the square with this ruler.



perimeter parallelogram = ____ tangram_length_units

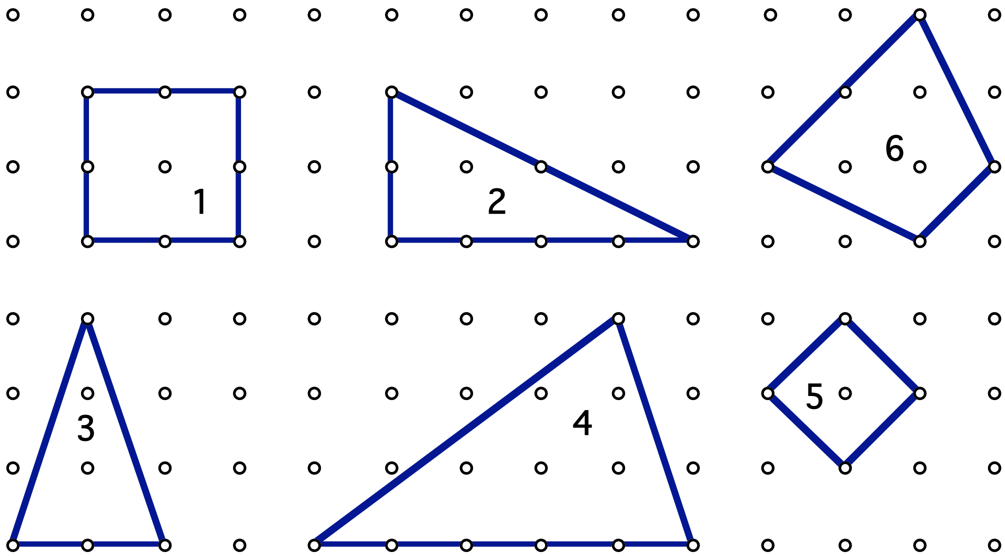
perimeter square = ____ tangram_length_units

8. The most common unit for measuring area is a square unit. For this problem, 1 square_unit is shown on the right. What is the area of each of the other polygons in square_units?

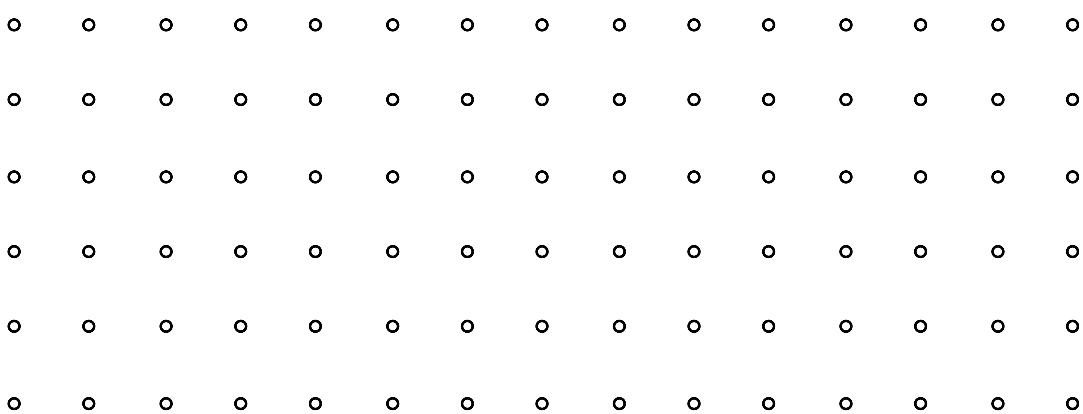


1 square_unit.

Do not use rulers or formulas except to check your work.



9. Find a square whose vertices are grid points that has twice the area of the first square, #1. Find a square that has twice the area of the other square, #5.



Standard units of length

The development of standard units of measurements has a long and colorful history: from kings with different sizes feet to the loss of NASA's Mars Orbiter because one team used metric measurements while another used English. There are many books and other sources available for all ages. And all are packed with interesting projects for a mathematical lesson if you know how to dig them out. Here are a few books to get you started. Your class might want to add to the list:

> The Librarian who Measured the Earth by Kathryn Lasky. This book for children contains a wealth of information about mathematics in ancient Greece. The book talks about "bematists" or people who were trained to walk with equal steps. In this way long distances could be measured with some accuracy.
>
> The Measure of All Things by Ken Alder. This book is about the two French surveyors who first determined the length of a meter.
>
> Measuring America by Andro Linklater. Introductory sections have some interesting discussions about why the United States did not adopt the Metric System.
>
> How Big is a Foot? by Rolf Myller. This delightful story, appropriate for children in grades 1-3, is about a King who wants a bed constructed for his wife. He makes measurements for the bed with his feet and gives the measurements to the carpenter who measures with his, much smaller, feet. The young reader will easily figure out why things went wrong before the King and carpenter resolve their conflict over the resulting small bed.

In elementary school, it is sometimes useful to start children learning about units by measuring a variety of classroom objects using units based on objects in the classroom. For example, children might make chains of paper clips and use the chain as a measuring tape to measure their desks in paper_clips_units. What are some considerations for a choice of unit? What are some things the children learn from this kind of experience that makes it desirable, rather than just measuring with standard rulers?

10. Assignment: In a second grade classroom, the children decide to measure the length of their desks using pencils as a unit of length. Everyone gets out all their pencils and lays them end to end on their desks. Mike and Mary are concerned because they get different numbers but they are pretty sure their desks are the same length. List some of the things that might have gone wrong for them and your suggestions to rectify the situation as a teacher. What are the important characteristics of a good unit of measurement?

The metric system of measurements came into being in the middle of the French Revolution by decree of the National Assembly of France and from the start was to be used "for all people, for all time." The French Academy of Sciences was asked to create a standard set of measurements for all measures and, in particular, for measurement of lengths. All of the measurements in this new system were to be based on measurements of the Earth. The unit of measure of length was called a **meter** and was declared to be one ten-millionth of the distance from the North Pole to equator. Two astronomers set out for destinations of Dunkirk and Barcelona to measure the world and determine precisely this distance. Eventually, a platinum bar with highly polished parallel ends was made and the meter defined precisely by the distance between the polished ends at a specified temperature (metal expands and contracts with temperature). In other words, they were very, very careful to be as precise as possible. Since that time the meter has been redefined several times to be even more precise, while not changing too much in length from this original determination. The latest change came in October 1983 when the meter was finally defined by the speed of light in a vacuum. This speed is constant and reproducible with the proper advanced equipment.

> The meter is the length of the path that light travels in a vacuum during a time interval of 1 / 299,792,458 seconds.

Of course, a second must also be defined precisely. The job of making, maintaining and ever increasing the accuracy of our standards of measurements is endless.

The metric system is based on the decimal system so it is useful for studying decimal numbers. A meter divides into 100 centimeters. A kilometer is 1000 meters. A millimeter is one-tenth of a centimeter. In the lower grades, rulers that measure whole centimeters can be used. When children have studied decimal numbers, they are ready to measure to the nearest millimeter. *Unless indicated otherwise, all measurements will be made to the nearest millimeter in this book.*

11. Assignment: Complete the following table. The facts mentioned in the previous paragraph have already been included in the table. From this information, you should be able to figure out the other entries. For example, the table indicates that 1 meter is equal to 100 cm, so you know that 1cm is 0.01 meter and can put that number in the appropriate box in the "Centimeter" row.

Unit	Millimeter	Centimeter	Meter	Kilometer
1 Millimeter =				
1 Centimeter =				
1 Meter =	1000 mm	100 cm	1 m	0.001 k
1 Kilometer =				

12. Assignment: What are some patterns you see in this table?

You should have access to both a meter stick and a ruler that measures centimeters. For this course, these will be our "standards." If you don't, it's easy to make both:

13. Assignment: Cut a strip from a piece of centimeter graph paper (provided in the back of the book). You can measure up to ____ (fill-in the blank) centimeters with this strip. Cut out more strips and tape them together end-to-end until you have enough to measure 100 centimeters or 1 meter. Use your tape to measure the total perimeter of this piece of paper.
Total perimeter of 8 1/2" x 11" piece of paper = _______

Compare your meter length with others or with a manufactured meter stick to see how much they differ. How accurate is your homemade meter?

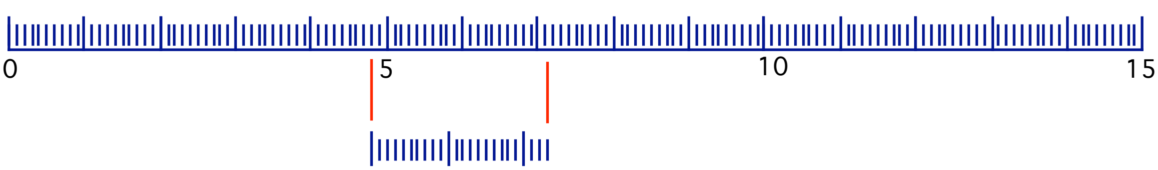
Practice with metric units

1. Whatever measuring device you use it is also handy to know how to estimate lengths. Using body parts just as people used long ago when getting started in the measuring business can help. Some people say that one centimeter is about the width of their little finger. Is your finger wider or narrower than a centimeter?

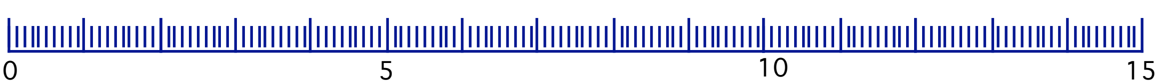
Estimate the length of each of your fingers in centimeters. Measure to the nearest whole centimeter. Draw a picture of your hand and label it with these measurements.

2. Mark the following lengths on this centimeter ruler: 3.5cm, 53mm, 1/10 m, 7.05cm. Sometimes it may be useful to measure out lengths that are given by fractions, not decimals. Mark the following lengths on the centimeter ruler: 1/12 meter, 3/8 centimeter, 4 7/8 cm, 3 1/3 cm.

3. In elementary school, the teacher may use a centimeter ruler to model addition of decimal numbers. In your own words state how this picture models 4.8 + 2.3 = 7.1



Use this ruler to model the subtraction problem: 12.4 - 9.6



The English System of measurements, unlike the Metric system, which was carefully planned in advance, evolved over thousands of years: from Babylonians to the Romans through centuries of Kings and Queens in Europe to the early American states. The earliest length measurements were based on body parts: a foot was the length of a person's foot, an inch the length of a man's thumb, and so on. Over the years rulers and governments declared certain lengths to be standard and these standards changed at the whim of each new government. For example, in the 16th century, Queen Elizabeth declared the mile to be 5280 feet. Since the time of the Romans the mile had been 5000 feet but the Queen wanted it to be exactly 8 furloughs, a measurement in common use at the time. Now the United States is the only country where the official measurement system is the English system rather than the Metric system. The National Institute of Standards and Technology in Washington, D.C. maintains the standards.

14. Assignment: Many stories, both factual and traditional, about the history of the English system can be found by searching the Internet. Find one that interests you and record it *in your own words,* as you might tell the story to children.

The foot, inch, yard, mile are all units of measuring length in the English system. In school, children will start measuring in whole inches and increase accuracy to halves, quarters, eighths, sixteenths as they become proficient with fractions and measuring to this accuracy. It is interesting to note that carpenters in this country use the English system and become skilled with added and subtracting in these fractions.

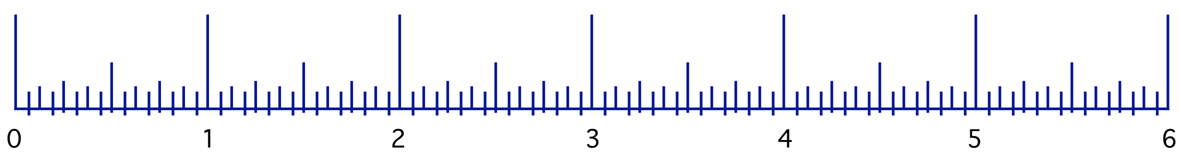
15. Assignment: Complete this chart which shows the relationship between the English units of length. Use fractions rather than decimals. Include your computations or proportional statements in the margins:

Unit:	Inch	Foot	Yard	Mile
1 Inch =	1 in.			
1 Foot =	12 in.	1 ft.		
1 Yard =		3 ft.	1 yd.	
1 Mile =		5280 ft.		

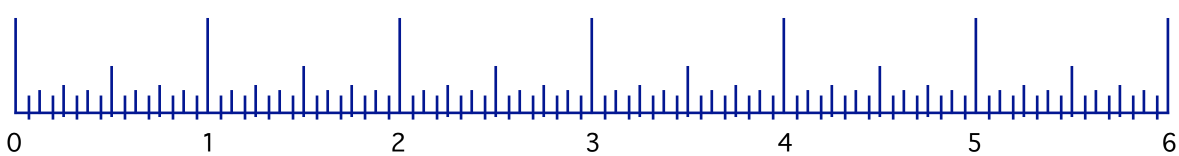
Practice with English units:

4. Estimate the length of your foot, arm length, thigh, in inches. Measure to the nearest one-half inch. Label an outline of the human figure with your measurements.

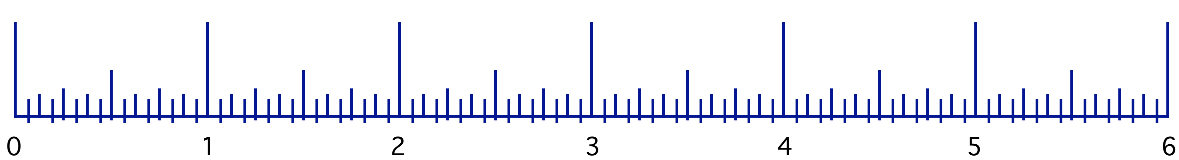
5. Find the following lengths on this inch ruler: 3 1/2 inches, 2 5/16 inches, 1/3 foot, 1/3 inch, 1/12 yard, 1/8 yard.



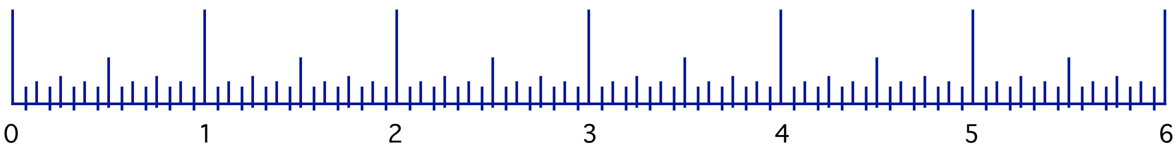
6. Measuring in the English system is a good opportunity to review fractions: what length is represented by the mark half way between 1/2 and 1 5/8?



7. Use rulers to model this arithmetic problem: 2 15/16 + 1 3/16.



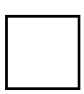
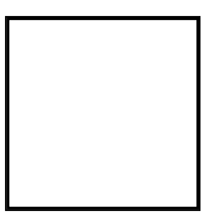
8. Think of another fraction problem that could be solved using rulers and solve it using this ruler.



9. Turn back to the quote at the beginning of the Unit, what do you think John Adams meant by “riveted to memory?” Which facts about length measurements, both metric and English do you have “riveted to memory?” Which ones should children “rivet to memory?” Discuss some ways to help remember these facts: How many centimeters in a meter? How many millimeters in a centimeter? How many inches in a foot? How many feet in a yard? How many feet in a mile? How many yards in a mile?

Standard units of area

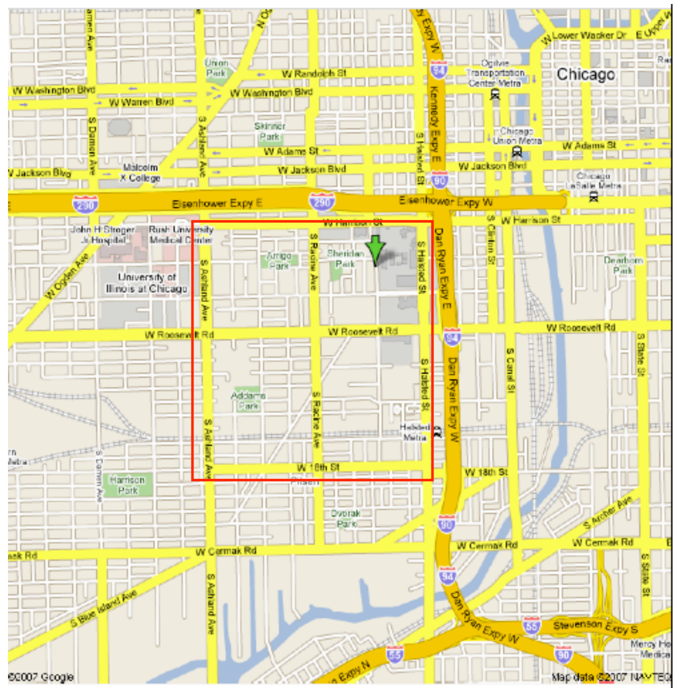
Standard units of area are squares. For any given unit of length there is a related unit for area: a square with side lengths equal to the given unit of length.

- This  is one *square centimeter* abbreviated sq. cm. or cm^2
- This  is one square inch sq. in. or in^2

One square foot is too big to draw on this page.

16. Assignment: Find a piece of paper that is large enough and cut out a square that measures one foot on each side. This will be your standard “square foot.” Fold it and include it with this page for future reference.

One square mile is of course really big, but we can get a feel for exactly how big it is by looking at a map. One square mile is shown on this map of Chicago.

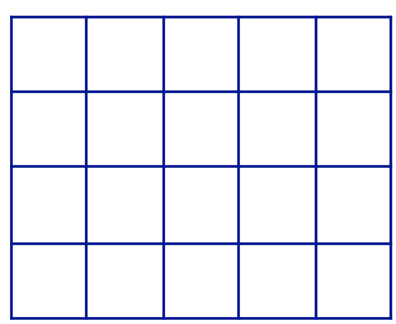


17. Assignment: Have you ever been to this area of Chicago? How long do you think it would take you to walk the entire perimeter of the square mile outlined on this map? Explain how you did this estimation.

Practice with square units

18. How many square inches does it take to fill a square foot? Include a diagram to explain your answer. You may use the square foot you made earlier if you like. How many square feet fit into a square that is one yard on each side? How many square feet are there in one square mile?

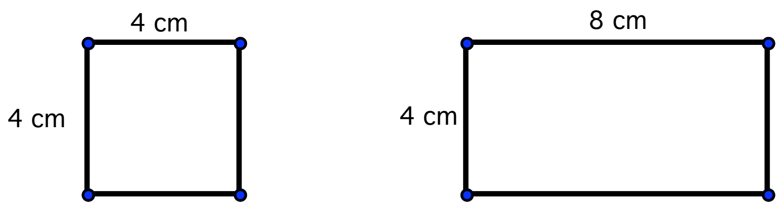
19. How many square feet are there in one square mile?



20. How many square centimeters are needed to make one square inch? One way to proceed is to use your English ruler to draw a square inch on centimeter graph paper.

Group problem: To end the lesson, here's a famous problem from a classroom situation that involves thinking about the different properties of area and perimeter. Deborah Ball, a mathematics educator who has done lots of work listening to children investigate mathematics, originally posed this problem. It is designed to help a teacher think about how she/he would help children explore mathematical ideas that she/he might not have encountered before. That is what your group will be doing as you think about this problem.

> Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you this picture to prove what she is doing:

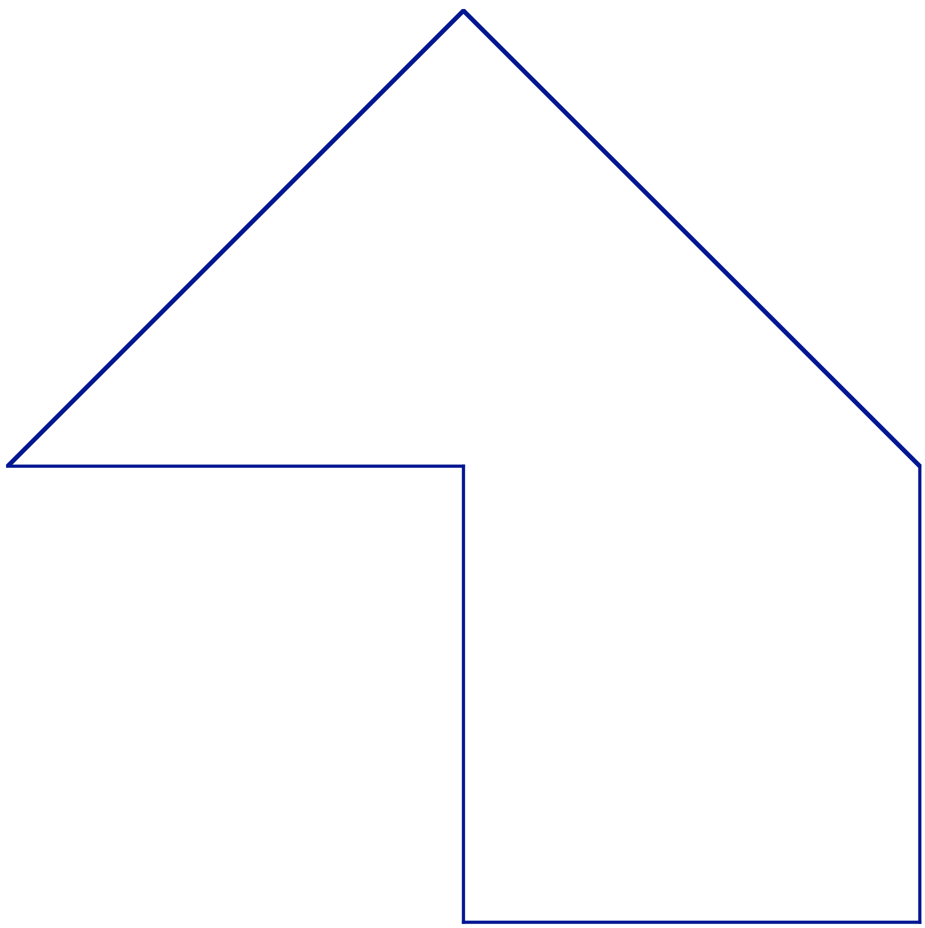


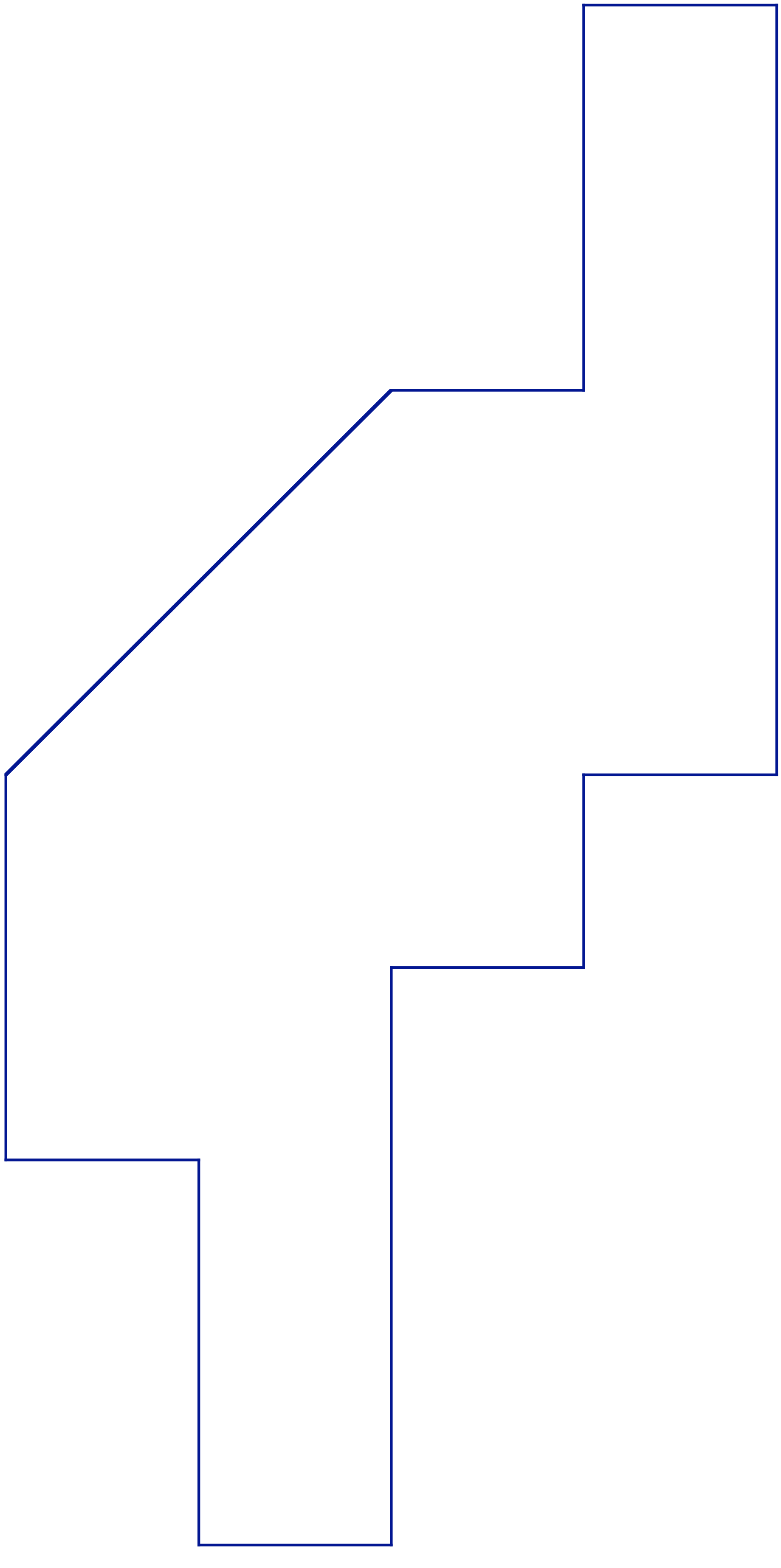
Perimeter = 16 cm
Area = 16 square centimeters

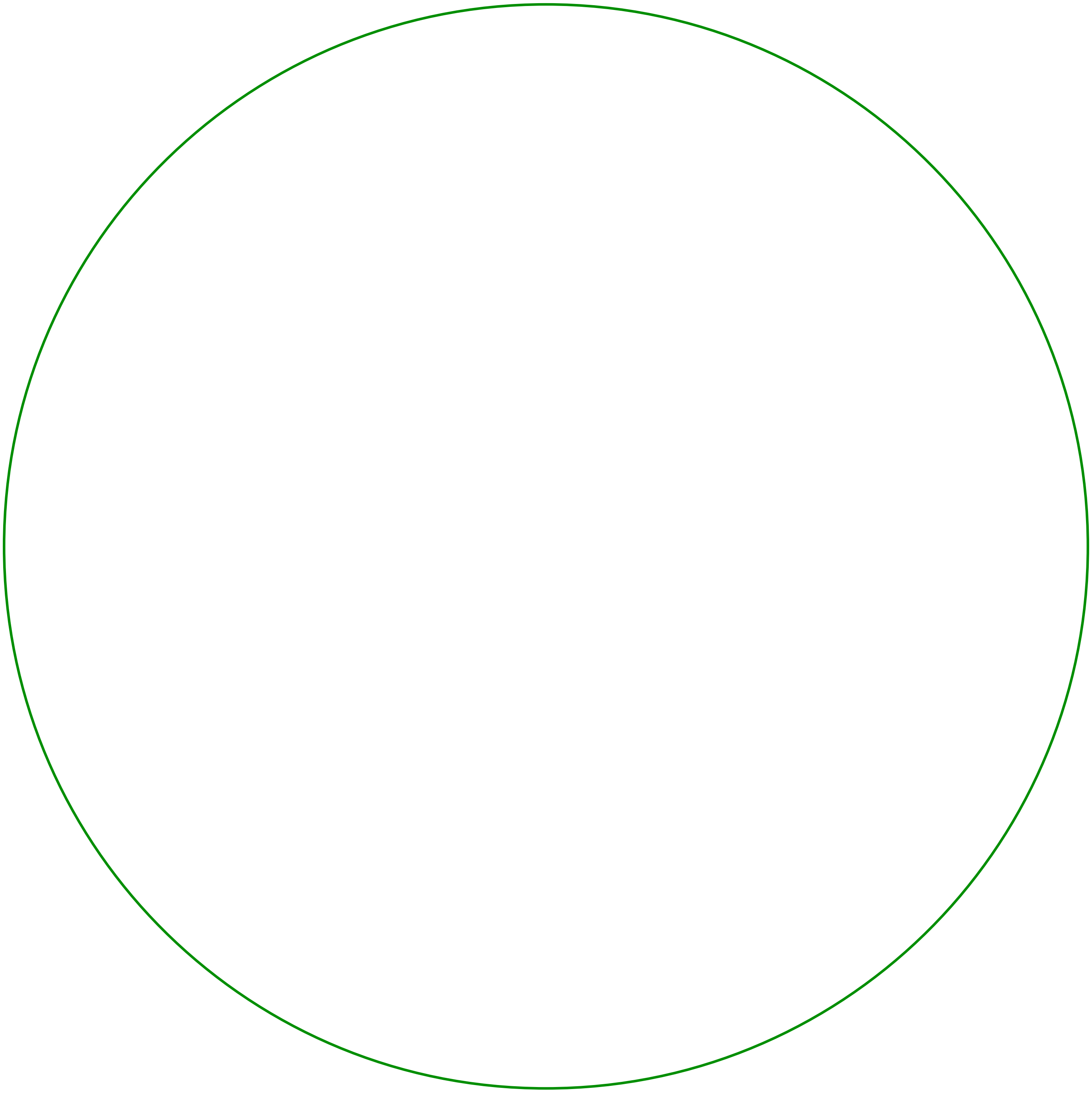
Perimeter = 24 cm
Area = 32 square centimeters

> How would you respond?[1]

[1] From LiPing Ma







Lesson 2 – Using Formulas to Solve Area and Perimeter Problems

Class Activity: Remembering Formulas

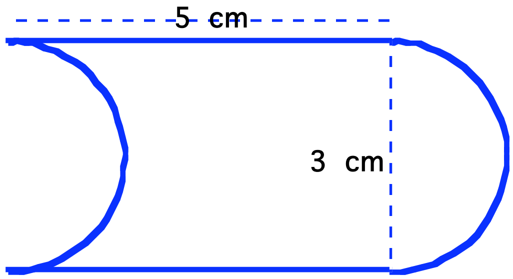
Together with the rest of your class, write down as many formulas for finding area and perimeter as you can remember from previous mathematics classes. As you work through the problems in the rest of this unit, you will want to refer to this page. You may also want to add formulas that you don't think of today.

Be sure to include a description of what is meant by each variable in your formula. You may wish to include pictures to help.

Class Activity: Three Area Problems

Find the area and the perimeter of each. You may use any of the formulas we've listed but don't forget to use your own reasoning skills. Be ready to explain your strategy. Using centimeter graph paper may be of help.

#1

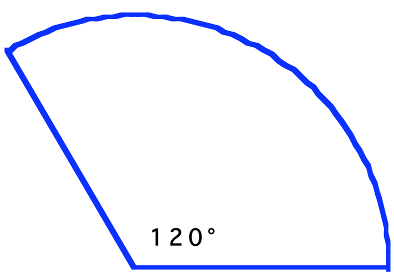


AREA =
METHOD:

PERIMETER=
METHOD:

The ends of this shape are half-circles. This is an exact drawing of the figure.

#2

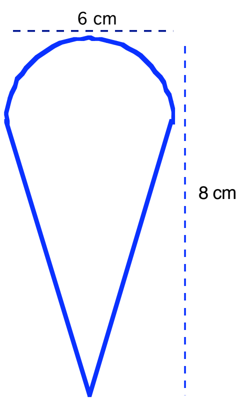


AREA =
METHOD:

PERIMETER=
METHOD:

This shape is part of a circle. It is called a **sector** of the circle. The radius of the circle is 3 feet. This is a scale drawing of the figure.

#3



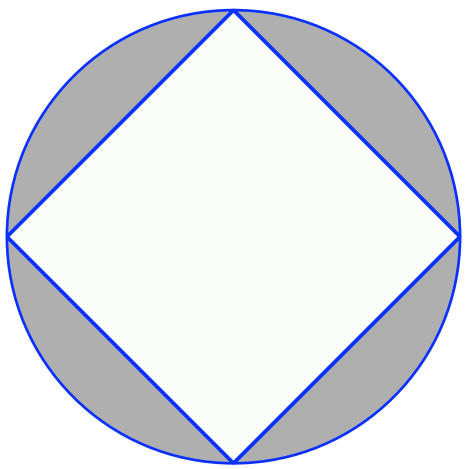
AREA =
METHOD:

PERIMETER=
METHOD:

This shape is an isosceles triangle with half of a circle on top. This is a sketch, it is NOT a scale drawing.

Discussion of estimate, accuracy and exactness.

Example: Four points are evenly spaced around a circle as shown. Connecting the points makes a square. The diameter of the circle is exactly 5 cm. What is the shaded area? Give your answer in three ways: an estimate, an answer that is accurate to 2 decimal places, and an exact answer.



Estimate: By overlaying the figure with centimeter graph paper, I estimate each "moonshape" has an area of roughly 1.25 sq. cm. My estimate of the area is 4 X 1.25 = 6 sq. cm.

Exact:
I want to find the area of the circle and subtract the area of the square. I know the diameter of the circle is 5 cm.

By decomposing the square into four right triangles that have legs 2.5 cm (half of the diameter of the circle) I conclude that the area of the square is $2 \times (2.5)^2$. The area of the circle is πr^2 or $\pi(2.5)^2$. I subtract the two values to get the area of the shaded region:

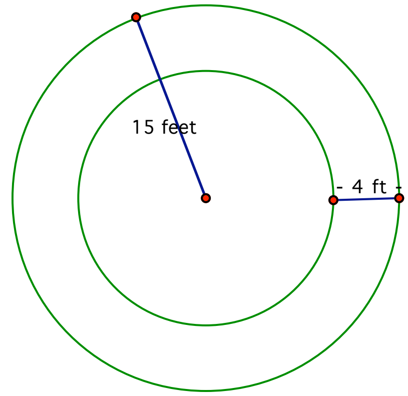
$$\pi(2.5)^2 - 2(2.5)^2 =$$
$$(2.5)^2(\pi - 2) =$$
$$6.25(\pi - 2)$$

The exact area is $6.25(\pi - 2)$ sq. cm.

Accurate to 2 decimal places: Using a calculator to compute i $6.25(\pi - 2) \approx 7.13$ sq. cm..

Group Project: Two Problems for the Marios' Garden Walk

The Marios are making a garden. They are working with a circular area that has a radius of 15 feet.

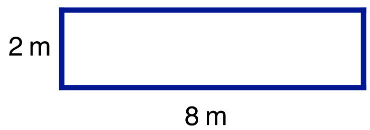


Part 1 They want to pour cement in a path that will be 4 feet wide all of the way around the garden. They need to buy the cement. The cement company can send out a mixer that contains enough cement to cover 30 square yards to the appropriate depth. Will that be enough cement?

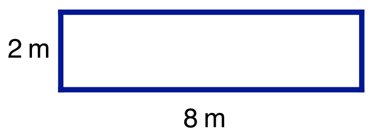
Part 2 They want to put a fence all the way around the garden on the inside of the path. The fencing material costs $76.49 per linear foot. How much will the fencing material cost?

Lesson 2 Problems:

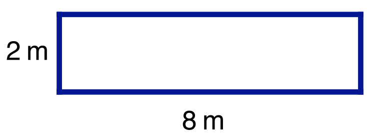
1. What are the dimensions of the square that has the same area as this rectangle?



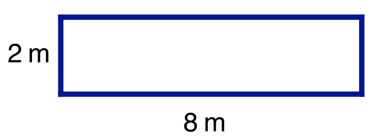
2. What are the dimensions of the square that has twice the perimeter as this rectangle?



3. What is the radius of the circle that has the same area as this rectangle?

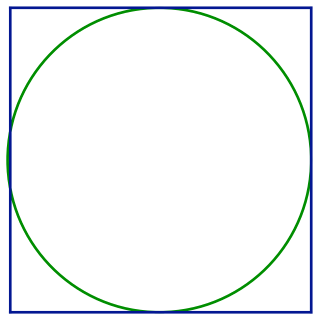


4. What is the radius of the circle that has a circumference equal to the perimeter of this rectangle?

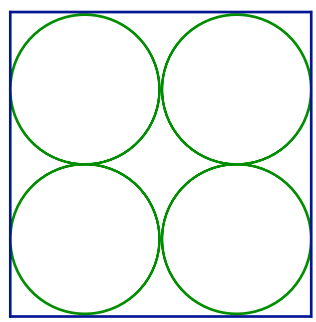


5. Maria is making cakes. Her 16″ cake requires 2 cups of icing and she wonders how much icing to make for the 18″ cake. Since 2″ is $\frac{1}{8}$th of 16, she figures the larger cake will require $\frac{1}{8} \times 2 = \frac{1}{4}$ cup more icing for a total of $2\frac{1}{4}$cups. Is this right? Explain. The cakes are round; the 16″ and 18″ refers to the diameters of the cakes and the icing covers only the tops of the cakes.

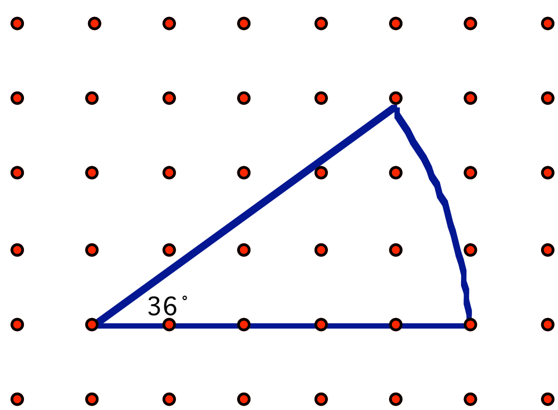
6. What is the percentage waste in cutting a circle from a square?



7. What is the percentage waste in cutting 4 circles from a square?

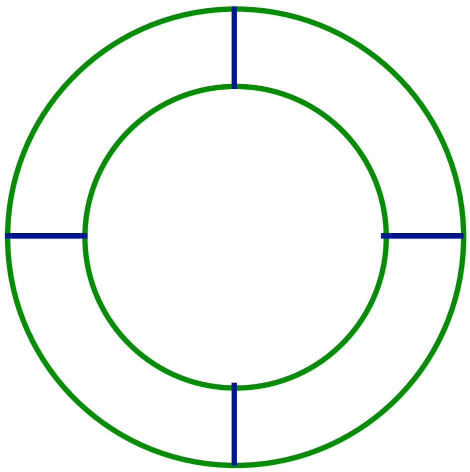


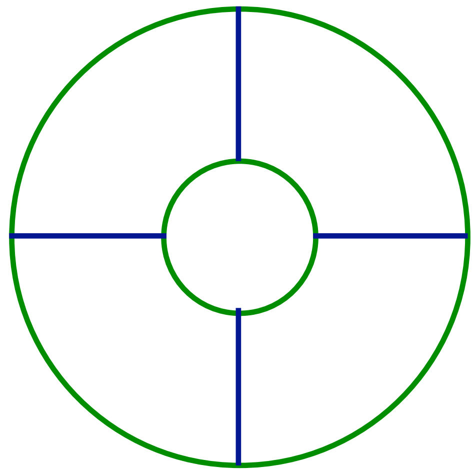
8. Find the area of this sector of a circle. Include an estimate, an answer correct to 3 decimal places, and an exact answer. The grid is a one centimeter grid.

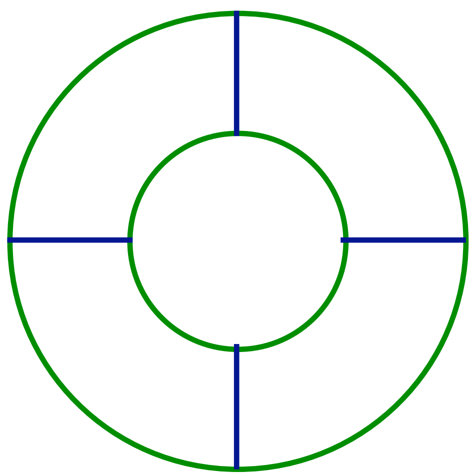


What is the perimeter of this sector?

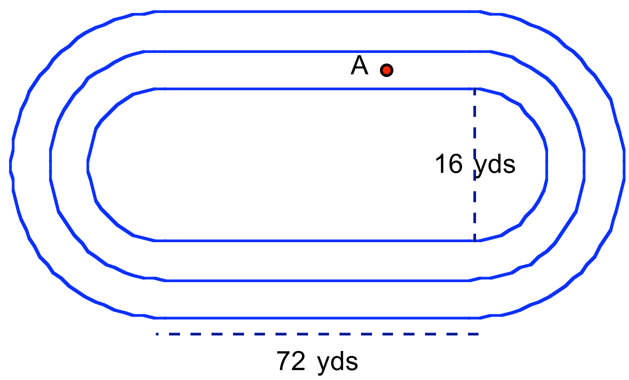
9. A group of sixth graders are discussing the Pharaohs' problems from the first activity in Lesson 1. They come up with the following design to divide Pharaoh III's circular land fairly for five children, but they are not sure what the exact diameter of the inside circle should be so that the area of each piece is the same. Can you help them? Which of these is closest to the right arrangement? What dimensions for the inner circle give the exact answer?





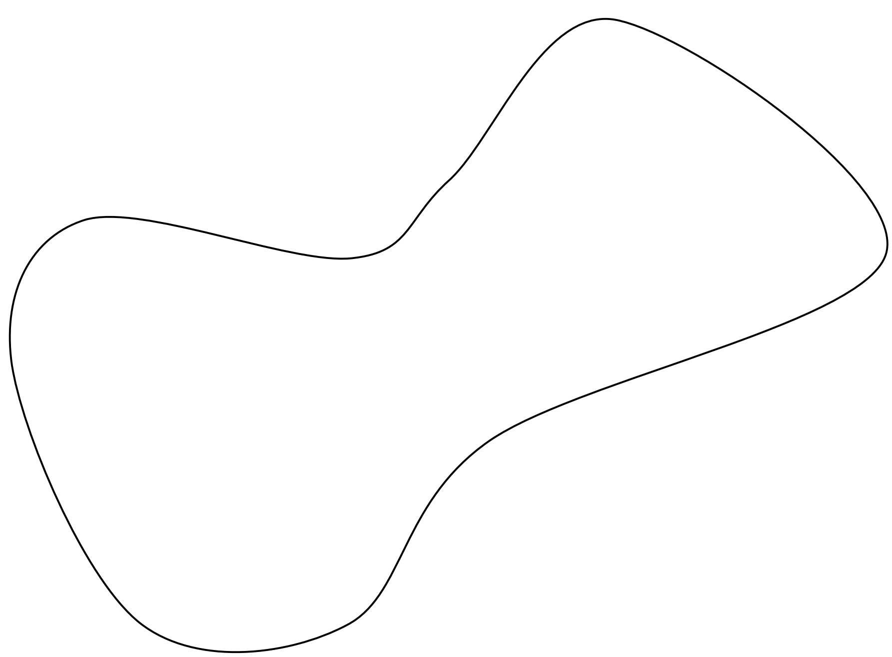


10. This race track is a rectangle with semi-circles attached at both ends. From this picture what are the dimensions of the rectangle? Each track is 3 feet wide and is exactly 3 feet all of the way around. What is the radius of the end semi-circles? Elementary children are taught to stay in the middle of the track. Sketch the path a runner should take. What is the distance around the track for the outside runner? What is the distance around the track for the inside runner? If the inside runner starts at point A to run a race that consists of exactly one complete lap, where should the outside runner start to run the same distance but finish the race at the same spot?



11. Here is a scale drawing of a patio built by Ace Patio Inc. It will be painted with Ace's Supreme All-Weather Patio Paint. The painters know that one gallon of paint will cover 120 square feet. How Many gallons will be needed?

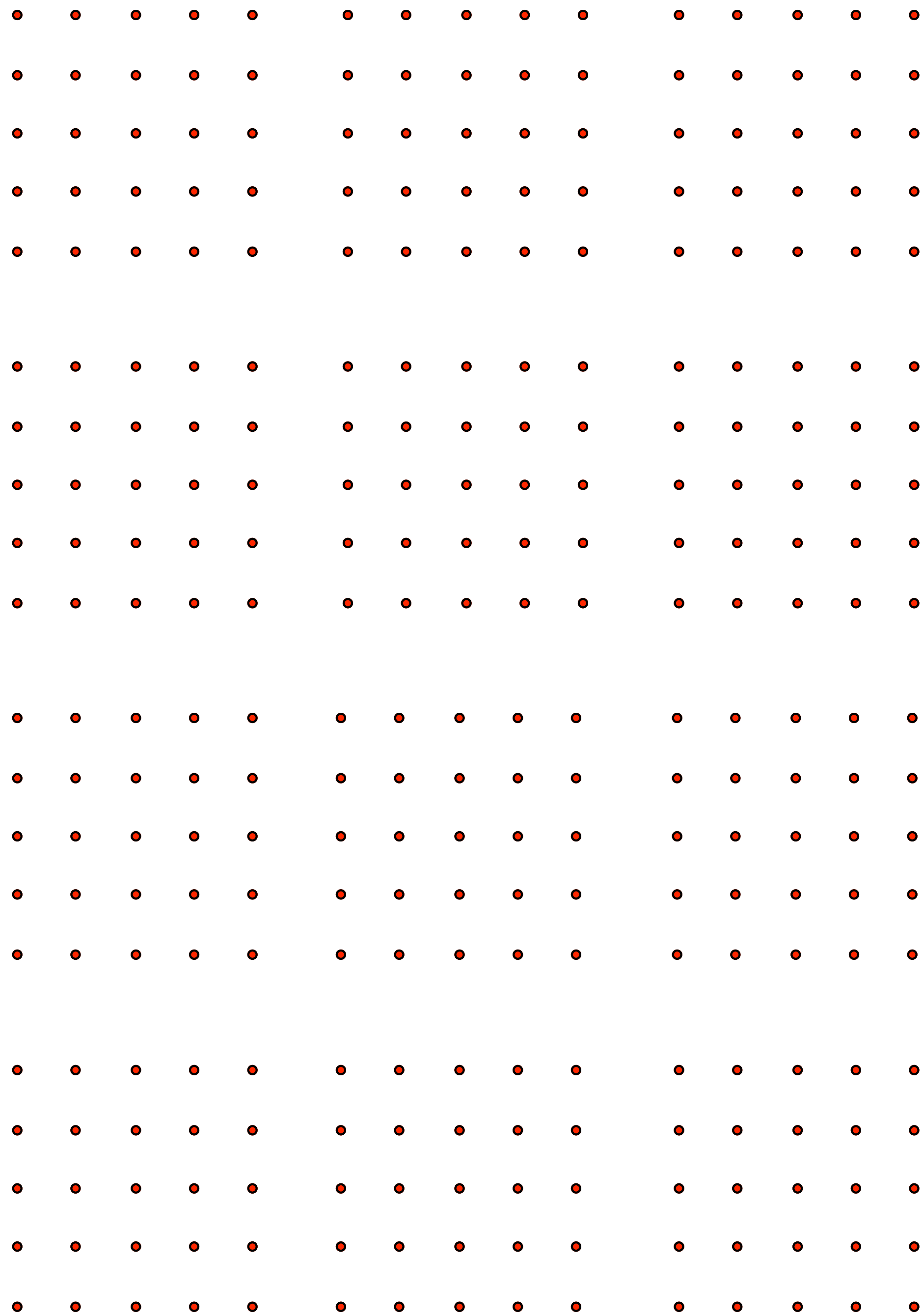
Scale: 1 cm : 7 ft



Lesson 3 – The Meaning of Area: Exploring Formulas

Class Activity: Exploring the Geoboard

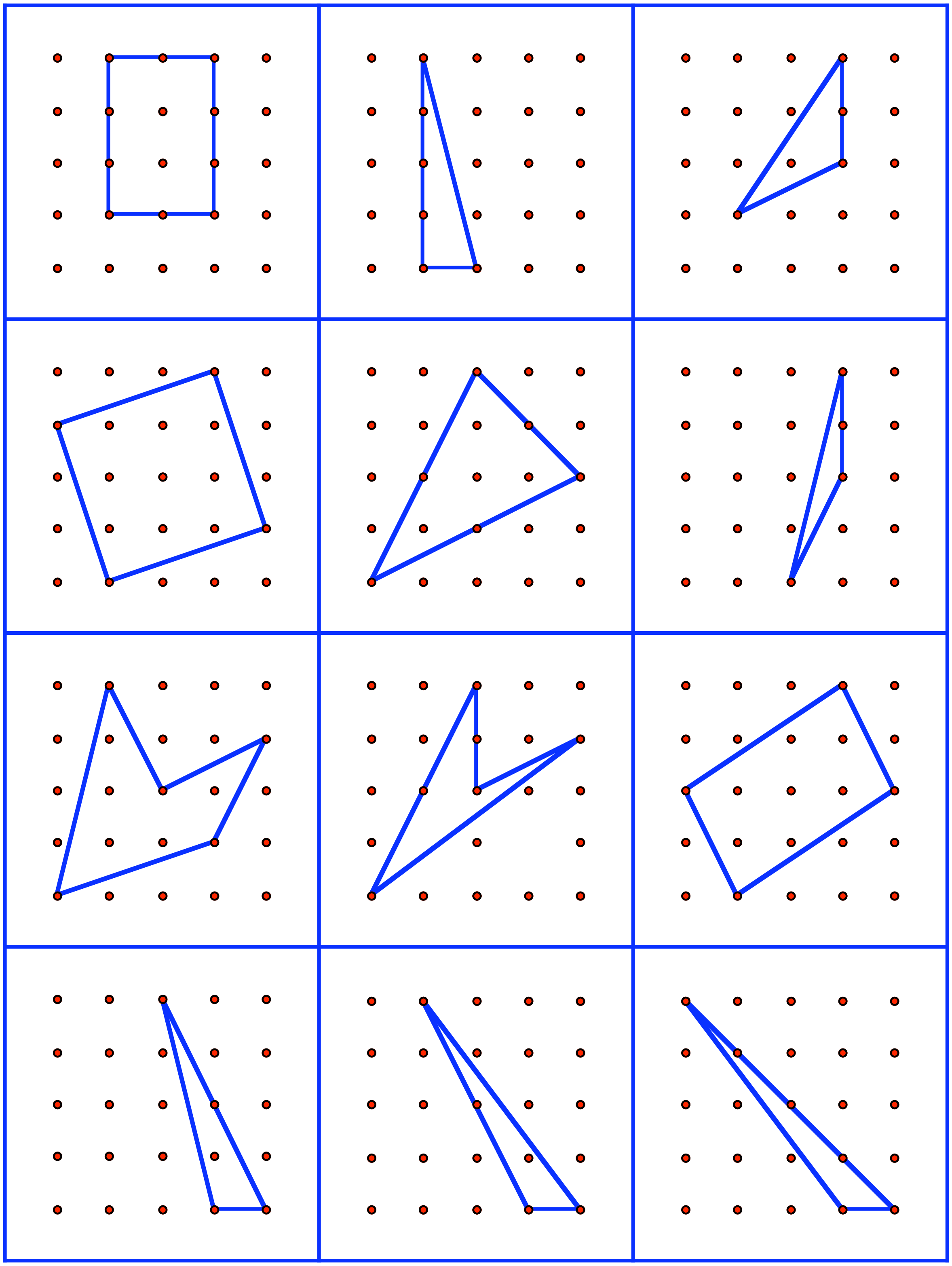
How many different squares can you find on a geoboard? Each vertex of the square must be one of the grid points of the geoboard.



Area Sq. units	Possible or Not
1	possible
2	
3	
4	
5	
6	
7	
8	
9	
10	possible
11	
12	
13	
14	
15	
16	

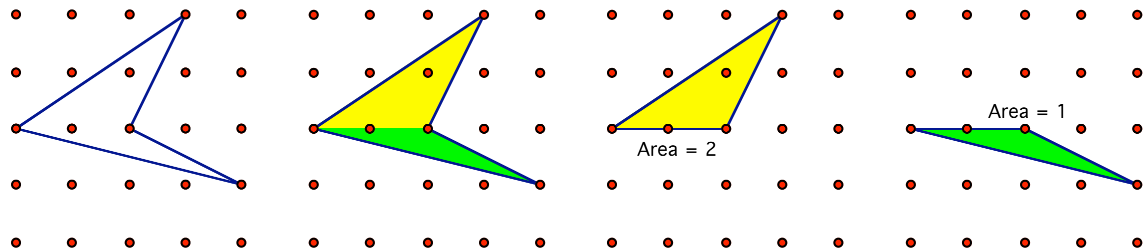
Group Project: Finding Areas on a Geoboard

Find the area of each polygon.



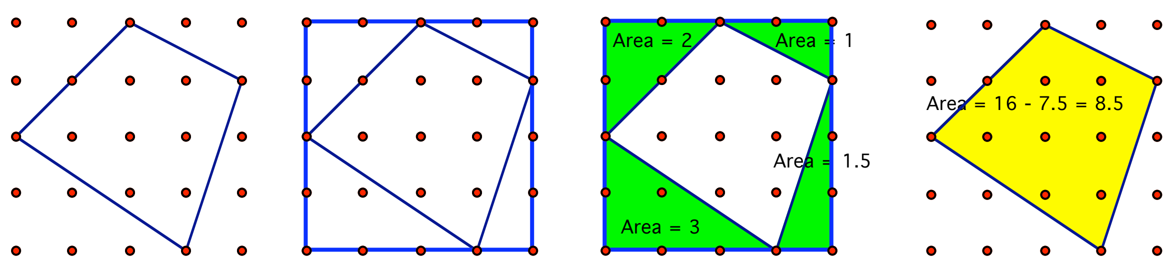
There are several strategies you may have used for finding the areas of the geoboard figures. Here are two useful strategies:

First strategy: Divide the figure into simpler shapes. Compute each component and add.



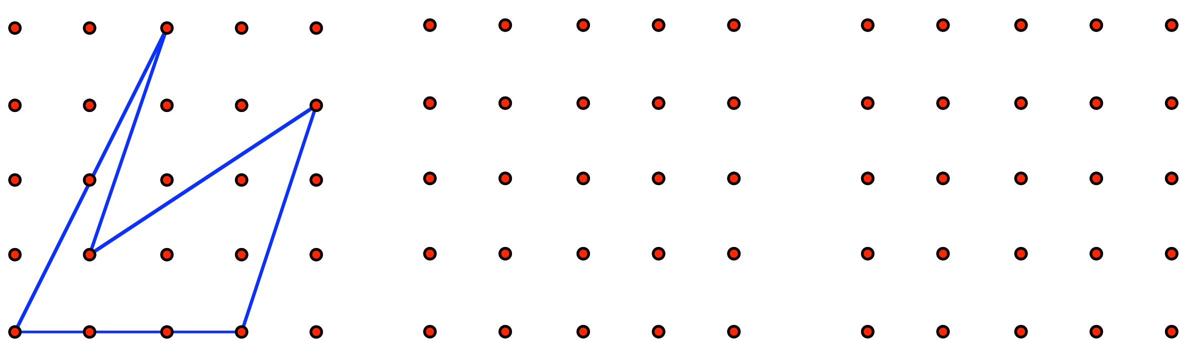
The total area of the polygon in the first grid is 3 square units.

Second strategy: Surround the polygon by a rectangle. Now you may want to divide the rectangle into simpler shapes and subtract some from the whole rectangle to get the area of the original polygon. This method is good to know because it can make the computations simpler. It is also an invaluable aid in understanding the formulas for the area.



Helpful hint: As you divide the shape into simpler regions it is a good idea to use extra dot paper. Copy each of the simpler shapes to the dot paper where you may view them without the surrounding confusion of the more whole problem.

1. Assignment: Try using the second strategy to find the area of this shape. You may want to use the extra geoboard grids for scratch work.



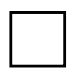
Understanding area by using definitions and postulates. In the first unit we saw how Euclid had presented definitions and postulates to begin a systematic study of geometry. We can also use this *axiomatic* approach for the careful study of area. In fact, this is an excellent way to learn more about how the axiomatic approach works.

Before we begin, think a little about how difficult it is to define what you mean by *area*. You might say something like this definition from The American Heritage Dictionary[1]

> **Area:** The extent of a planar region or of the surface of a solid measured in square units. The extent of a planar region or of the surface of a solid measured in square units.

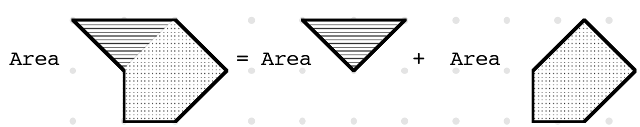
This definition explains the intuitive idea but offers little help in how to actually compute or determine the number of square units in any kind of shape. It is possible to define area and give basic rules at the same time by giving one definition and 2 postulates. These statements assume you know about linear measure (units of length and measuring lengths).

Definition: The area of a square that measures one unit on each side is 1 *square unit*, abbreviated sq. unit or unit^2

Area  = 1 square unit

Postulate 1: Congruent shapes have the same area.

Postulate 2: The area of the union of two non-overlapping regions is the sum of the areas of the two regions.



We have been using this definition and these postulates throughout this unit without explicitly talking about them.

2. Explain how you used the ideas presented in this definition and postulates in solving the Pharaohs' problem in Lesson 1.

3. Show how this definition and postulates can be used to justify the first and second geoboard techniques.

Children develop an intuitive understanding of them as they experiment with finding areas in practical ways. By the time one begins a systematic study of area in high school these postulates should seem obvious.

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[1] The American Heritage® Dictionary of the English Language, Fourth Edition Copyright © 2004, 2000 by Houghton Mifflin Company.

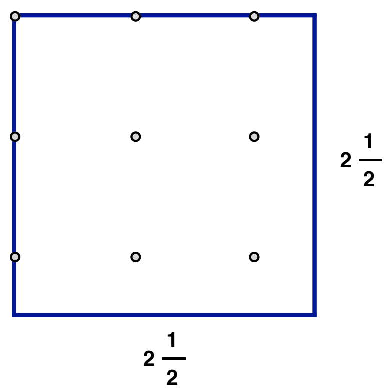
Proving that area formulas work. We are going to use these postulates to derive the common formulas for the area of parallelograms, and triangles. We start with rectangles and proceed systematically to more complex figures.

Theorem: The area of a rectangle that has dimensions b units by h units is b x h square units.

Proof: This proof can get quite complex if we allow b and h to be any real number but is quite easy to see if we just consider positive integers and think about geoboards and what we know about the meaning of multiplication. The rectangle that has dimensions b by h can be considered to be b columns of h squares in each row. The meaning of multiplications tells us this is b groups of h objects. Each object is one square unit. Postulate 2 says the total area is the sum of the areas of the squares, or b x h square units

Class discussion:

What if b and/or h are fractions? For example, what is the area of this square?

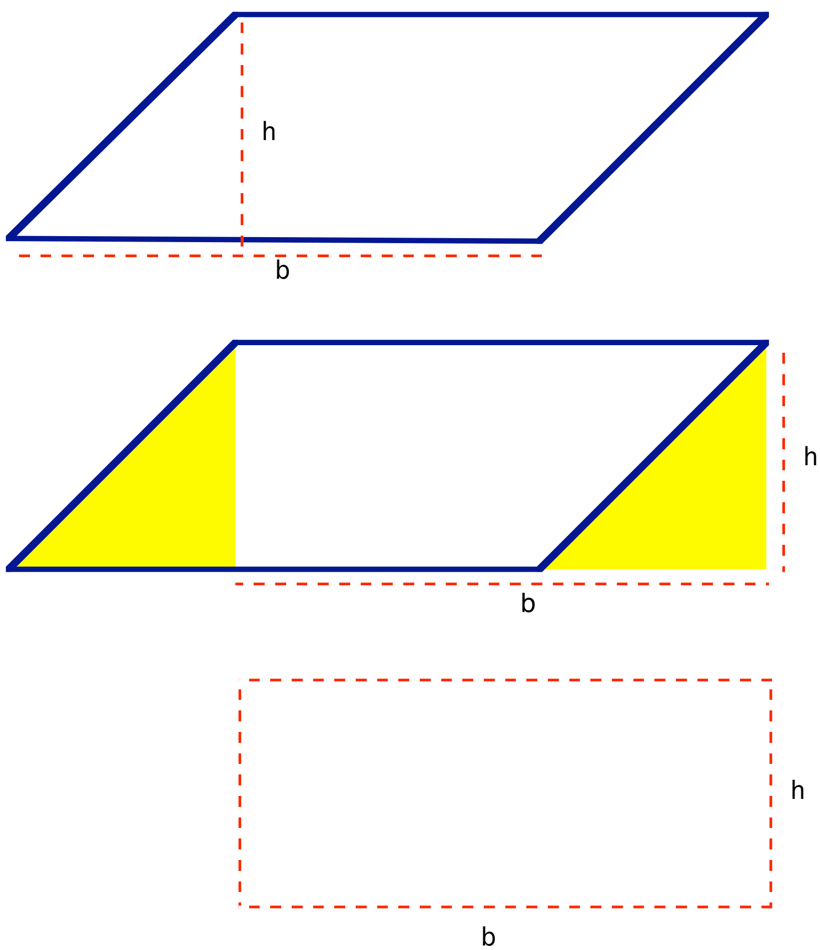


4. Show that the area of this square is $2\frac{1}{2}\times 2\frac{1}{2}$.

Once we have shown why the area of a rectangle is base times height, we can show that the area of a parallelogram is also base times height but we must be careful about what we mean by “base” and “height”

Caution: Take particular care to understand what is meant by the height of a parallelogram

The following pictures suggest a proof. The original parallelogram is divided into two regions. The shaded triangle is translated to the other side of the parallelogram. The resulting rectangle has the same area as the original parallelogram. And since the area of the rectangle is b x h, that must also be the area of the original parallelogram.

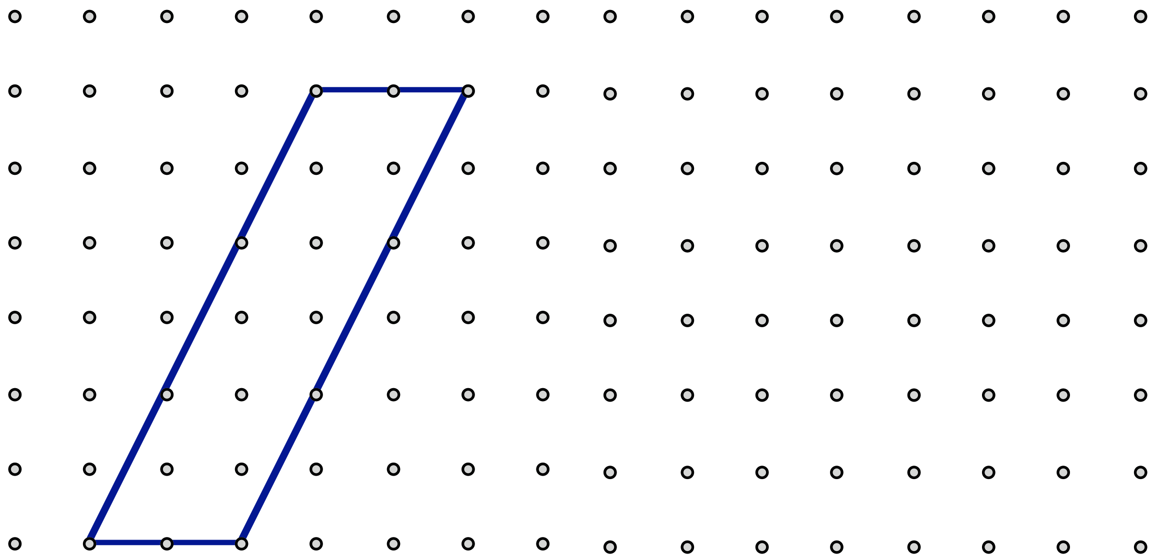


5. Assignment: Explain how the postulates, and previously proved theorems, were used in this proof.

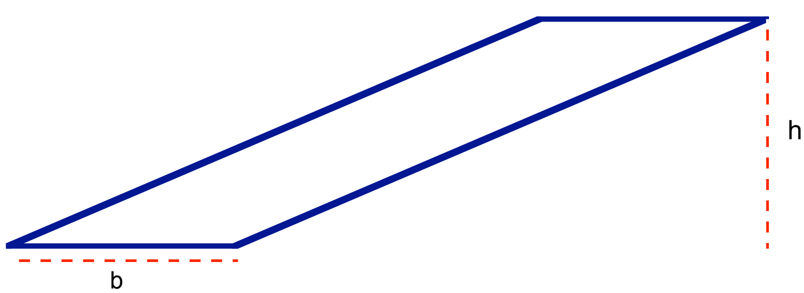
There is something special about the parallelogram we chose to demonstrate this proof of the formula. Here are some other parallelograms to consider.

6. Show how to cut and rearrange each parallelogram to change it into a rectangle that has the same base and the same height.

For this one you may use the extra grid space to show your rearrangement.



For this next one, you may want to make a copy of the parallelogram so you can cut and paste the parts into a rectangle.



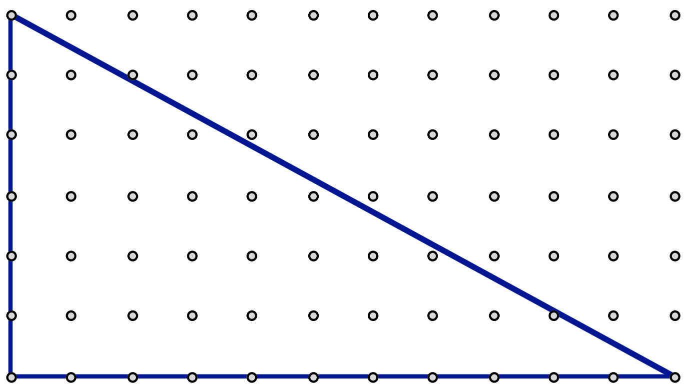
As you work on these problems, try to establish a method that would work to show that the area of any parallelogram is b x h.

7. Can you think of a parallelogram that you might not be able to rearrange into a rectangle? If so, draw it here and explain.

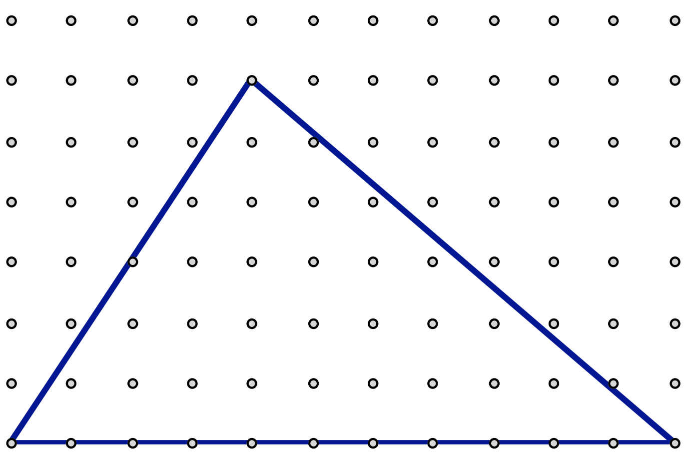
Group Activity: Formula for Triangles

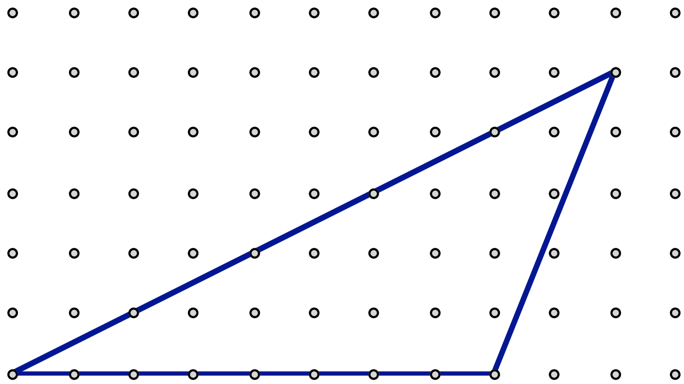
Show how to think of each triangle as one-half of a parallelogram (it could be a rectangle or square) that has the same base and height and so prove the standard formula for the area of a triangle, $\frac{1}{2}b \cdot h$.

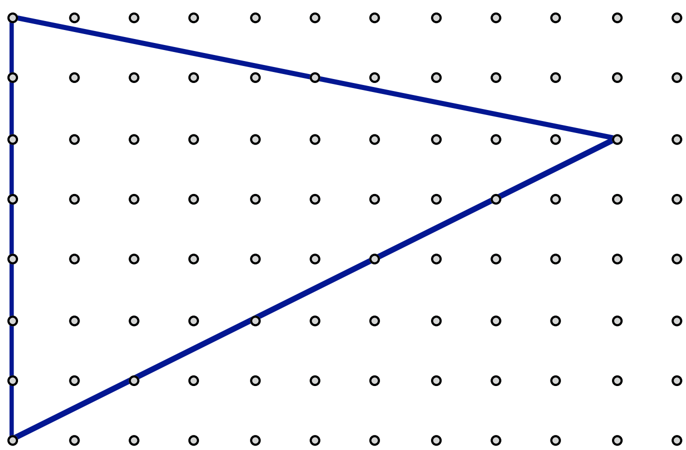
1) Label the base of each triangle with a "b". Label the height with a "h"
2) Draw a parallelogram that has the same base and height.
3) Explain how the postulates and the formulas for the area of a parallelogram are used to show that the area of the triangle may be expressed as $\frac{1}{2}b \cdot h$.



Using the second geoboard strategy, we can see that two congruent the right triangles make the entire rectangle. So the area of one of the right triangles is $\frac{1}{2}$ the enclosed rectangle. Since the area of the rectangle is $b \cdot h$, the area of the triangle must be $\frac{1}{2}b \cdot h$.



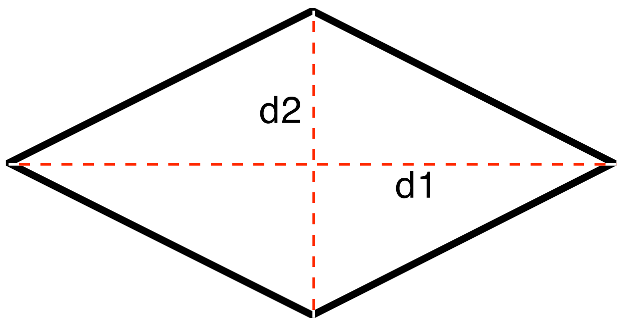




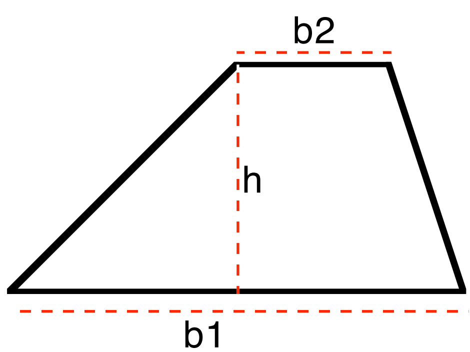
Formulas for other shapes.

For each of the next problems, cut the quadrilateral into pieces and rearrange the pieces to form a new parallelogram with the appropriate dimensions. You may consider using two copies of the quadrilateral in order to form a rectangle that has twice the area.

8. Find a formula for the area of a rhombus in terms of the two diagonals.



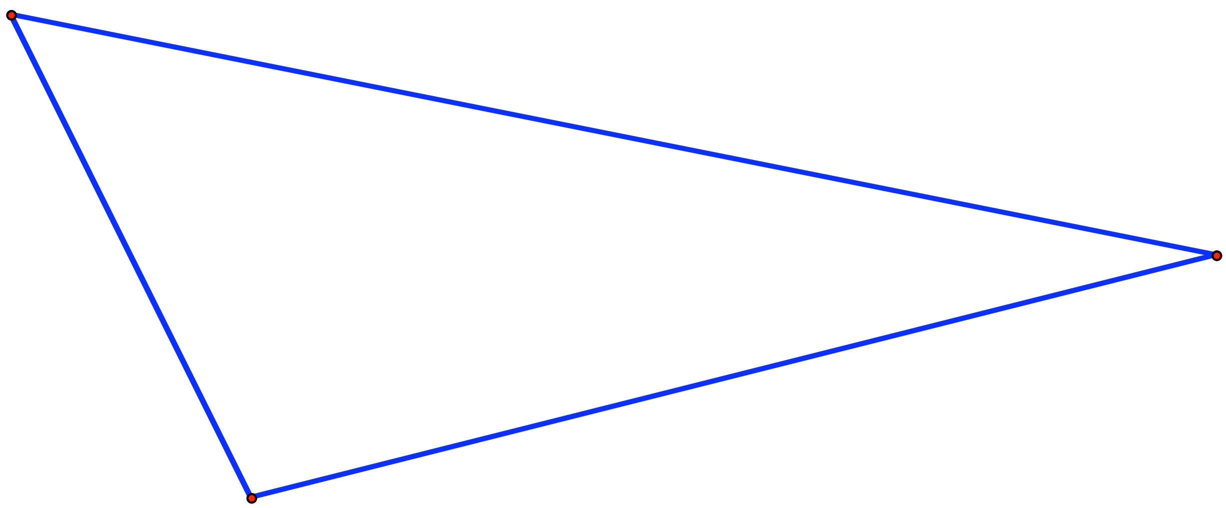
9. Find a formula for the area of a trapezoid in terms of the two bases and the height.



Class Activity: Altitude Construction, An Aside

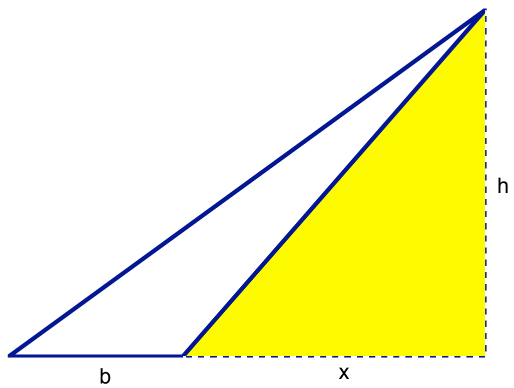
When computing areas of triangles we often need to construct the height, or the altitude, of the triangle. There are always three altitudes to a triangle, one for each side. Sometimes the altitude may be outside of the triangle.

Construct all three altitudes for this triangle.



Using algebra to justify the standard formula for the area of a triangle.

Consider the following justification for the standard formula for the area of a triangle when the height of the triangle is outside the triangle.

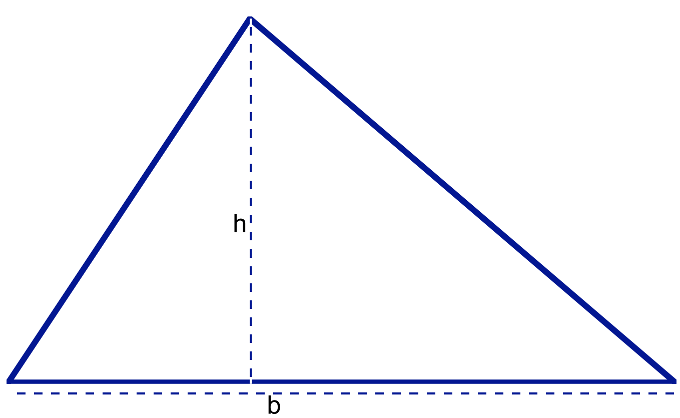


We see two right triangles in this picture by considering the shaded triangle as shown.
The area of the larger triangle is $\frac{1}{2}(b+x)h$.
The area of the shaded triangle is $\frac{1}{2}xh$.

So the area of the original triangle is the difference of the two areas:

$$\begin{aligned} &\tfrac{1}{2}(b+x)h - \tfrac{1}{2}xh \\ &= \tfrac{1}{2}bh + \tfrac{1}{2}xh - \tfrac{1}{2}xh \\ &= \tfrac{1}{2}bh \end{aligned}$$

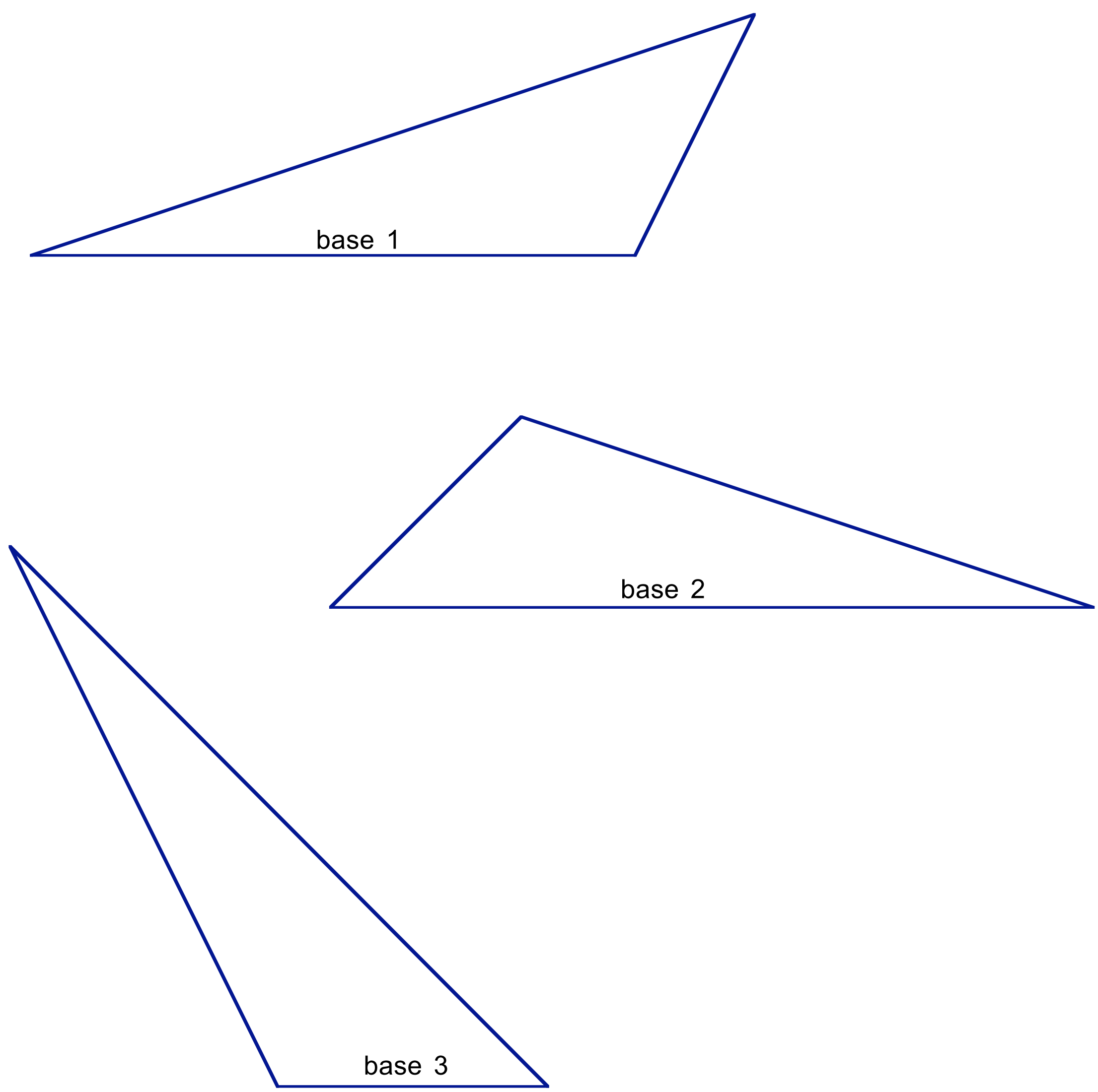
10. Explain how the algebraic expression shows that the area of this triangle is $\frac{1}{2}b \cdot h$. In particular, what is x?



$$A = \frac{1}{2}x \cdot h + \frac{1}{2}(b - x) \cdot h = \frac{1}{2}b \cdot h$$

Lesson 3 Problem

1. Convince yourself that these triangles are all congruent by moving a copy of one onto the other two. Since they are all congruent to each other, they each have the same area. Find the area in three different ways by using the standard formula with each of the three bases. In each case you will need to construct and measure “h.”



2. Show that the area of a rectangle that is $2\frac{1}{3}$ inches by $1\frac{1}{4}$ inches is $2\frac{1}{3} \times 1\frac{1}{4}$ by sketching the rectangle and dividing it into smaller rectangles of equal area. Be sure to explain how your picture demonstrates how to multiply fractions.

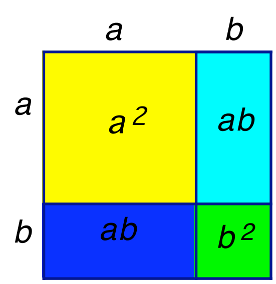
Lesson 4 – Understanding the Pythagorean Theorem

In this lesson we see how to combine several different concepts to prove one of the most famous theorems of all time, The Pythagorean Theorem. We use algebra and concepts of area to show something about lengths of sides of triangles. It is really the meeting of lots of different ideas and is so much fun that mathematicians for centuries have looked around for more proofs of it. At one time, in order to get a degree in mathematics, one had to create an original proof. We'll do one as a group activity and discuss several more. Your instructor may very well have a different proof for you to consider.

Before we get started on the proof, let's review what we'll need to remember from algebra.

Example: Explain how the picture shows this algebraic identity.

$$(a+b)^2 = a^2 + 2ab + b^2$$



Of course, we could prove the identity using algebra:

$$\begin{aligned}(a+b)^2 &= (a+b)\cdot(a+b)\\ &= a\cdot(a+b)+b\cdot(a+b)\\ &= a\cdot a+a\cdot b+b\cdot a+b\cdot b\\ &= a^2+a\cdot b+a\cdot b+b^2\\ &= a^2+2ab+b^2\end{aligned}$$

1. **Assignment:** Label each line in this derivation showing whether the commutative law of multiplication or the distributive law is being used.

2. **Assignment:** Draw a picture that shows how "FOIL" works to multiply two binomials.

$$(a+b)\cdot(d+f) = a\cdot d + a\cdot f + b\cdot d + b\cdot f$$

Group Project: A Proof of the Pythagorean Theorem

Step 1 Cut out four congruent right triangles. You might want to use 3X5 cards to make sure you have right angles. Label the legs of each triangle ***a*** and ***b***. Label the hypotenuse ***c***. Do NOT make isosceles right triangles – you'll want to be sure you can tell the difference between side ***a*** and side ***b***.

Step 2 Arrange the four triangles to make a large square with a smaller square hole in the middle. Tape this figure to a sheet of paper.

Step 3 Write your answers to the following questions in terms of ***a***, ***b***, and ***c***.

i. What is the area of the square hole?

ii. What is the total area of the four triangles?

iii. What is the area of the large square?

Step 4 Write an equation relating the three areas from Step 3. Simplify.

Step 5 Discuss both side length and angle in your answers to the following questions

i. How can you be sure that your figure is a square?

ii. How can you be sure that the hole is a square?

Challenge: There are two different ways to make the figure in step 2). Find the other way and redo steps 3), 4) and 5) for this new figure.

Converse of the Pythagorean Theorem:
If the three sides of a triangle satisfy $a^2 + b^2 = c^2$, then the triangle is a right triangle. The legs have length, a and b, and the hypotenuse has length, c.

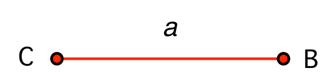
3. **Assignment:** State the Pythagorean Theorem. Put a box around the theorem. How does your statement differ from the above statement of the Converse of the Pythagorean Theorem?

Proof of the converse of the Pythagorean Theorem: Suppose you have a triangle with sides that measure a, b, c and it is true that $a^2 + b^2 = c^2$. This is the starting point of our proof; we must show that the triangle in question is a right triangle.

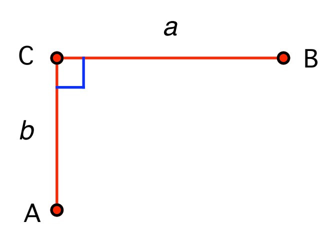
Suppose you want to convince a friend that it must be a right triangle, but you are talking to her on the phone and you don't really trust her with a protractor, so you can't just construct the triangle with her and have her measure the angle.

So you need to tell her what to do step by step:

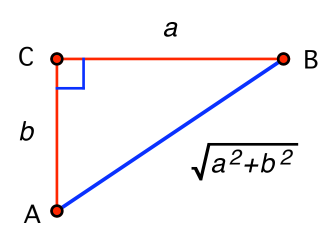
"Draw a line segment CB that has length a.



At C draw the line perpendicular to CB and mark off a segment that has length b. Call it CA.



Connect the points A and B."



What will your friend observe?

1 -- She has made a right triangle.
2 -- One base has length a; one base has length b.
3 -- She could measure the hypotenuse but she doesn't need to because she knows the Pythagorean theorem. So she knows that the hypotenuse has length $\sqrt{a^2 + b^2}$. Of course this must be equal it c because you told her that $a^2 + b^2 = c^2$.
4 -- On the other hand, your friend's triangle must be congruent to yours because it has the same three side lengths: a, b, c. That's SSS. So your original triangle must be a right triangle as well.

And that proves it: if $a^2 + b^2 = c^2$ then the triangle with sides a, b, c is a right triangle.

The next step is to see the Converse of the Pythagorean Theorem is action:

4. Assignment: Construct a triangle with sides of the given lengths and measure all the angles:

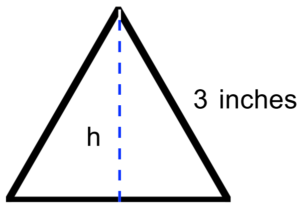
a) 7 cm, 7 cm, 10 cm b) 3 cm, 4 cm, 5 cm c) 4 cm, 7 cm, 8 cm

Apply the Converse of the Pythagorean Theorem or the Pythagorean Theorem to determine precisely which are right triangles.

More Algebra and Special Triangles

Using algebra and the Pythagorean Theorem we can find the area and the perimeter of some special triangles.

5. Assignment: Find the exact area of an equilateral triangle, given that one side has length 3 inches.



We'll start by finding the length of the altitude so we can use the common formula $\frac{1}{2}bh$ for the area of the triangle. Let h be the length of the altitude as shown in the picture. The altitude forms a right triangle. Since the altitude bisects the base of an isosceles triangle, the other leg of the right triangle is 1.5 inches.

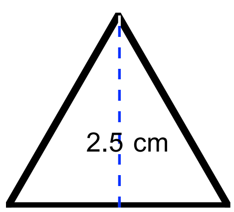
$$\begin{aligned}(1.5)^2 + h^2 &= 3^2\\ 2.25 + h^2 &= 9\\ h^2 &= 9 - 2.25\\ h^2 &= 6.75\\ h &= \sqrt{6.75}\end{aligned}$$

Now we can find the exact area of the equilateral triangle that has base of 3 inches and a height of $\sqrt{6.75}$

$$\begin{aligned}\text{Area} &= \frac{1}{2}bh\\ &= \frac{1}{2}\cdot 3\cdot\sqrt{6.75}\end{aligned}$$

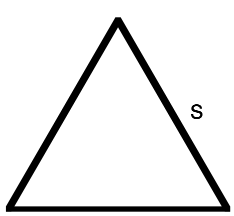
The approximate area is 2.598 square inches.

6. Assignment: Find the exact perimeter of an equilateral triangle that has an altitude of length 2.5 cm.



This time we need to find the length of the side but we know the height of the triangle. Let s be the length of the side. You can complete this one:

7. Assignment: Find a formula for the area of an equilateral triangle in which each side has length s.



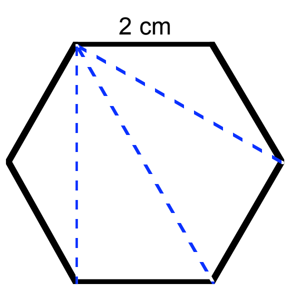
Lesson 4 Problems

1. Draw a picture to illustrate the Distributive Law:

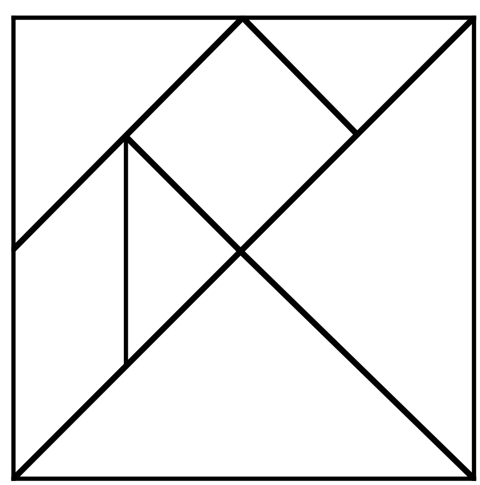
$$c \cdot (a + b) = c \cdot a + c \cdot b$$

2. Find the exact area of a square, given that the diagonal of the square has length 5 cm.

3. Find the exact lengths of all the diagonals in this regular hexagon.



4. Find the exact lengths of all sides of all the tangram pieces if the overall square has dimensions 4 inches by 4 inches. Label the lengths in this picture.



Identifying Identities

Review of rules for using square roots and a few other things

An **identity for real numbers** is an equation involving mathematical expressions that holds true when any real numbers are substituted for the variables. For example, $a+b=b+a$, is an identity that because commutative rule of addition holds for all real numbers.

On the other hand, $a+b=a$, is not an identity because it is only true if $b=0$. And $\sqrt{a^2}=a$ is not an identity because it is not true if any negative number is substituted for the variable a.

Which of the following are identities? For each identity, give a convincing argument that the equation is always true. For each equation that is not an identity, give examples when the identity is false.

1. $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$

2. $(a+b)^2=a^2+b^2$

3. $\sqrt{4ab}=2\sqrt{ab}$

4. $2(a\cdot b)=(2a\cdot 2b)$

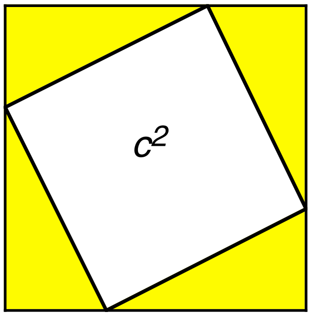
5. $3\cdot(a+b)=3\cdot a+3\cdot b$

6. $(a+b)^2=a^2+2a\cdot b+b^2$

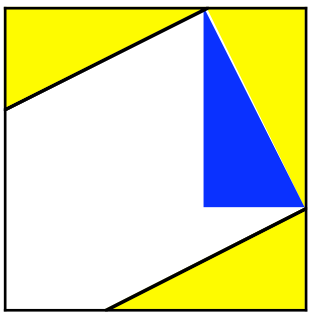
7. $4(a\cdot b)=(2a\cdot 2b)$

8. $\sqrt{a^2+2a\cdot b+b^2}=a+b$

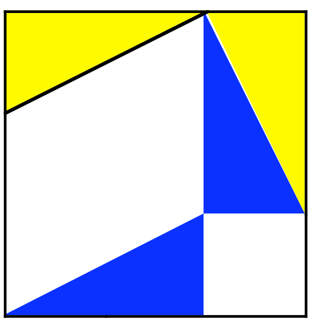
Another way to demonstrate the Pythagorean Theorem



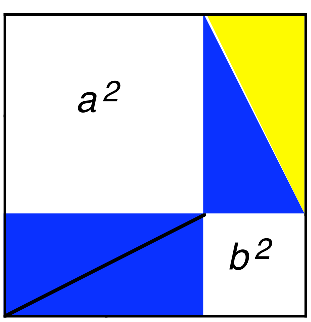
Start with the same configuration as you did in the lesson.



Slide the bottom left triangle up to meet the top right triangle as shown.



Slide the bottom right triangle to the left corner.



Slide the top left triangle down to join the bottom triangle.

8. Explain how this rearrangement proves the Pythagorean Theorem.

Lesson 5 – Reasoning about Circles

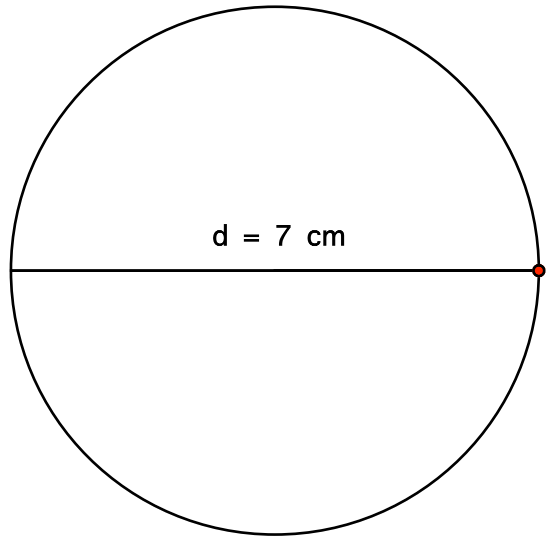
Class Activity: Circumference of a Circle

Anya has a garden and she wants to make a scale drawing of it. The garden is in the shape of a perfect circle. She measures the diameter of the garden. It is 7 feet. She measures the circumference of her garden. It is 22 feet.

She draws this scale drawing of her garden.

What is the circumference of her scale drawing?

Answer without measuring or calculating.



Diameter	Circumference
14 feet	
3.5 inches	
28 miles	
65 centimeters	
1 foot	
	6.6 feet
	17 meters
D units	

1. **Assignment:** Using only Anya's measurements determine the circumference or the diameter of each of the circles described in the table.

Many people do these problems using a proportional statement like:

$$\frac{C}{D} = \frac{22}{7}$$

2. **Assignment:** In your own words, explain why this proportion works.

Conclusions:
Because Anya was measuring, the circumference measurement may not be exact. Nonetheless, it is clear from our procedures that the ratio of the diameter of a circle to the circumference of the circle will always be the same number no matter what circle we consider or what units we use.

Mathematicians realized this fundamental principle long ago and gave a name to the number one gets by dividing the circumference by the diameter. They called it π (Pi, the Greek letter).

$$\boxed{\frac{\text{Circumference}}{\text{Diameter}} = \pi}$$

Interesting facts about π

If Anya's measurements were all we had to go on we would conclude that

$$\pi = \frac{22}{7} \approx 3.14$$

Not only are Anya's measurements not exact but it is impossible to be as exact, for mathematicians have discovered that pi is an irrational number, the decimal expansion of pi neither repeats nor stops. Last count someone found over one million digits. Here's a lot of them:

3.14159265358979323846264338327950288419716939937510582097494459
2307816406286208998628034825342117067982148086513282306647093844
6095505822317253594081284811174502841027019385211055596446229489
5493038196442881097566593344612847564823378678316527120190914564
8566923460348610454326648213393607260249141273724587006606315588
1748815209209628292540917153643678925903600113305305488204665213
8414695194151160943305727036575959195309218611738193261179310511
8548074462379962749567351885752724891227938183011949129833673362
4406566430860213949463952247371907021798609437027705392171762931
7675238467481846766940513200056812714526356082778577134275778960
9173637178721468440901224953430146549585371050792279689258923542
0199561121290219608640344181598136297747713099605187072113499999
9837297804995105973173281609631859502445945534690830264252230825
3344685035261931188171010003137838752886587533208381420617177669
1473035982534904287554687311595628638823537875937519577818577805
32171226806613001927876611195909216420199 . . .

3\. **Assignment:** How many digits are displayed?

4\. **Challenge:** Is the number displayed larger or smaller than pi?

5\. **Assignment:** Use the internet to find four interesting facts about pi and write them here in your own words, as you would tell a child:

Class Activity: Diameter vs Circumference

Complete this chart by measuring several circles. Use any tools you wish but do not use a calculator.

Object	Circumference	Diameter

Use this data to find an approximate value for π.

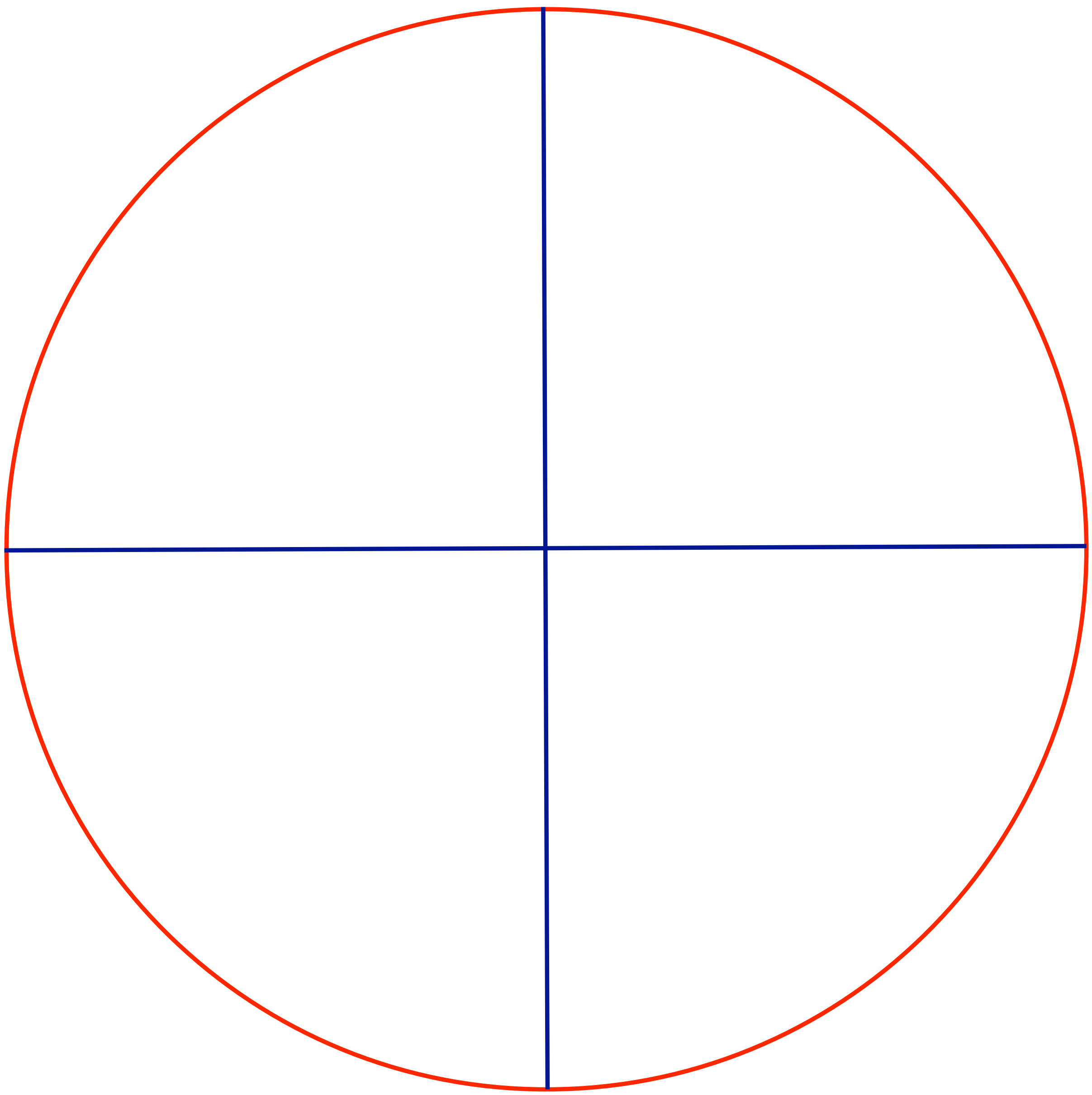
Group Project: "Tip to Tail" Tale, the Area of a Circle

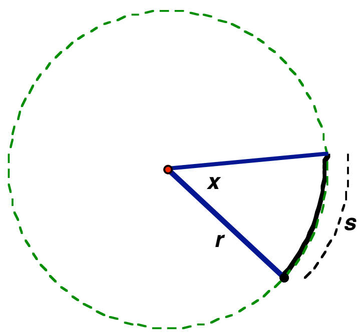
As shown on the next page, this is an activity that might be done in elementary school to help children understand the formula for the area of a circle:

$$A = \pi r^2$$

i. Color around the circle red. Color all radii blue.
ii. Cut out the four sectors. Place the four sectors "tip to tail" and estimate the area as if your figure were a rectangle.
iii. Draw another radius in blue bisecting each of the sectors and cut on the bisector. Place these eight sectors "tip to tail" and estimate the area again.
iv. Repeat with 16 sectors and again with 32 sectors

Write a paragraph explaining how the "tip to tail" tale you understand the formula for the area of a circle.





Areas and Arc lengths of sectors of circles
More formulas

A circle sector is like a piece of pie. It is determined by an angle, ***x***, and a radius, ***r***.

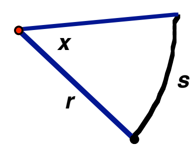
Here is a picture of a circle sector. Usually, the full circle is not included in a picture, but you can always sketch it in if it will help you understand a problem.

In this picture, ***s*** marks the length of the arc of the sector.

Our job in this lesson is to find formulas for **s** and a formula for the area of the sector.

First, use proportional reasoning to find a formula for the area of a circle sector in terms of ***r*** and ***x***. Call the area of the sector ***A***.

This should get you started: $\frac{x}{360} = \frac{A}{\text{area of full circle}} = \frac{A}{\pi r^2}$

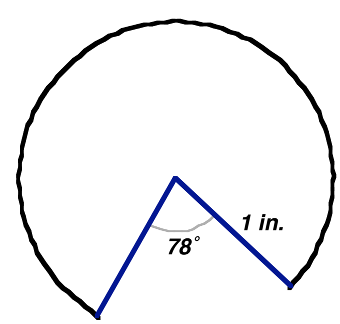


Area of sector = ***A*** *=*

Then, use proportional reasoning to find a formula for ***s*** in terms of ***r*** and ***x***.

Length of curve of the sector = ***s*** *=*

6. **Assignment:** Do your formulas still work for "Pacman" type sectors like this one? Explain why or why not. What is the total perimeter of this sector?



Lesson 5 Problems

1. Find circumference of a circle that has a diameter of length 11 cm. Give an answer accurate to 5 decimal places.

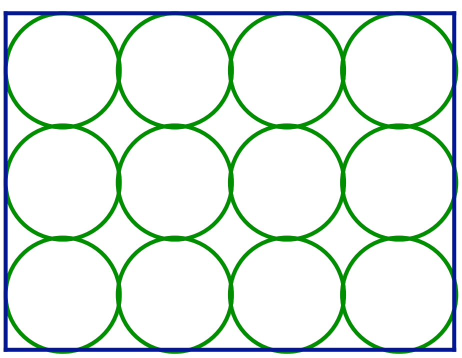
2. Find the diameter of a circle that has a circumference of length 22 inches. Give an answer accurate to 5 decimal places.

3. School children know how to make a circle by joining hands and stretching out as far as possible. Thirty first-graders want to make such a circle in their rectangular classroom, which is 32 feet by 40 feet. On the average each child's arm spread is 4 feet. Will their circle fit? What is the greatest number of children whose circle will fit in the room?

4. Find area of a circle that has a diameter of 7.4 meters.

5. Find the circumference of a circle that has area 15 square centimeters.

6. Find percent waste when twelve circles are cut from a rectangular sheet of metal. The radius of each circle is 2 cm.

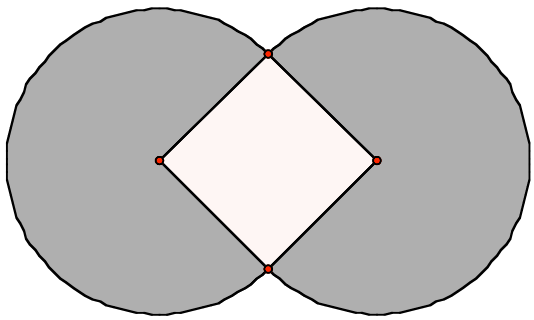


7. If you double the radius of a circle what happens to the diameter? The circumference? The area?

8. **Challenge:** Children's circle: Many children's games involve making a circle with some of the children while the rest gather in the middle of the circle. Suppose you want at least 16 square feet per child in the center. Suppose you have 30 children. What is the maximum number of children that you can put in the circle and still allow for this much room for each child?

9. Use centimeter graph paper and quarter-inch graph paper to find the area of a circle that has a radius of 10 centimeters. Compare your estimates with the exact value you get by using the formula.

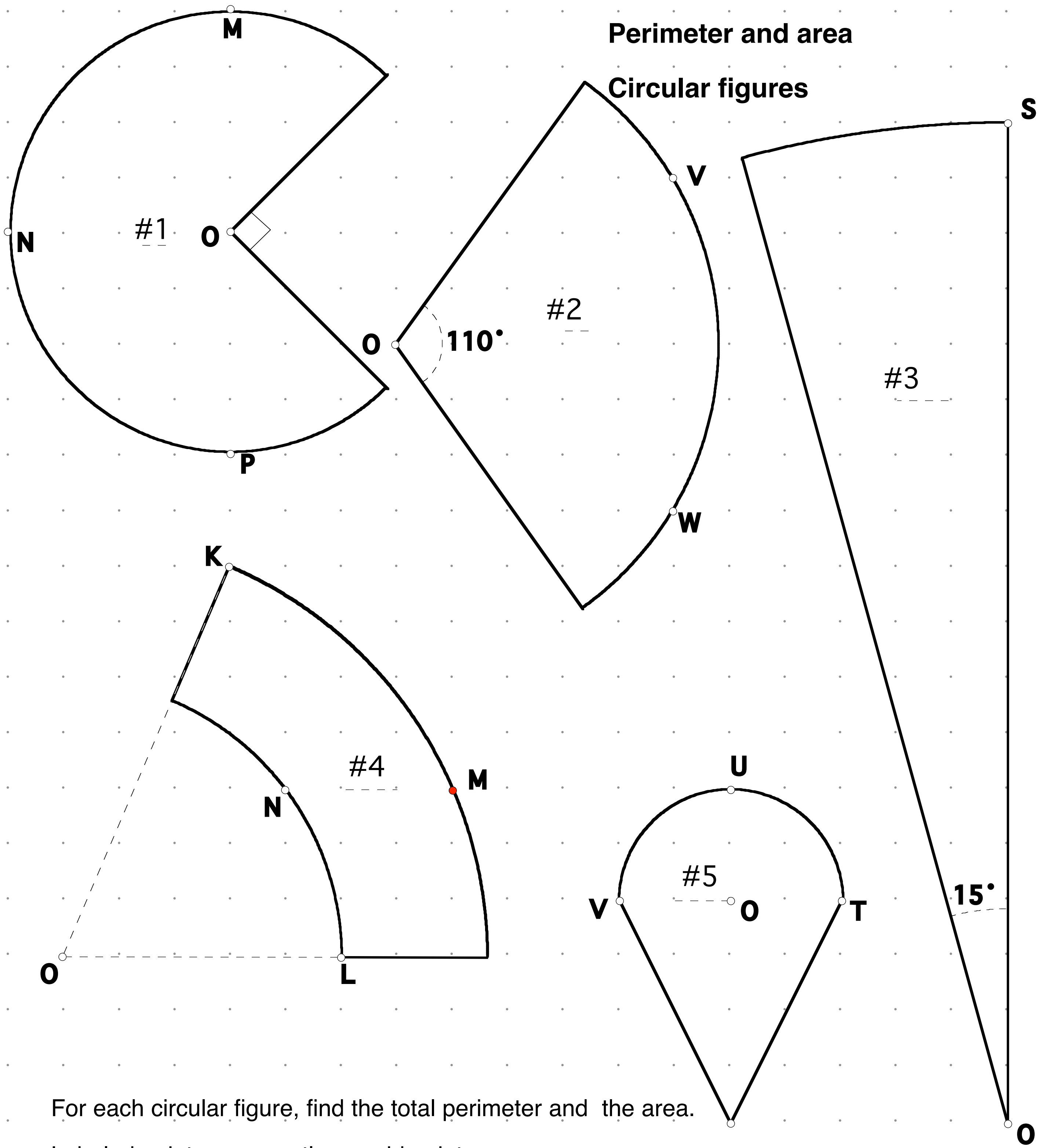
10. The area of the square shown below is 16 cm^2. Two of the vertices of the square are located at the centers of the circles. The other two vertices are on the intersection points of the two circles. Find the area of the shaded region.



11. Find the area and parameter of all the figures on the next page.

Perimeter and area

Circular figures



For each circular figure, find the total perimeter and the area.

Labeled points are exactly on grid points.

Express your answers exactly and
as a decimal number, correct to five decimal points

Lesson 6 – Grazing Goat Geometry: Putting it all Together

This lesson which concludes the unit on measurements of length and area is based on lessons from a middle school mathematics workbook called *Maneuvers With Circles.* This workbook is one of a series, *Maneuvers With Mathematics* developed by David Page and Kathyrn Chval and Phil Wagreich at the University of Illinois at Chicago as part of the NSF call to write exemplary school mathematics in the wake of the NCTM Standards..

Before we begin a word about calculators and accuracy
Maneuvers Wiith Mathematics was among the first school mathematics projects to integrate the use of calculators in a middle school curriculum. Among other things, the authors devised a method of using the calculator to make the exercises “self correcting.” By providing “calculator” boxes for the answers, the students can check to see if their answers are possibly correct: in most cases, the book provides the location of the decimal point and two of the digits in the answer. Most likely, if these three things agree with a student’s answer the answer is correct and the student can proceed to the next problem.

Introduction to Goat Geometry

It’s time to put together some of the ideas we have been discussing about shapes and area. David Page believed that a good imagery scenario worked as well if not better than a real world problem in helping children (and many adults) to understand mathematics. Imagine this scenario. A farmer uses a long rope or tether to tie his goat, Billy, to a stake so that Billy can graze. While Billy eats the grass, the farmer notices that the amount of grass Billy eats is really a geometry problem.

1. **Assignment:** Use the your geometry tools to picture the following grazing goat problems.

The grass Billy can eat if the farmer ties Billy in the middle of a field with a 15 foot tether.

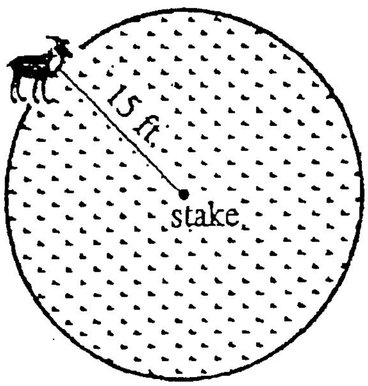
The grass Billy can eat if the farmer ties Billy to a fence with a 15 feet tether.

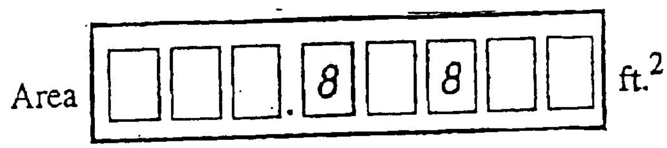
The grass Billy can eat if the farmer ties Billy to a corner of a barn with a 15 feet tether.

Here are some pictures suggested in *Maneuvers with Circles* for these three problems

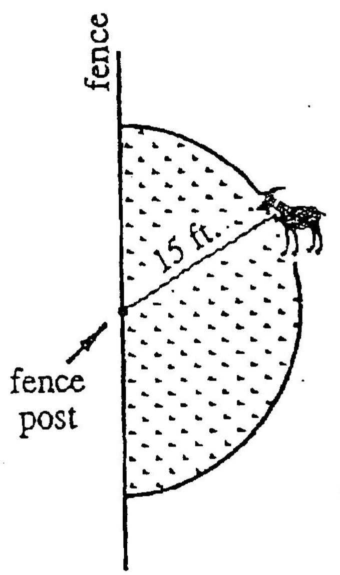
2. Assignment: Complete the computations for these problems.

a)

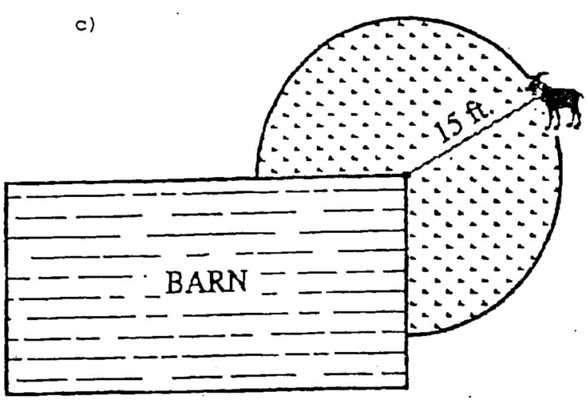


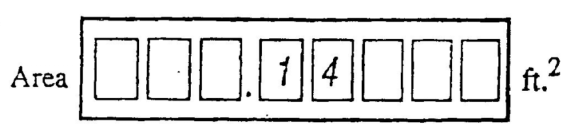


b)



c)

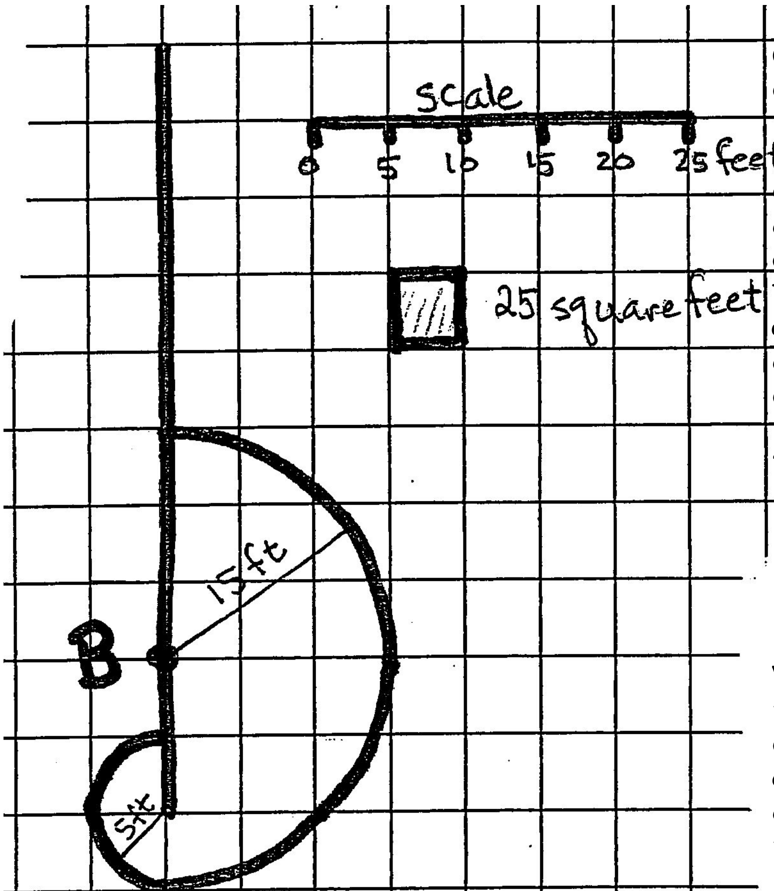




Sample problem: Note that only 1 picture is provided for problem b), but there could be many different pictures. What if the fence is 50 ft. long and Billy is tied 10 ft. from the end of the fence with his 15 feet long rope?

3. Assignment: Decide where Billy could eat and use your compass to make a picture. Finally compute the maximum grazing area for Billy.

Solution:



Our drawing includes a scale. We chose 5 feet for every one centimeter.

We draw the fence to be 10 centimeters (50 feet) and tether Billy at point **B**, 2 cm (10 feet) from one end.

Then with a compass we carefully draw the boundaries of Billy's grazing area, which consists of two half-circles.

4. Explain how we chose the centers and radii for the two circles.

We **estimate** the area, by counting the centimeter squares. Each square centimeter represents a 5-foot square or 25 sq ft. Since we get approximately 15 squares, we estimate the area to be

15 squares X 25 sq ft per square= 375 sq ft

We get a more **accurate** answer by using the formula for circles and computing with a calculator:

$$\frac{1}{2}\pi 15^2 + \frac{1}{2}\pi 5^2 = 392.699\ sq.ft.$$

5. Assignment: Can you get this same answer using your calculator?

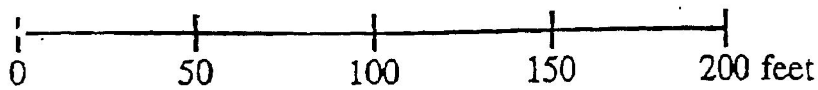
Next we simplify the expression to get the **exact** answer: $\frac{1}{2}\pi 15^2 + \frac{1}{2}\pi 5^2 = 125\pi$

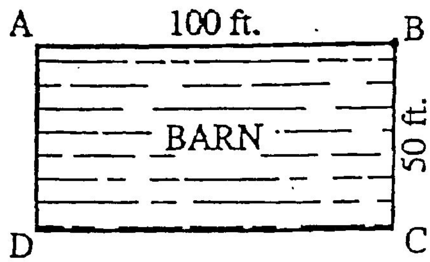
Finally, we notice that our estimate is within 25 square feet of the exact answer: $125\pi - 375 < 25$

6. Assignment: What do you get on your calculator for $125\pi - 375$?

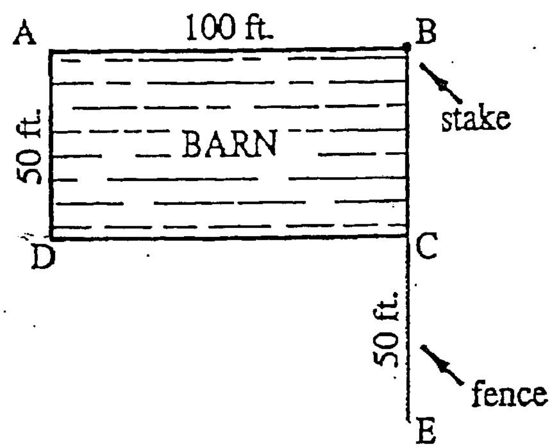
Since, with the scale we used, that's within one square centimeter, our answer checks with our initial estimate. We feel confident about our solution.

1. **Assignment:** If Billy is tied to point B on the barn below with a rope 150 feet long. Use your compass to draw Billy's path and then calculate his maximum grazing area.



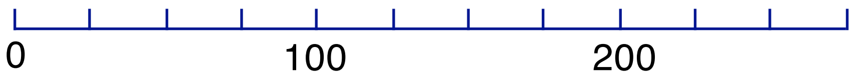


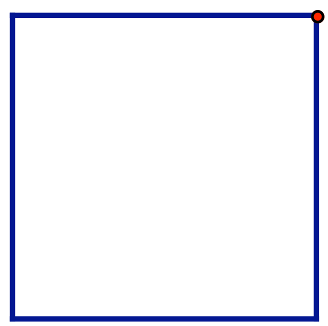
2. **Assignment:** Now there is a fence 50 feet long connected to barn. If Billy is tied to B with a 150 rope feet long, will his maximum grazing area be the same as, less than, or greater than the area in problem 1? Justify your answer.



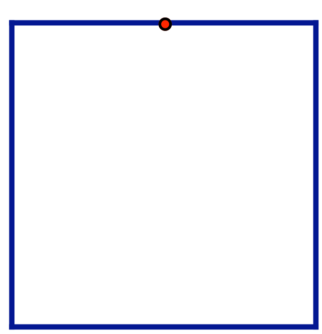
Lesson 6 Problems:

1. Billy is tied to the corner of a barn with a rope 200 feet long. The barn is a square that is 100 feet on each side. Use your compass to draw Billy's path and then calculate his maximum grazing area.

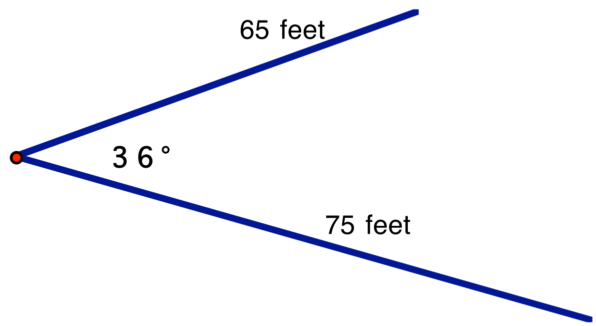




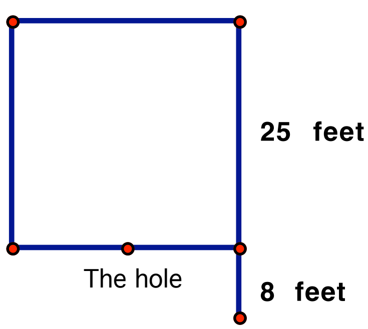
2. Billy is tied to the midpoint of the same barn with a rope 200 feet long. Use your compass to draw Billy's path and then calculate his maximum grazing area.



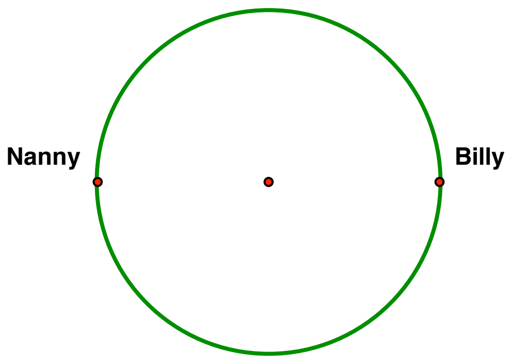
3. Two fences meet forming an angle that measures 36˚. Billy is tethered at angle vertex on the outside of the angle. His tether is 20 feet long. The farmer would like to graze his other goat, Nanny, on the inside of the angle. He wants to make sure that each goat gets the same amount of grass. What length should he use for Nanny's tether?



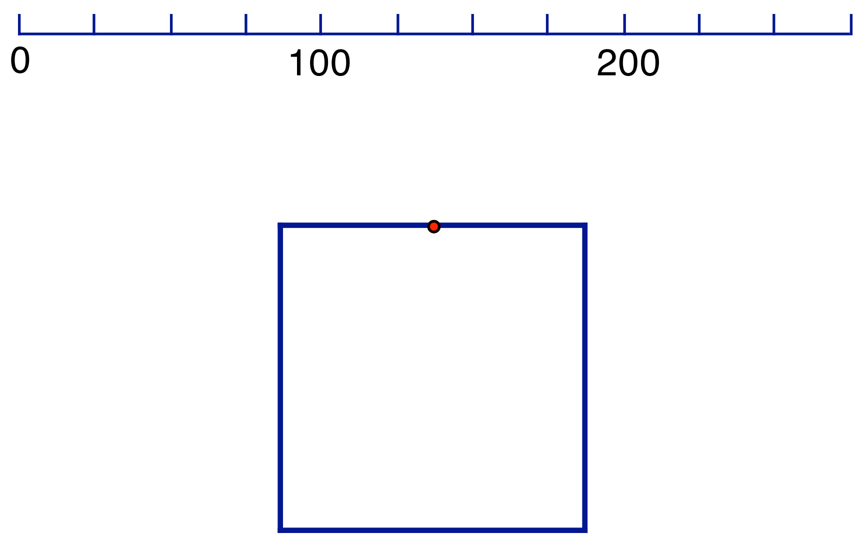
4. Billy is tethered to the end of a fence that extends out from one corner of Mr. McGregor's square garden, as pictured below. While happily grazing he finds a hole in the fence half-way between the two corners just big enough for him to slip inside. Are there any places in the garden where Billy won't be able to graze? Explain. The garden is 25 feet square; the extension is 8 feet long; and Billy's tether is 35 feet.



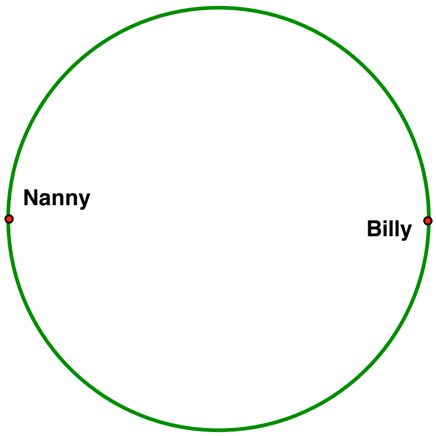
5. The farmer has a circular silo that has a diameter of 30 feet. He tethers Nanny to the outside of the silo with a rope that is 15 feet long. He tethers Billy exactly opposite Nanny. What is the longest rope he can use so that Billy and Nanny don't cross paths?



6. Billy is tied to the midpoint of a barn with a rope 250 feet long. The barn is a square that is 100 feet on each side. Use your compass to draw Billy's path and then calculate his maximum grazing area.



7. The farmer has a circular pen with a 50-foot diameter. He tethers Billy to one end of a diameter of the pen and he tethers Nanny to the other end of that diameter. Both are inside the pen. Both have tethers that are 25 feet long. Draw the maximum grazing area of the two goats. What area is left ungrazed?



8. Design your own grazing goat problem so that the maximum grazing area is between 900 sq. ft. and 1000 sq. ft. Include a proper picture and the solution as well as the statement of the problem.

Unit #2 Problems

1. George is building a large model airplane in his workshop. If the door to his workshop is 3 feet wide and 6 ½ feet high and the airplane has a wingspan of 7.1 feet, will George be able to get his airplane out of the workshop?

2. Jason has an old trunk that is 16 inches wide, 30 inches long, and 12 inches high. Which of the following objects would b\he be able to store in his trunk? Explain your answers.

a) a telescope measuring 40 inches
b) a baseball bat measuring 34 inches
c) a tennis racket measuring 32 inches

3. A rectangle whose length is 3 cm more than its width has an area of 40 square centimeters. Find the length and the width of the rectangle.

4. Given are the lengths of the sides of a triangle. Indicate whether each triangle is a right triangle, an acute triangle, or an obtuse triangle.

a) 70, 54, 90	b) 63, 16, 65	c) 24, 48, 52
d) 27, 36, 45	e) 48, 46, 50	f) 9, 40, 46

5. Which is the better buy – a 9 centimeter diameter circular pie that costs $7.00 or a square pie 9 centimeters on each side that costs $9.00?

6. A restaurant chain has sold over 80 billion hamburgers. A hamburger is about one-half inch thick. If the moon is 240,000 miles away, what percent of the distance to the moon is the height of a stack of 80 billion hamburgers?

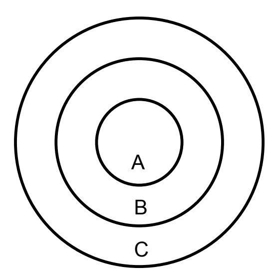
7. Suppose that every week the average American eats one-fourth of a pizza. The average pizza has a diameter of 14 inches and costs $8.00. There are about 250,000,000 Americans, and there are 640 acres in a square mile.

   a) About how many acres of pizza do Americans eat every week?
   b) What is the cost per acre of pizza in America?

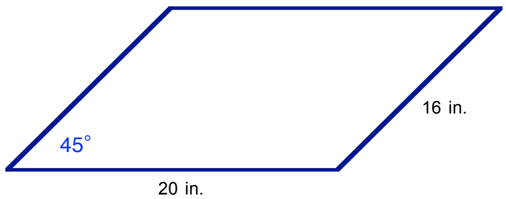
8. In the dartboard shown, the radius of circle A is 1, of circle B is 2, and of circle C is 3. Hitting A is worth 20 points, hitting region B is worth 10 points, and hitting region C is worth 5 points.

   a) Is this a fair dartboard? Justify your answer.

   b) What point structure would make it a fair board if region A is worth 30 points?

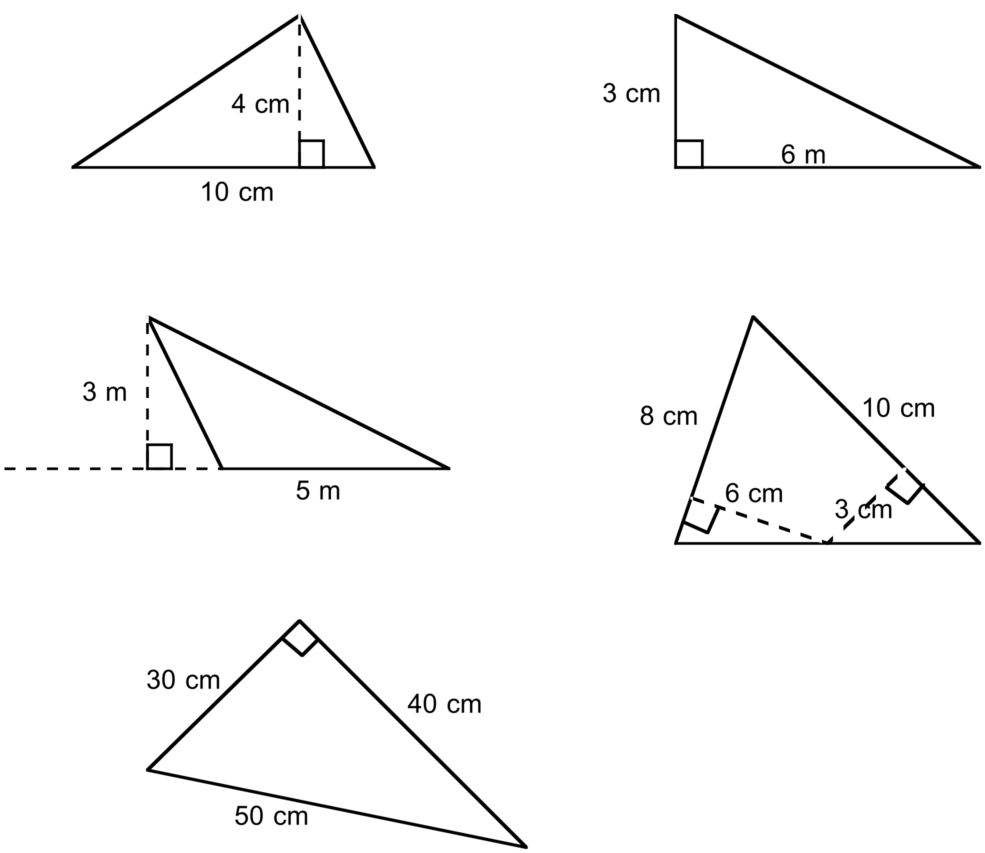


9. Larry says the area of a parallelogram can be found by multiplying length times width. So the area of the parallelogram below must be 20 x 16 = 320 sq. in. Do you agree? If not, what could you do to give Larry an intuitive feeling about its area? Is it possible to find the exact area in this case? Explain.

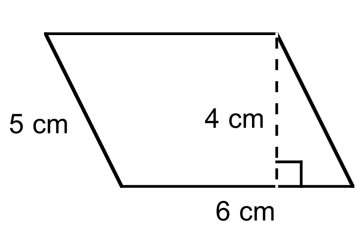


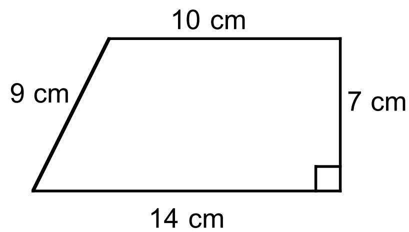
10. A student has a tennis can containing three tennis balls. To the students' surprise, the perimeter of the top of the can is longer than the height of the cen. The student wants to know if this fact can be explained without performing any measurements. Can you help?

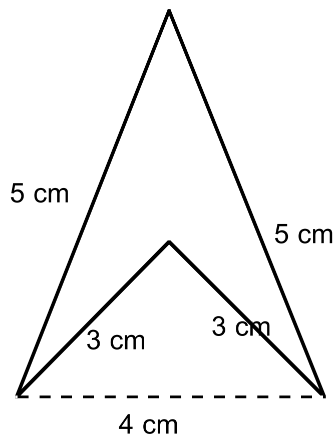
11. Find the area of the indicated triangles. They are NOT necessarily drawn to scale.

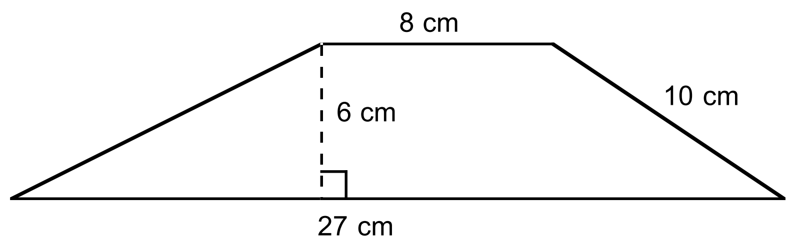


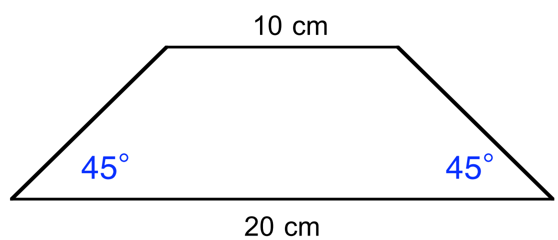
12. Find the area of each of the following quadrilaterals:







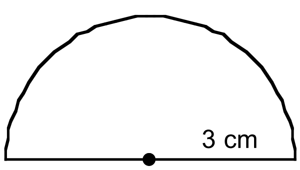


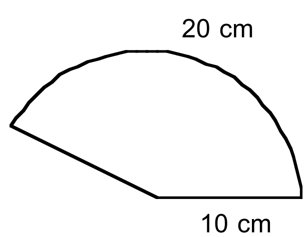


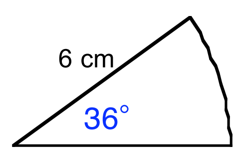
13. Which room costs more to carpet:

a) Room dimensions: 6.5 m by 4.5 m and carpet cost = $13.85 per square yard
b) Room dimensions: 15 ft by 11 ft and cost = $30 per square meter

14. Find the area of each of the following. Leave your answers in terms of π. Which of these drawings drawn to scale?





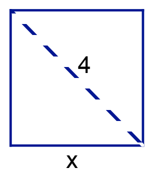


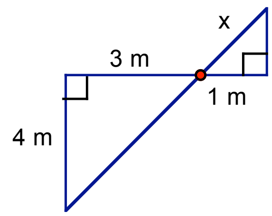
a) an equilateral triangle with side length s.
b) a regular hexagon with side length 4 inches.

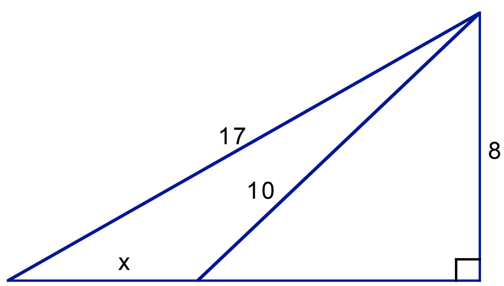
16. The screens of two television sets are <u>similar</u> rectangles. The 20-in set costs $400, and the 27-in set with similar features costs $600. The dimensions given are the diagonal lengths of the sets. If a customer is concerned about the size of the viewing area and is willing to pay the same amount per square foot, which is a better buy?

17. Find x in each of the following:

a square with diagonal of length 4:







18. Two cars leave a house at the same time. One car travels 60 km/hr north, while the other car travels 40 km/hr east. After 1 hr, how far apart are the cars?

19. Two airplanes depart from the same place at 2:00 p.m. One plane flies south at a speed of 376 km/hr, and the other flies west at a speed of 648 km/hr. How far apart are the airplanes at 5:30 p.m?

20. Starting from point A, a boat sails due south for 6 mi, then due east for 5 mi, and then due south for 4 mi. How far is the boat from point A?

21. A wire 10 m long is wrapped around a circular region. If the wire fits exactly, what is the area of the region?

22. Suppose a wire is stretched tightly around Earth. The radius of Earth is approximately 6400 km. If the wire is cut, its length is increased by 20 m, and the wire is then placed back around the earth so that the wire is the same distance from Earth at every point,m could you walk under the wire

Unit #3 3D Geometry and other Applications

Table of contents for Unit #3

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Lesson 2 - Classification of 3D shapes This lesson includes a variety of ways of constructing and naming 3D shapes.

Lesson 3 - Surface Area

Lesson 4 - Volume
The students will solve a puzzle and use algebra to construct a proof of the Pythagorean Theorem. The converse of the Pythagorean Theorem is also proven. More Pythagorean puzzles are given.

Lesson 5 - Putting it all together: Two Activities
This lesson contains two activities toexplore the relationship between surface area and volume.

Lesson 1 – Indirect Measurement

Indirect measurement covers many different techniques that might be used to measure distances or lengths that cannot be measured directly. For example, how would you go about measuring the height of a tree if you couldn't climb to the top of it?

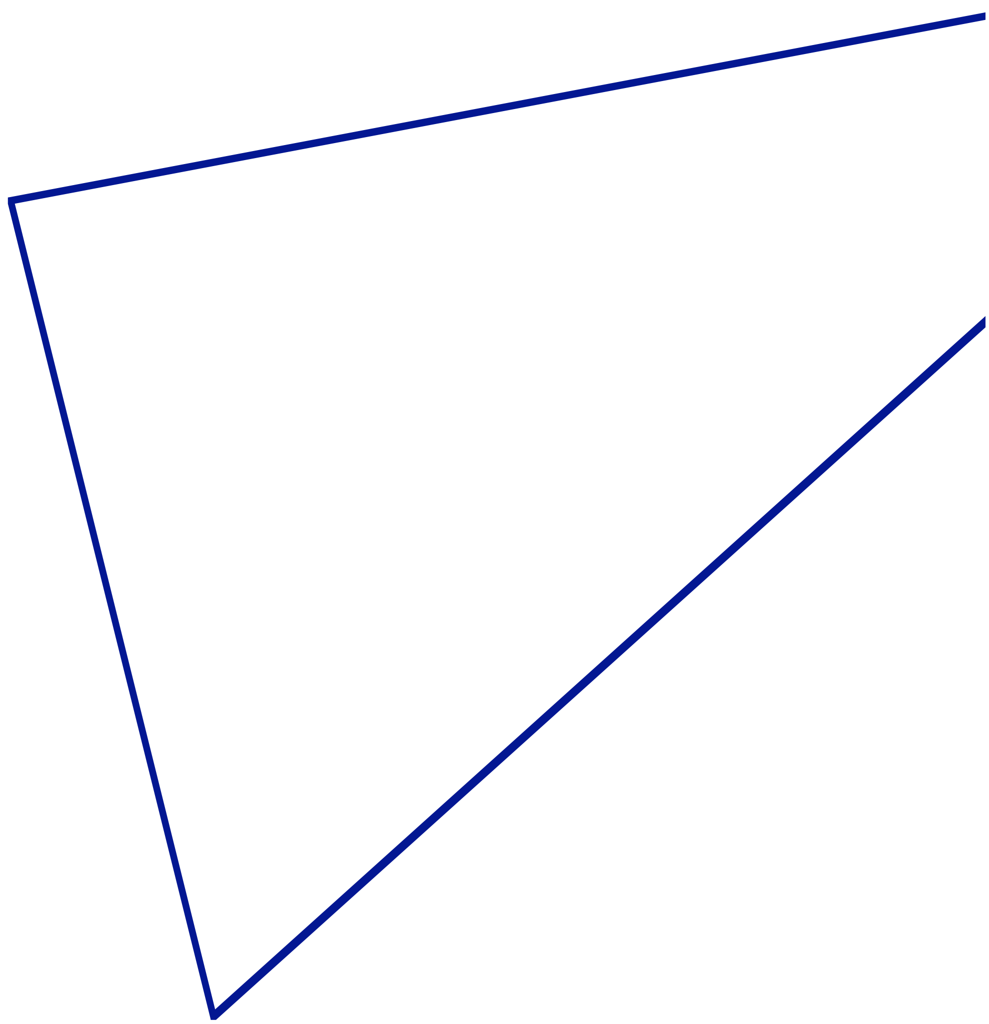
The first technique we will study is triangle trigonometry. You may or may not have studied trigonometry before but it is actually quite easy to understand once you have a good sense of similar triangles.

The second part of this lesson is an experiment from the TIMS (Teaching Integrated Math and Science) Labs. In this lab you will make an instrument for measuring heights that you otherwise might not be able to measure.

In both case our experiences with proportional reasoning and reasoning about similar figures, particularly triangles, will be invaluable.

Here is another problem from Didactique to get us started:

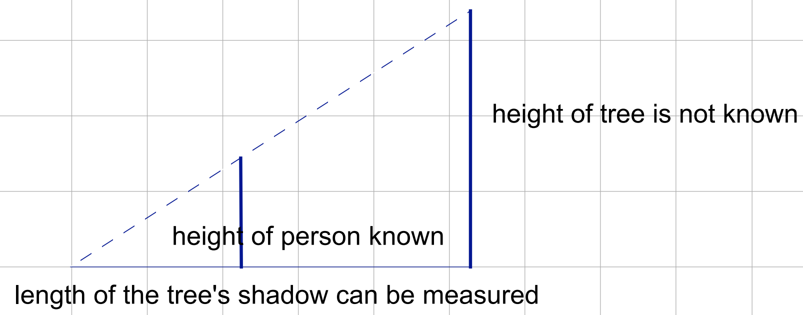
1. **Assignment:** Without measuring anything outside of this piece of paper, find the lengths of each side of this triangle. You can only see part of the triangle on this page.



Indirect Measurements

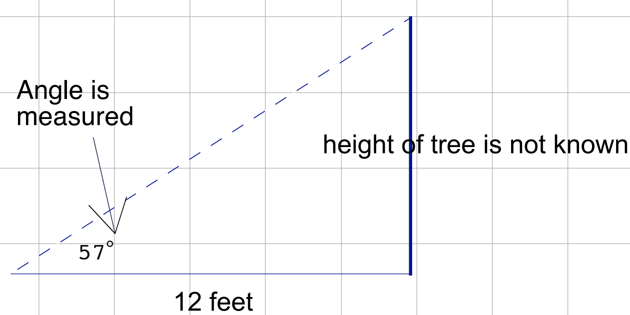
In this first problem we show how to find the height of a tree indirectly, by measuring shadows and solving proportions.

warning: drawings on this page may NOT be to scale.



1. Suppose the tree's shadow is measured to be 10 feet and the person's shadow is 3 feet. The person is 5 feet tall. How tall is the tree?

Another way to find the height of a tree is to measure angles:



Distance from tree where angle measurement is made can be measured.

2. Make a scale drawing of this situation and find the height of the tree.

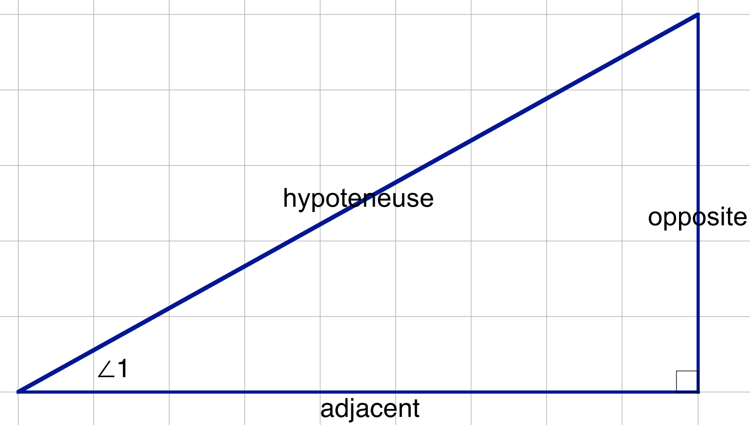
Trigonometry is Easy!

Definitions of trig functions:

$$\sin(\angle 1) = \frac{\text{opposite}}{\text{hypoteneuse}} = \frac{\quad}{\quad}$$

$$\cos(\angle 1) = \frac{\text{adjacent}}{\text{hypoteneuse}} = \frac{\quad}{\quad}$$

$$\tan(\angle 1) = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{9} \sim 0.5556$$



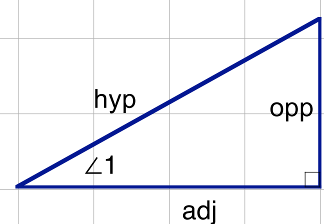
Reminder “Sock-ah-to-ya” or SOH-CAH-TOA:

Sin - Opposite over Hypotenuse

Cos - Adjacent over Hypotenuse

Tan - Opposite over Adjacent

Any right triangle that has an angle congruent to ∠1 is similar to this triangle, so that the ratios of corresponding sides will be the same. Therefore, any right triangle with an angle congruent to ∠1 may be used to compute the ratios:

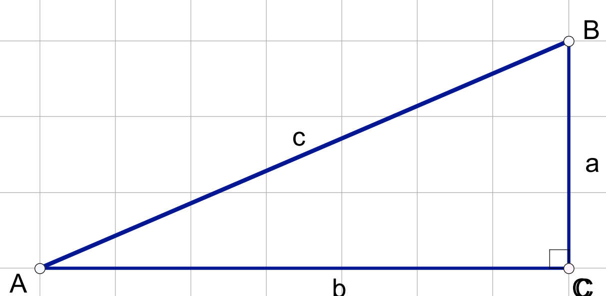


$$\sin(\angle 1) = \frac{\quad}{\quad}$$

$$\cos(\angle 1) = \frac{\quad}{\quad}$$

$$\tan(\angle 1) = \frac{2.2}{4} \sim 0.55$$

Here's another way to give the same definitons with letters. Notice how the side opposite an angle is labeled with the lower case letter. This is a widely used convention



$$\sin(\angle A) = \frac{a}{c} \qquad \sin(\angle B) = \frac{b}{c}$$

$$\cos(\angle A) = \frac{b}{c} \qquad \cos(\angle B) = \frac{a}{c}$$

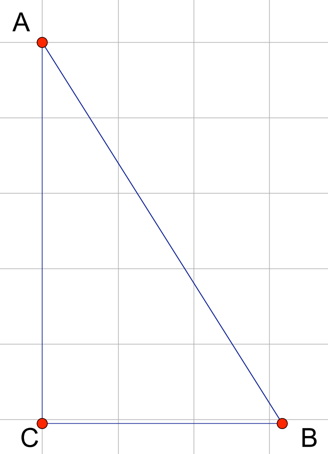
$$\tan(\angle A) = \frac{a}{b} \qquad \tan(\angle B) = \frac{b}{a}$$

Notice that we are not defining trig function for ∠ C, the right angle.

1. What are some of the relationships you notice among these six functions?

Trigonometry Practice

The triangles sketched below are NOT drawn to scale. If possible, make a scale drawing and estimate the answer first. Then use trigonometry to find the answer rounded to the3 decimal places. Finally, use the Pythagorean theorem to check your answers.



$\angle A = 65^\circ$

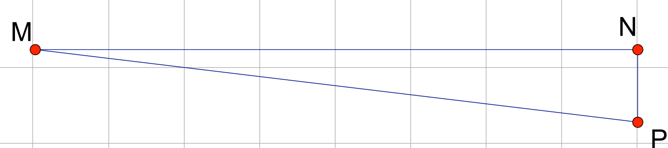
$\angle B =$

$\angle C = 90^\circ$

$a =$

$b =$

$c = 8$ cm



$\angle M = 22^\circ$

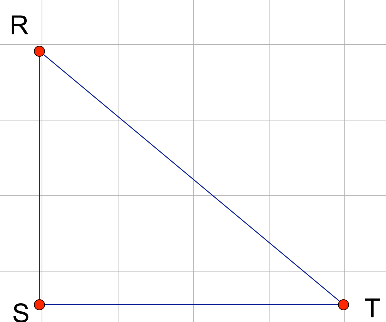
$\angle P =$

$\angle N = 90^\circ$

$m =$

$n =$

$p = 3$ feet



$\angle S = 90^\circ$

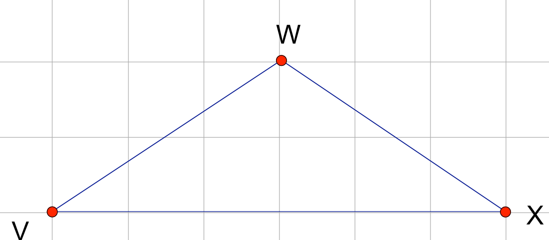
$\angle R =$

$\angle T =$

$t = 4$ cm

$r =$

$s = 5$ cm



$w = 10$ cm

$x = 6$ cm

$v = 6$ cm

More Trigonometry Practice

In problems 1- 4, sketch the indicated triangle, then use trigonometry to find the missing parts exactly.

1. $\angle A = 37°$

   $\angle B =$

   $\angle C = 90°$

   a = 20 inches

   b =

   c =

2. $\angle M = 60°$

   $\angle P =$

   $\angle Q = 90°$

   m =

   p =

   q = 3 feet

3. $\angle S = 90°$

   $\angle R =$

   $\angle T =$

   t = 7 cm

   r = 3 cm

   s =

4. $\angle A =$

   $\angle B =$

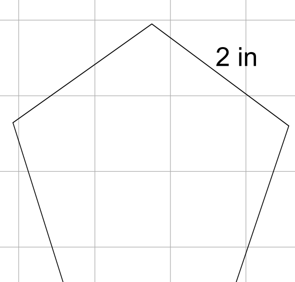
   $\angle C =$

   a = 7 cm

   b = 3 cm

   c = 3 cm

5. **Challenge:** Find the length of a diagonal in a regular pentagon that has sides 2" in length.



Trigonometry Word Problems

Draw a diagram and write an trigonometric equation to solve each problem.

1. The angle of elevation of a 15-ft. ladder is 70°. Find out how far the base is from the wall.

2. If a ladder 11 meters long rests with one end against the wall of a building and with the other end on the ground 4 meters from the foot of the wall, what angle does it make with the ground?

3. Determine the height of a tree if it casts a shadow 7 m long on level ground when the angle of elevation of the sun is 50°.

4. When a fence post 1.7 meters high casts a shadow 2.3 meters long, what is the angle of elevation to the sun?

5. How many feet of cable will it take to anchor a guy wire to a 30-ft pole if the angle of elevation is 28°?

Trigonometry Word Problems *continued*

Draw a diagram and write an trigonometricequation to solve each problem.

6. A rectangular plot of ground is 28 meters wide and 77 meters long. If a diagonal path is laid out across the plot, what angle does it make with the longer side?

7. As a plane takes off, it flies 1500 ft along a straight path and rises at an angle measuring 22°. What is its vertical rise when it has flown 1500 ft? At the same time, how far has it moved horizontally?

8. What is the angle of elevation to the sun when the shadow cast by a post is two-thirds as long as the post?

9. A gutter cleaner wants to reach a gutter 40 ft above the ground. It is dangerous to tilt a ladder more than 15° from the vertical. Find the length of the shortest ladder that can be used to reach the gutters.

10. A jet plane cruising at 450 mph climbs at an angle measuring 13°. Determine how much altitude the jet gains in 5 min.

Name______________________________ Date _________________________

View Tube

Picture

Draw a labeled picture of the experimental setup.
Call the distance from the meterstick to the eye L.
Call the height of the field of view H.

1. Which is the manipulated variable? _________________________________
2. Which is the responding variable? _________________________________
3. What variables shoul be held fixed during the experiment? _____________

____

____

____

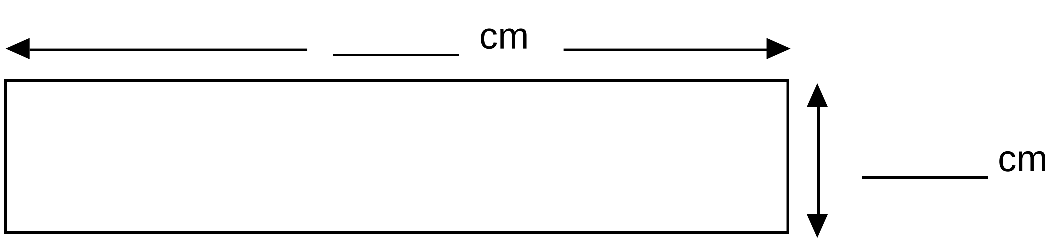
Data Table

Work with your partner to gather your data.
Record your data in the table below.

Table I

L in ______	H in ________________			
	Trial 1	Trial 2	Trial 3	Average
100				
200				

Measure the length and diameter of your view tube.
Record the measurements on the picture below.



View Tube Dimensions
Side View

Graph

Plot your data on a sheet of graph paper. Remember to label each axis and write in proper units. Draw a best-fit curve for your data.

4. On which axis (horizontal or vertical) did you plot H?

   ____

5. On which axis (horizontal or vertical) did you plot L?

   ____

6. Is the curve a straight line? ______________________________

   Why or why not? ______________________________

   ____

   ____

Comprehension Questions

7. Is it important that the same person look through the view tube for the whole experiment?

   ____

   Why or why not? ______________________________

   ____

   ____

8. Use your graph to predict how much of the meterstick you could see if you stood 150 cm from the meterstick.

   ____

   Check your prediction. How much can you see? ______________________________

   Was your prediction close? ______________________________

9. Use your graph again. Tell how many centimeters you could see if you stood 5 meters away.

   ____

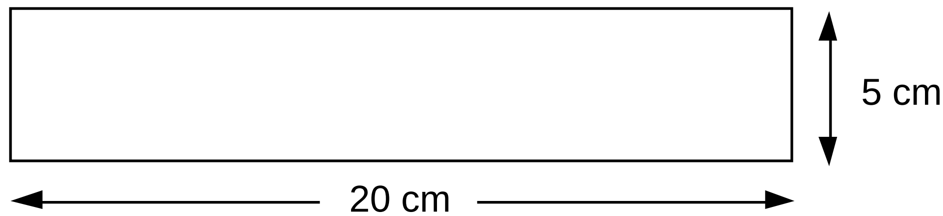
10. If you can see 80 cm of the meterstick, predict about how far away you are from the meterstick.

____

Check your prediction. How far are you from the meterstick? __________

Was your prediction close? ______________________________________

11. Sally has the view tube shown below.
On your graph, plot what you think her curve will look like.



SIMPLE PROPORTIONAL REASONING

12. Use your data and proportional reasoning to predict how many cm you would see if you were 6 meters from the meterstick. Show all your work.

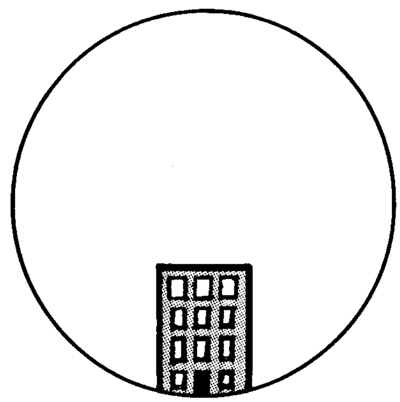
____

13. At the zoo you find that you can just see all of the giraffe when you step back 18 meters from the fence where the giraffe is standing.

    How tall is the giraffe? Show your work.

    ____

14. You want to know how far you are from the public library. You look through your view tube and see that the library covers only $\frac{1}{3}$ of the field of view of the view tube as shown at the right. The library has 4 stories and each story is about 4 meters tall.



    Use your data and proportional reasoning to predict how far you are from the library. Show your work.

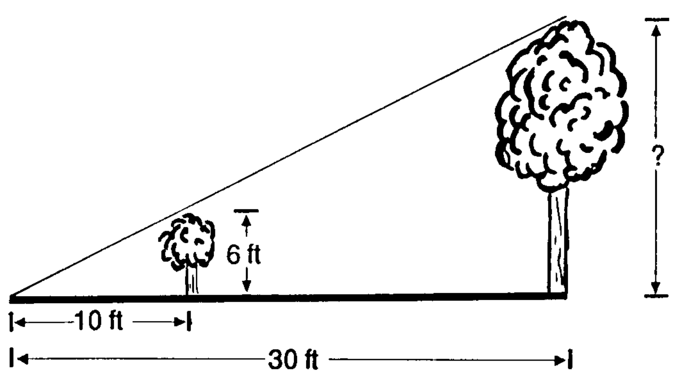
    ____

15. Brian can see 25 cm of his meterstick through his view tube when he is 1 meter from it. Brian goes to the aquarium with his class. If Willie the Whale is 15 meters long and Brian can just see all of Willie in his view tube, how far is Brian from Willie? Show your work.

____

16. Use your experience with the view tube to find the height of the tall tree in the picture below. Show your work.

____



17. You have a new view tube with the same length as your toilet paper roll but it has twice the diameter.

    a. What happens to your new field of view?

    ____

    ____

    b. How much would you see at 1, 2, and 3 meters?

    1 m ____________ 2 m ____________ 3 m ____________

18. You have a new view tube with the same diameter as your toilet paper roll but it has twice the length.

    a. What happens to your new field of view?

    ____

    ____

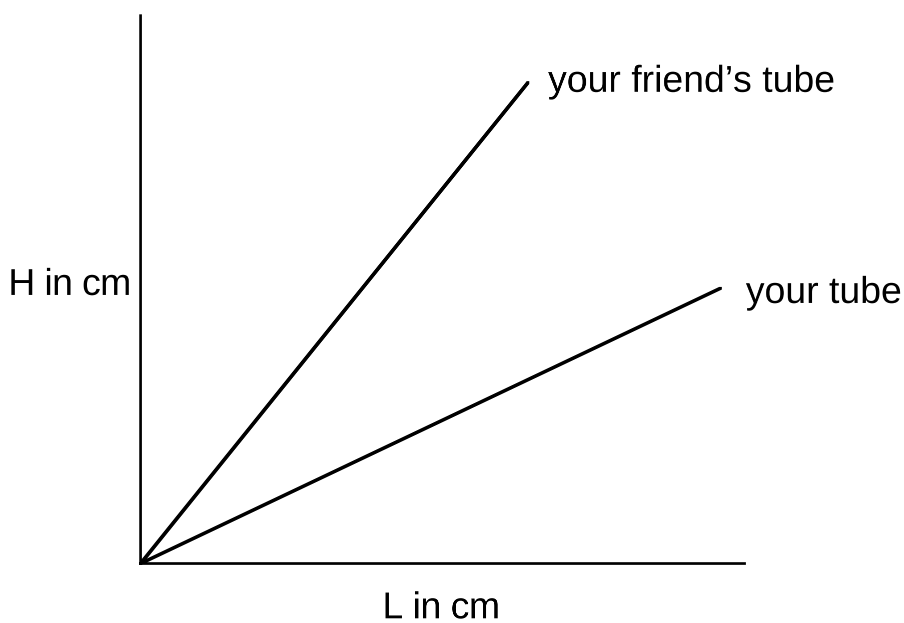
    b. How much would you see at 1, 2, and 3 meters?

    1 m ____________ 2 m ____________ 3 m ____________

19. You have a new view tube with twice the length and half the diameter of your toilet paper roll. How much would you see at 1, 2, and 3 meters?

1 m _______________ 2 m _______________ 3 m _______________

20. A friend tells you that he has a view tube too.
The graph of the curves for both your view tubes is shown below.



How might the dimensions of your friend's view tube differ from yours?

____

____

____

____

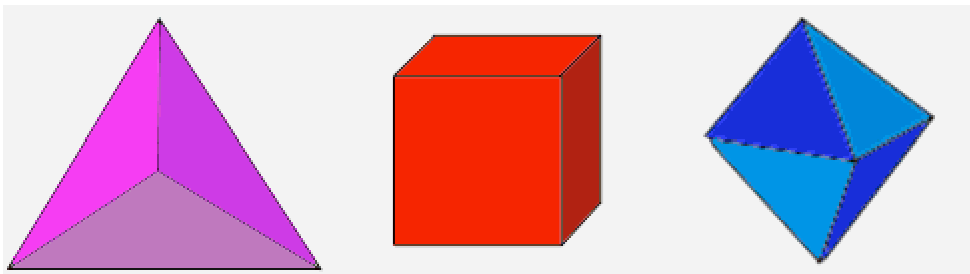
Lesson 2 Classification of 3D shapes

Regular and semi-regular polyhedra

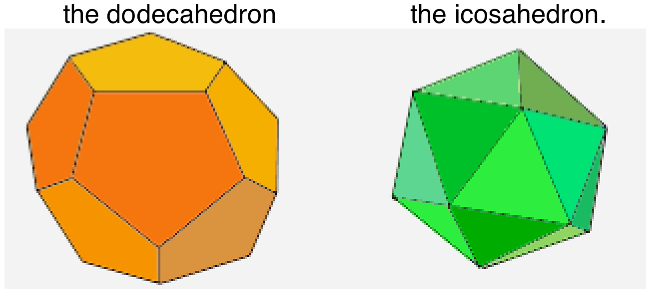
Do not hesitate to do some research to complete this lesson. There are many web sites that have good pictures and animations of ***regular*** and ***semiregular polyhedra***.

There are only 5 *regular polyhedra.* They are also called ***Platonic solids***

the tetrahedron the cube the octahedron,



the dodecahedron the icosahedron.


[1]

1. Write a definition for a regular polyhedron.

2. Construct your own regular polyhedra models.

   a) Construct a closed tetrahedron using 4 equilateral triangles.
   b) Construct a closed cube using 6 squares.
   c) Construct a closed octahedron using 8 equilateral triangles.
   d) Construct a closed dodecahedron using 12 regular pentagons.
   e) Construct a closed icosahedron using 20 equilateral triangles.

3. What happens when you try to construct a regular polyhedron using regular hexagons? What about a regular 7-gons?

[1] Pictures from www.math.hmc.edu/funfacts hosted by Harvard Mudd College Math Department

4. For each solid, place a fingertip on a vertex and try to trace or "redraw" all the edges without lifting your finger and without tracing any edges more than once. For which solids can you successfully do this?

   Tetrahedron cube octahedron dodecagon icosahedron

5. Color your Platonic solids so that no two faces with a common edge have the same color. What is the smallest number of colors you need to color the following Platonic solids?

   Tetrahedron cube octahedron

Sometimes open or "see through" models can allow you to see polyhedra properties more easily.

6. Use wire, straws, marshmallows, gummy bears, raisins, toothpicks, etc. to create an open model for the 5 regular polyhedra. Be creative!

   a) Construct an open tetrahedron.
   b) Construct an open cube.
   c) Construct an open octahedron.
   d) Construct an open dodecahedron.
   e) Construct an open icosahedron.

7. Use either set of models to complete the table below:

Solid	Number of faces	Number of vertices	Number of edges
Tetrahedron			
Cube			
Octahedron			
Dodecahedron			
Icosahedron			

8. Do you see any patterns in the table above? Write some observations, conjectures, etc. here.

The platonic solids have fascinated mathematicians for a long time. Johann Kepler was one of the early mathematicians who studied solids systematically. However, he was not above mysticism. In his 1619 book, *Harmonice Mundi*, Kepler assigned each solid to one of the classical elements: earth, air, fire, water, and the universe.

9. Use your own reasoning to match each platonic solid to one of the elements. Explain your thinking.

Tetrahedron

Cube

Octahedron

Dodecahedron

Icosahedron

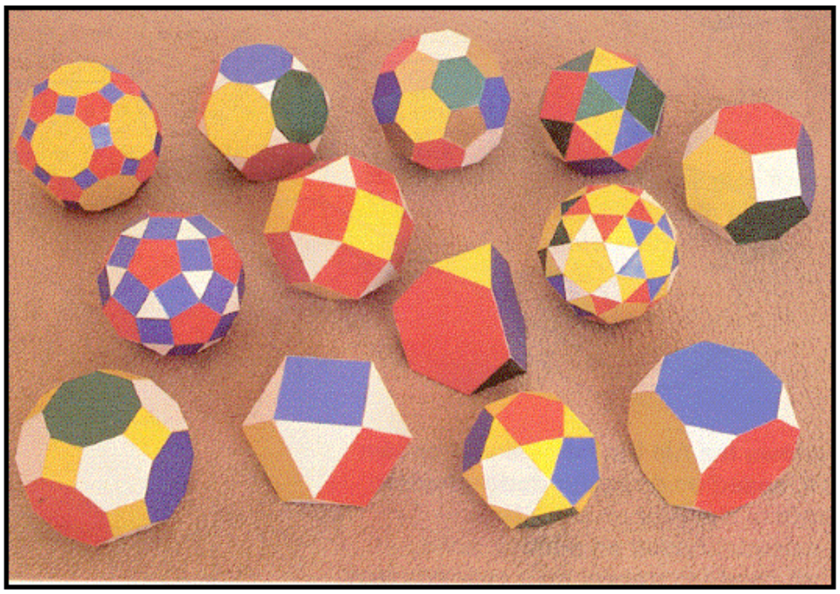
A *semi-regular polyhedron, also called Archimedean solid,* is a polyhedron which has faces that are copies of 2 or more regular polygons. The polygons are always in the same arrangement around each vertex.

There are 13 *semiregular polyhedra,* also called ***Archimedean solids***.

10. **Research question:** Pick two examples from the 13 Archimedean solids, shown here. For each of your choices, do the following

    a. Find the name for the solid.

    b. Decide how many of each regular polygon are needed to construct the solid. Construct these polygons and use tehm to make the solid

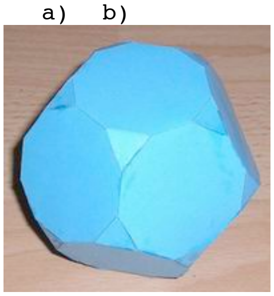
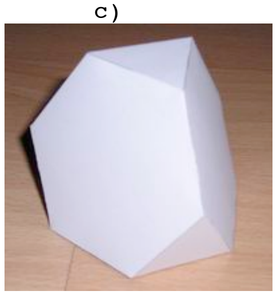
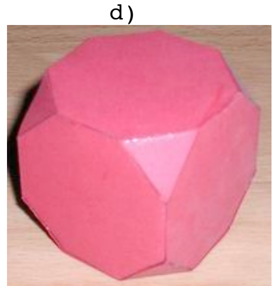
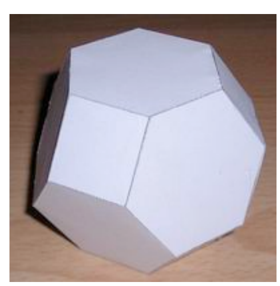


from http://www.ul.ie/~cahird/polyhedronmode

The 4 Archimedean solids shown below can be obtained by truncating Platonic solids. For example, if each corner of a tetrahedron is cut off, we obtain a truncated tetrahedron having 4 hexagons and 4 triangles as faces, b).

11. Identify which Platonic solid is truncated to obtain each of the following semiregular polyhedra:

a) b) c) d)

from http://www.korthalsaltes.com/archimedean_solids_pictures.html

Vertex Notation for Archimedean Solids

Every Archimedean solid can be given a name by listing, in order, the regular polygons that appear at each vertex. For example, the first solid listed in this table consists of triangles (notated by a **3**) and squares (notated by **4**). At each vertex there are two triangles and two squares and they alternate around the vertex, so the name is **3-4-3-4**. Notice: **3-4-3-4** is not the same as **3-3-4-4**, but it is the same as
4-3-4-3.

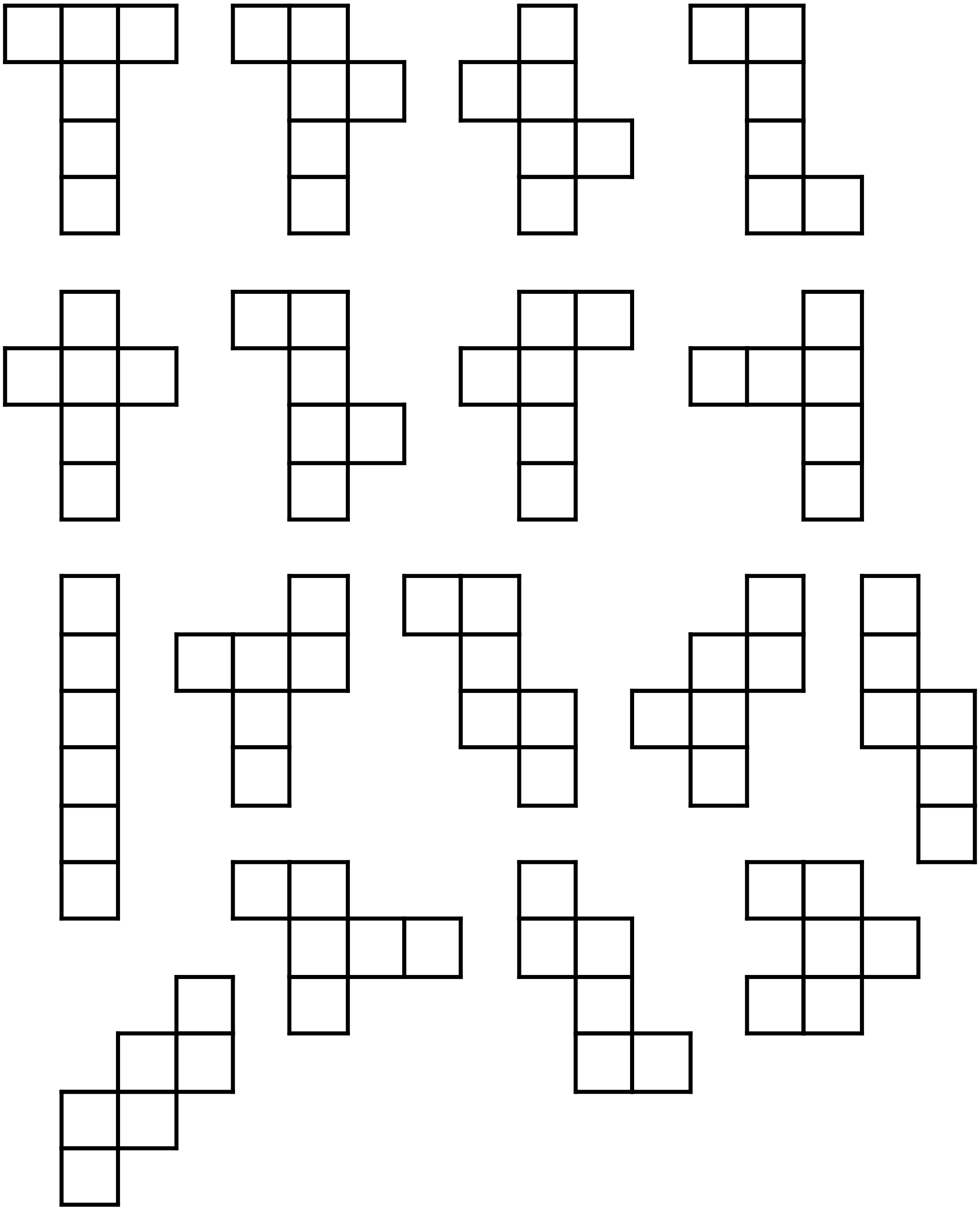
12. Complete this table. For each Archimedean Solid in the pictures on the previous page, label it with the correct vertex notation.

Number of sides of polygons used	3	4	5	6	8	10	sum of angles at one vertex
interior angle of polygon	60°	90°					
3-4-3-4	2	2					300°
3-5-3-5	2		2				336°
	1			2			
		1		2			
	1				2		
			1	2			
	1					2	
	1	3					
		1		1	1		
	1	2	1				
		1		1		1	
	4	1					
	4		1				

13. What is the significance of the sum of the angles in the last column?

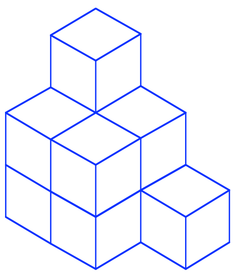
Building 3D shapes with nets

Some of these figures are nets for cubes and some are not.
Cross off duplicate nets. In this case, duplicate means the two nets are congruent.
Cross off figures that are not nets for a cube.



Building 3D shapes with unit cubes

Each person should have a set of centimeter cubes and a piece of centimeter graph paper .
If you need to complete this project at home, you can buy a box of sugar cubes to use instead of the centimeter cubes. You will have to make your grid paper with the correct size for sugar cubes.
Here are two different shapes, both made from unit cubes, represented in two dimensions in different ways. Construct both shapes from unit cubes and answer the questions



2D drawing for solid 1

3	2	1
3	2	2
2	1	1

floor plan for solid 2

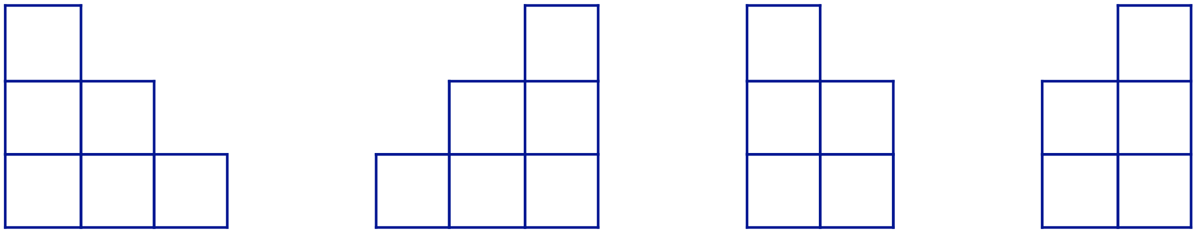
14. How many cubes does it take to build **solid 1**?

15. Show the floor plan for **solid 1**.

16. How many cubes does it take to build **solid 2**?

17. Draw a 3D picture of **solid 2.**

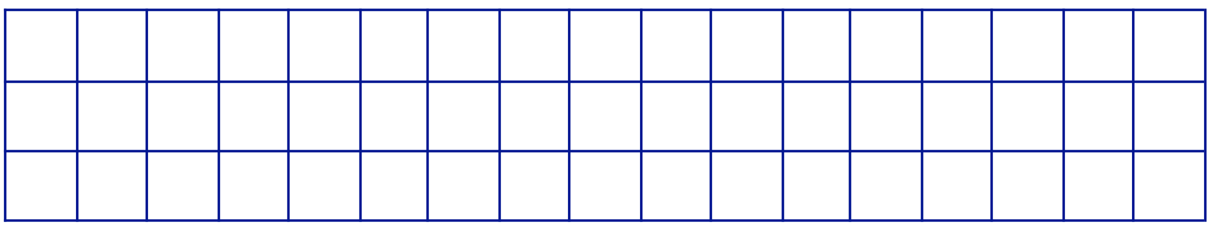
These are the **side views** of **solid 1**



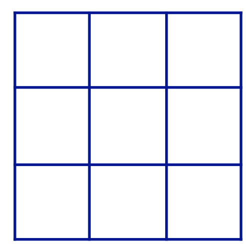
FRONT BACK LEFT RIGHT

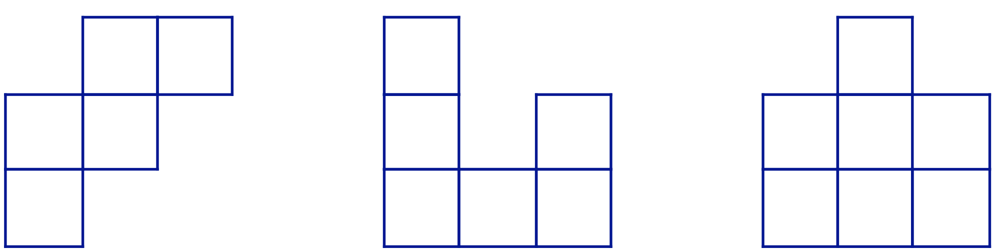
To confirm that these are correct, rotate the model you've made of **solid 1** so that it is oriented as indicated by these views and compare them to what you see. Notice that "FRONT" "BACK" LEFT" and "RIGHT" may be subjective.

18. Sketch all four side views for **solid 2**



19. Make the floor plan for a structure given the top and two side views





TOP FRONT RIGHT

20. Show the Front, Back, Left and Right views of the structure made from this floor plan.

3	2	
4	3	2
1	1	
1		

Activity: The Blue Solids
Part I – Describing 3D shapes

Sketch each of the Blue Solids next to its name.

Cylinders and Prisms A cylinder has two bases that are congruent and parallel to each other. The rest of the surface can be described as all the lines that connect corresponding points of the two bases. If the bases are polygons, the shape is called a prism.	cube large rectangular prism small rectangular prism large triangular prism small triangular prism hexagonal prism cylinder
Cones and pyramids A cone or pyramid consists of a base shape and all the lines that connect the base to a given point. If the shape is a	square pyramid triangular pyramid cone
Spherical Shapes In this set just the sphere and the hemisphere.	sphere Hemisphere

Activity: The Blue Solids
Part II -- Telephone geometry

For homework you will be asked to make nets for some of the blue solids. In order to do this at home, without the blue solids in front of you, you will need certain “critical” measurements.

Telephone geometry refers to that essential information that you would tell a classmate who called to ask you to describe one of the solids. In each case you need to give enough details so that the classmate could make the solid at home.

Knowing the “critical” lengths should allow you to completely reconstruct the solid. For example, the critical lengths for a right rectangular prism are the lengths of each edge: height, length and width. For the cone, you might give the radius of the circle base and the height from the center of the circle to the vertex.

21. For each blue solid, determine a set of “critical” lengths that determine the solid. Measure each one and record them in your sketches on the previous page.

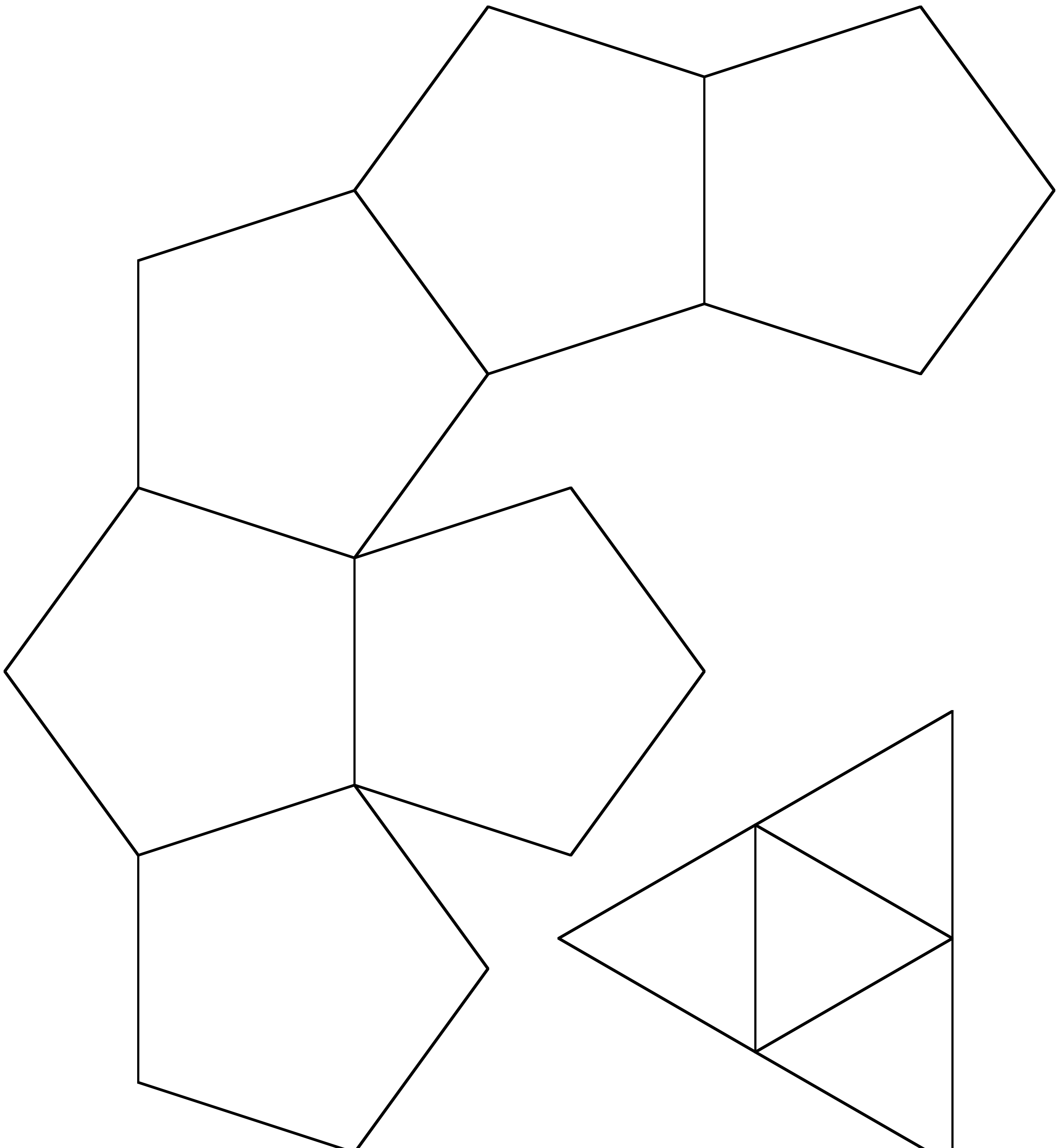
Part III – Height

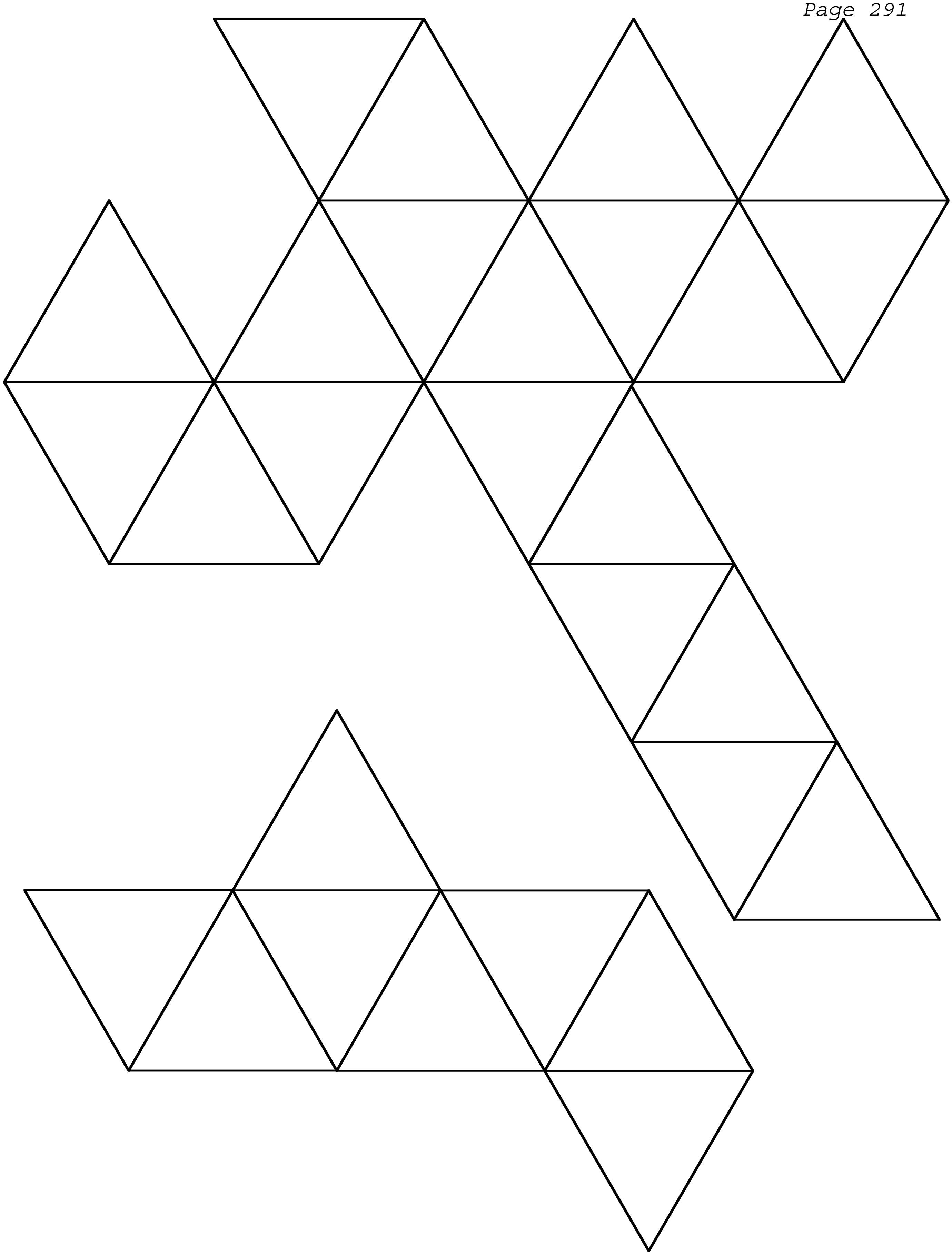
The height of a solid is a critical measurement that is useful in describing a solid, but it might not always be obvious which is the height. The height, or how tall something stands, will depend on which face the solid is sitting on. You cannot measure the height of a person if they are lying in bed, because we normally thinking of a person’s height when they are standing. For a cylinder or prism, height usually refers to the height when the prism is sitting on one of the bases. This will be very important when we look at formulas for volume. In the case of rectangular prisms the height will depend on which rectangle is on the ground.

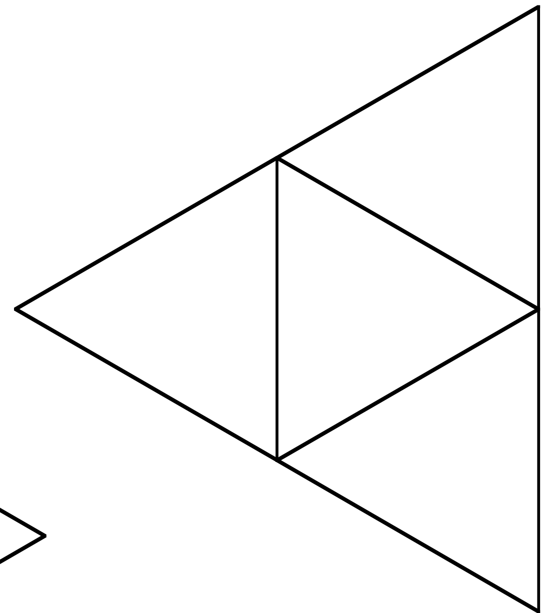
It is easy to measure the height of a prism because it is just the length of one of the edges.

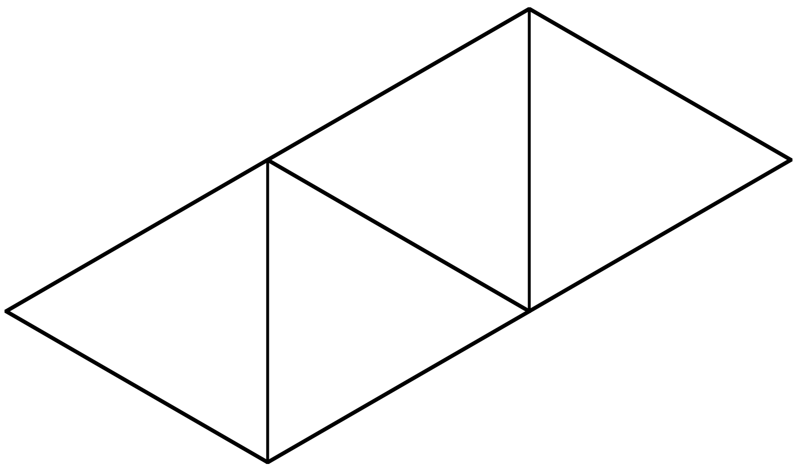
It is difficult to measure the height of a pyramid because the line segment you’d like to measure goes through the solid. We could measure indirectly to see that the height of each blue solid is 5 centimeters if it is sitting on the appropriate base. We could also compute the height using other measurements.

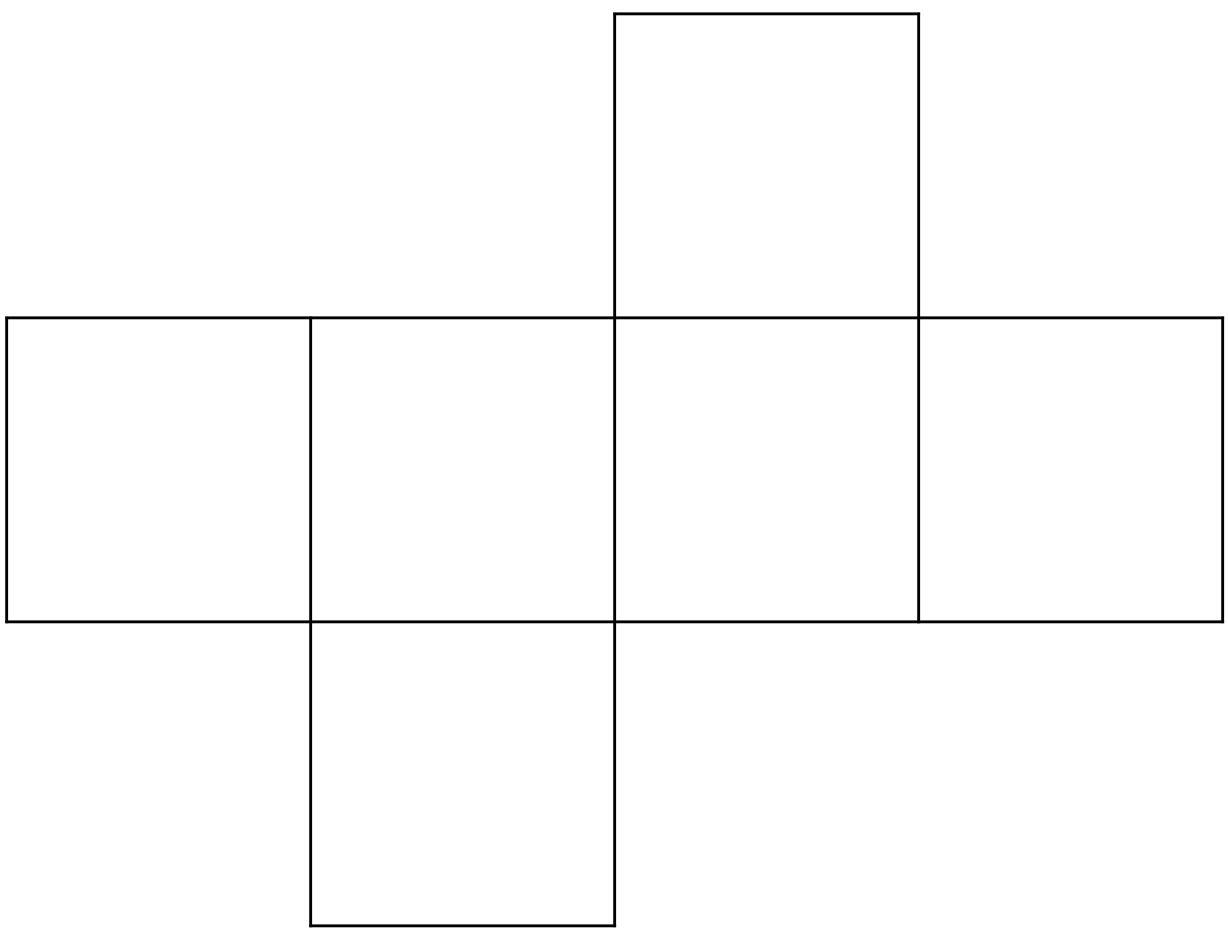
22. Compute the height of each pyramid by first measuring the “slant height” and then using the Pythagorean theorem. Note that there is more than one “slant height” (either one will work – but the calculations will be different) so label your picture carefully to explain your work each solid. Also note that in each case your answer should be “close” to 5 cm.











Lesson 3 – Surface Area

Blue Solids Opening:

You may want to use this list of all of the Blue Solids for reference in this project.

large rectangular prism

small rectangular prism

cube

large triangular prism

small triangular prism

hexagonal prism

cylinder

square pyramid

triangular pyramid

cone

sphere

hemisphere

Conjecture:

For each of these questions, provide some explanation for your answers. When are you sure you are right? What might you do to better determine the correct answers.

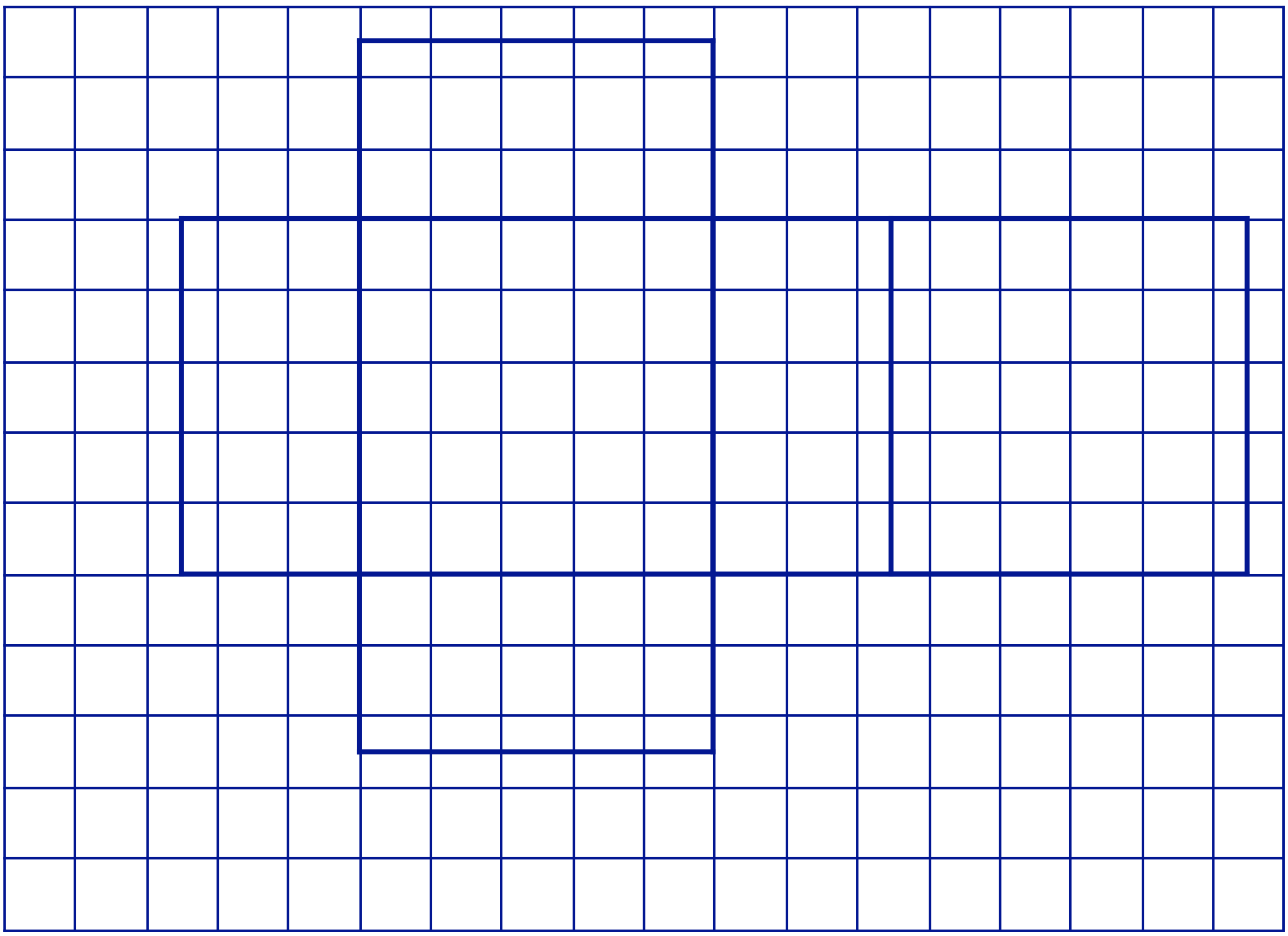
Which **prism** took the most blue plastic to make? The least?

Which **pyramid** took the most blue plastic to make? The least?

List the **blue solids** in order of the amount of blue plastic was used to make it.

Group Project:
Nets for Blue solids

Cut out this net and fold it into a solid. Which Blue Solid have you made?



As a group, make an exact replica of each of the other Blue Solids, using centimeter graph paper to make a net.

Formulas for Surface Area: Class discussion

For each shape, draw a picture and give a formula for the surface area of the shape. In each case, explain why the formula is correct.

Rectangular solid

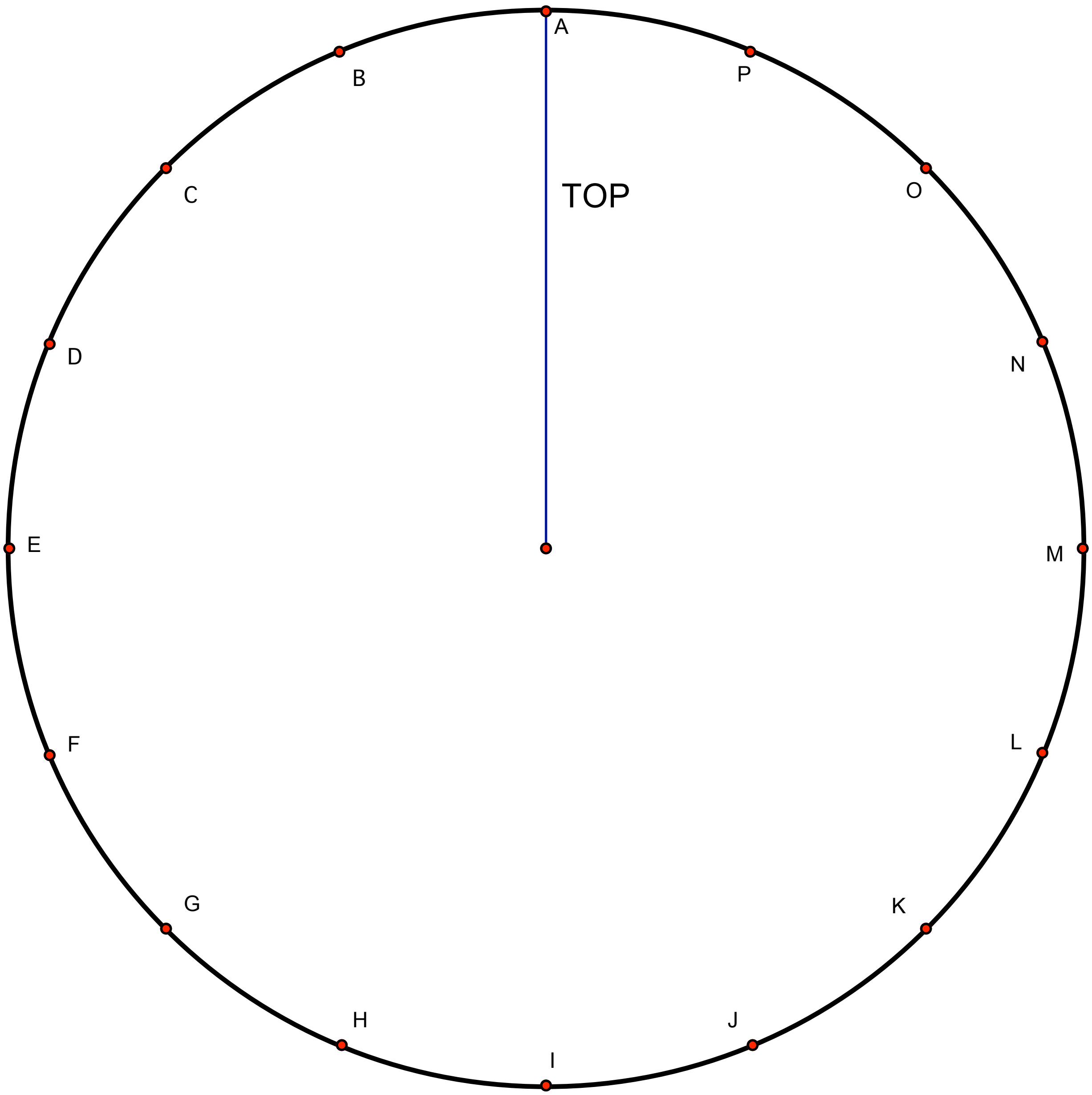
Cylinder

Sphere and hemisphere

Class Activity:
Surface area of a cone

Cut out the circle.
Cut along the radius.

Slide the top of the circle around from A to B to C and so on to make different cones.



Class Activity continued: Surface area of a cone

Work as a class to make as many of the different cones as you can. The very narrow cones are difficult to construct without cutting off some of the excess paper.

1 - Color the outside of the cone. The colored part is called the *lateral surface area*. Open it up and compute the area of the sector using the formulas we developed in Unit 2.
2 - Make the cone again. Color around the circular base of the cone. Open it up and compute the partial circumference sector using the formulas we developed in Unit 2.

Record everyone's answers in this chart.

	Circumference of base	**Radius of base**	**slant height**	**lateral surface area**	**height of cone**	**Volume of cone**
A						
B						
C						
D						
E						
F						
G						
H						
I						
J						
K						
L						
M						
N						
O						
P						
Formula		r	l	lsa=	h	V=

Class Activity continued:
Surface area of a cone

Use this space to record a formula for the surface area of a cone based. Include pictures and descriptions of each of the variables in your formula.

Lesson 4 – Volume

Blue Solids Opening:

You may want to use this list of all of the Blue Solids for reference in this project.

large rectangular prism

small rectangular prism

cube

large triangular prism

small triangular prism

hexagonal prism

cylinder

square pyramid

triangular pyramid

cone

sphere

hemisphere

Conjecture:

For each of these questions, provide some explanation for your answers. When are you sure you are right? What might you do to better determine the correct answers?

Which **prism** holds the most water? The least?

Which **pyramid** holds the most water? The least?

List the **blue solids** in order of which hold the most water.

Units for measuring volume and capacity

On this page, list units that are helpful for computing volume and capacity. Write them here along with some good unit ratios for converting between them.

1. Go to the webpage, http://www.onlineconversion.com/volume.htm, or some other source, to see if there are others you want to add to your list. Find two volume or capacity measurements you have not heard about before. Report here on which units you chose. What things are they used to measure? Write down the conversion factor, or unit ratio, that relates the two units.

2. Use dimensional analysis or some other method to do these conversions. You may have to do some research about the units involved and add to the unit ratio list you have started. Show your work.

12 ft^3 = _______ in^3

10 cm^3 = ______ cubic meters

26 pints = _______ gallons

24 tablespoon = ________ teaspoons

24 yd^3 = _______ gallons

10 tablespoons = ________ cups

1000 in^3 = ________ Liters

2 quarts = ________ Liters

Rectangular solids:
Volume Formulas for prisms and cylinders

3. Recall the definition of a prism and write it here:

A **rectangular prism** is a prism that has a rectangular base. A **right rectangular prism** is a rectangular prism in which the base is perpendicular to the other faces.

Construct each of these right rectangular prisms with unit cubes and find the volume. There is room to show your calculations.

Length	Width	Height	Volume
3	2	4	
4	3	2	
2	5	3	
2	4	6	

4. What observations can you make?

Use your observations to find the volume of each of these rectangular prisms:

Length	Width	Height	Volume
17	25	10	
2.5	5.7	4	
l	**w**	**h**	

Notice that the last row gives a formula for the volume of a rectangular prism using length (**l**) width (**w**) and height (**h**).

A rectangular prism can have three different bases.

5. Find the area of each base of a 2 x 3 x 5 rectangular prism. Use the area of the base and the corresponding height to find the volume of this prism in three different ways. Explain why you get the same answer each time in terms of basic properties of numbers.

Base	B, Area of base	h, Height	B x h

Another formula for the volume of a rectangular prism, using the area of the base, **B**, and the height, **h**, is **Volume = Bxh**.

6. Explain how the formula you wrote in the previous problem may be applied to determine the volume of a cylinder.

The volume of any prism is the area of the base times the height.

If you use 125 centimeter cubes to make a 5cm x 5 cm x 5 cm cube it will be congruent to the cube in the blue solid set, but it will not fit *inside* that cube, because the blue plastic takes up space. In other words, the volume of water that the cube will hold is less than the volume we get by multiplying the lengths of the sides on the outside.

7. Measure the insides of the cube to find an accurate measure of how much water the cube will hold. Test your measurement by measuring how much water it actually holds.

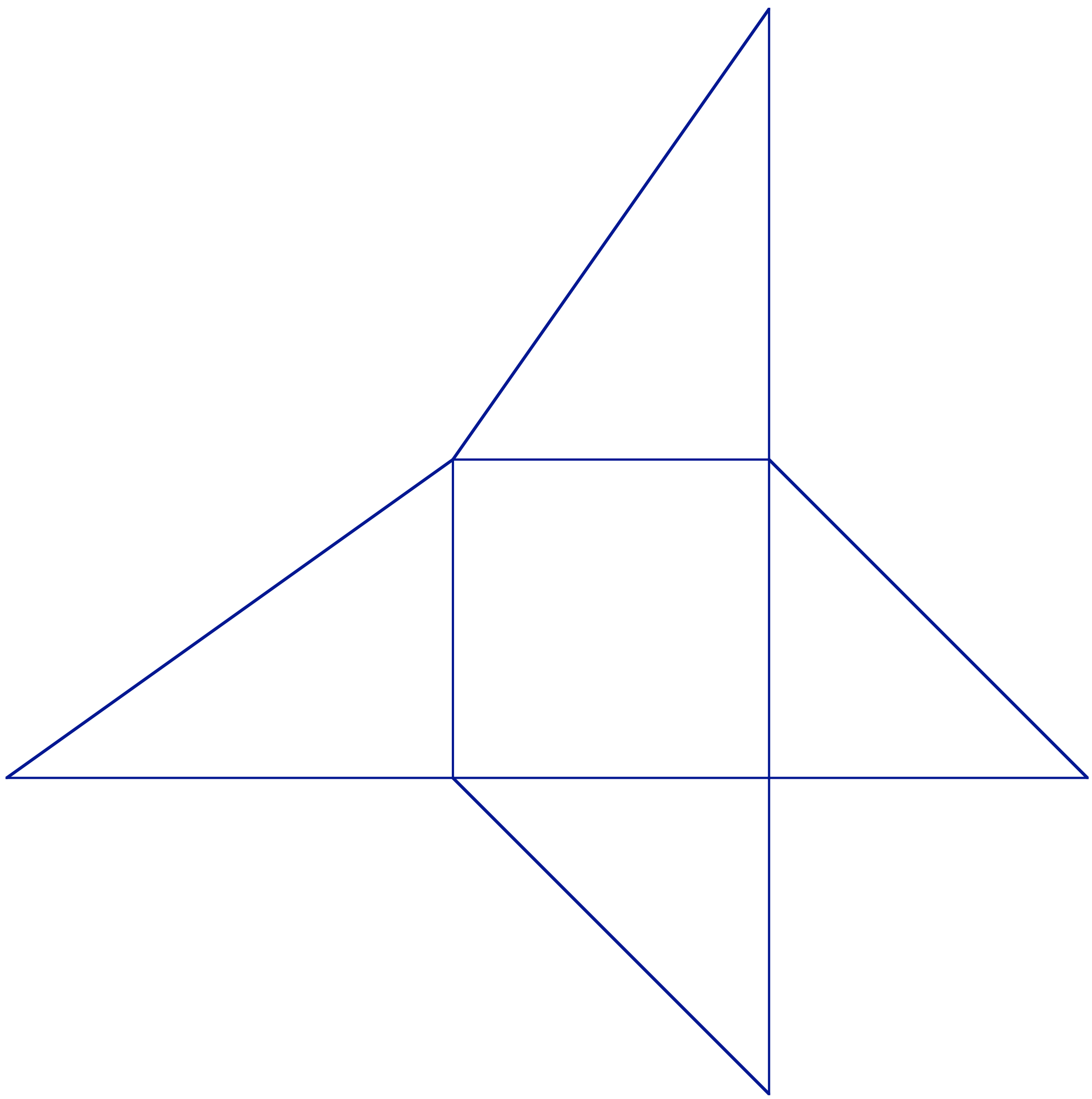
The volume of a pyramid, or cone, is 1/3 base times height: Two demonstrations

Find a prism and a pyramid that have congruent bases and congruent heights. Do this experiment:

8. How many times can you fill the pyramid with water and pour it into the prism without overflowing the prism?

Pyramid Puzzle:

9. Use this net to make three congruent pyramids. Put them together to form a cube. What is the volume of the cube? What is the volume of one of the pyramids? Explain how this demonstrates the formula for the volume of a pyramid.



10. Find the volume of all of the pyramids in the Blue Solid set.

11. Find the volume of all of the cones from the activity in Lesson 3. Which cone has the greatest volume?

Formulas for Problem Solving

On this page, write down all of the formulas for volume and surface that you think would be for solving problems.

Practice Problems
Include pictures to help with all of these problems

1. Find the volume of a sphere with a circumference of 15 cm.

2. Find the volume of a hemisphere with a surface area is 18 ft^2.

3. Sketch each house (given as **height** times **length** times **width**) and find the total volume.

    a. a house with no roof: 20 ft x 18 ft x 8 ft

    b. same house with a rectangular pyramid roof: 12 ft total height

    c. same house with a triangular prism roof with a equilateral triangular base, each side has length 8 feet.

    d. same house with a semi-cylindrical roof, with circular diameter of 8 feet.

4. Find the volume of a conical tent (tepee) with slant height of 12 ft if the circumference of the base 27 ft.

Lesson 5 – Putting it all together: Two Activities

Group Project: Roll a Cylinder

For this activity, you will need 2 sheets of 8.5 in x 11 in stiff paper, tape, and popcorn, rice, or cereal.

Roll each sheet of paper so that the edges meet. Tape the edges carefully to form 2 open-ended cylinders, one with a height of 11 in and the other with a height of 8.5 in.

1. Make a conjecture: Will both cylinders hold the same amount of popcorn?

Place the taller cylinder on a flat surface and fill it with popcorn, etc. Now place the shorter cylinder over the taller one. Lift the taller cylinder and allow the contents to fill the wider cylinder.

2. What did you discover? Was your conjecture correct?

3. Use the formula for the volume of a cylinder to explain your result.

Cut one sheet of paper in half the long way. Tape the halves together along the uncut edges so you first have a 4.25 x 11 rectangle, and then a cylinder that is 4.25 inches high.

Cut one sheet of paper in half the short way. Tape the halves together along the uncut edges so you first have a 5.5 x 8.5 rectangle, and then a cylinder that is 5.5 inches high.

4. What is the ratio of the new volumes to the volumes of the original cylinders?

5. Imagine cutting each sheet of paper in half again. If you used one entire sheet of paper to make a cylinder 2.75 in high, what would be its volume?

6. Will cutting a sheet of paper in half always result in a larger volume?

7. What about cutting a piece of paper into four equal squares? Halving each dimension?

Group Project: Comparing Volume to Surface Area

You will need 8 sheets of centimeter graph paper, scissors, and tape for this activity.

8. Make an open box by cutting 1 square from each corner of a sheet of centimeter graph paper. Be careful not to overlap the paper. Make more boxes by cutting 2x2, 3x3, etc. squares from other sheets of graph paper.

9. Use your boxes to complete the following chart

Dimensions Of Squares Cut Off	Area of base	Altitude	Surface Area	Volume
1x1				
2x2				
3x3				
4x4				
5x5				
6x6				
7x7				
8x8				

10. What did you notice about the surface area as you cut off larger squares?

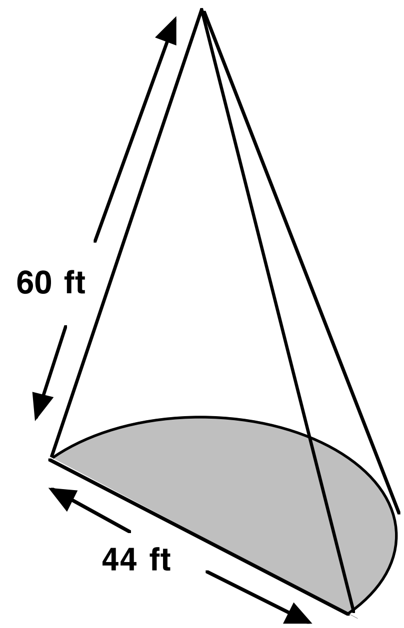
11. What did you notice about the volume as you cut off larger squares? Was you conjecture correct?

12. Use the formula for the volume of a rectangular prism to explain what you observed.

13. If you were a manufacturer of cereal boxes, how might you use the results of this activity to minimize costs and maximize profits?

Problems on Surface Area and Volume

1) Suppose this object is to be filled with cement. How many cubic feet of cement are needed?



After the cement dries the object will be placed on its half circle base and painted. How many gallons of paint will be needed if one gallon covers 24 sq feet?

The diameter of the base of the cone is 44 ft. The slant height of the cone is 60 ft.

Suppose instead the object is placed with the triangular side on the ground. Is there more or less exposed surface area?

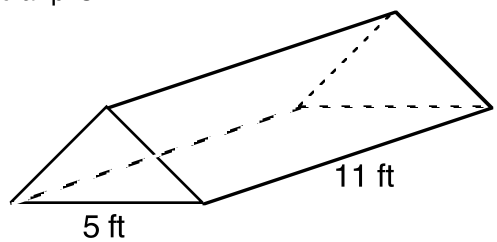
2) Convert the following measurements:

25 in^3 = _______ cm^3

45 km^2 = ________ cm^2

33 yd = ________ ft

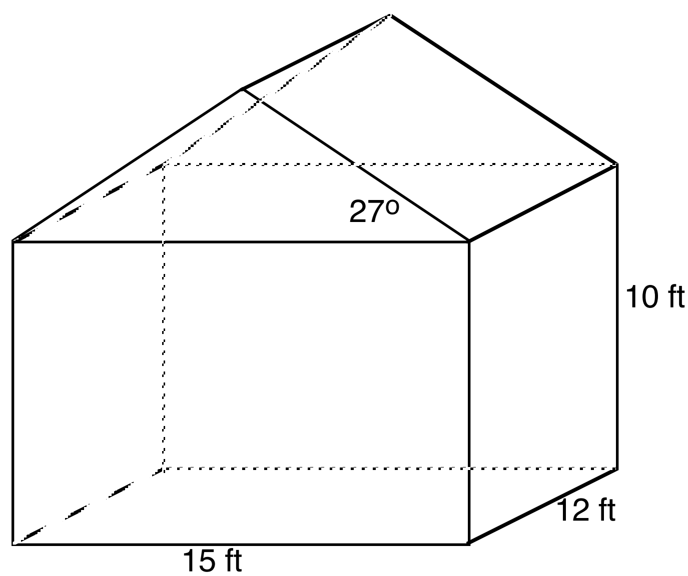
5) Find the volume of this triangular prism:



The triangular ends are right isosceles triangles.

6) This sketch shows plans for a storage shed.

a -- What is the volume of the shed? Include the space under the roof.

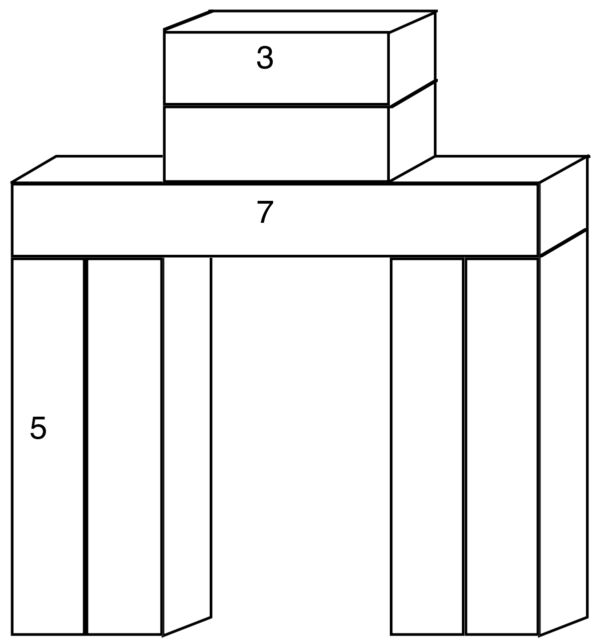


b -- The roof will be covered with asphalt sheeting that comes in bundles. Each bundle costs $43 and will cover 20 sq yd. If each bundle costs $43, how much will it cost to buy the roofing material.

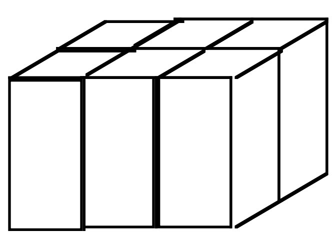
3) Build the Cuisinaire-rod bridge pictured below and answer the following questions.

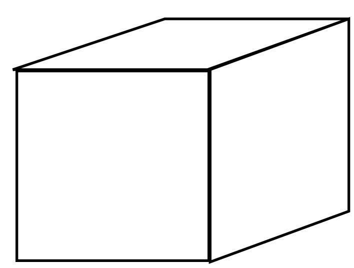
What is the volume in cm^3?

What is the total exposed surface area?

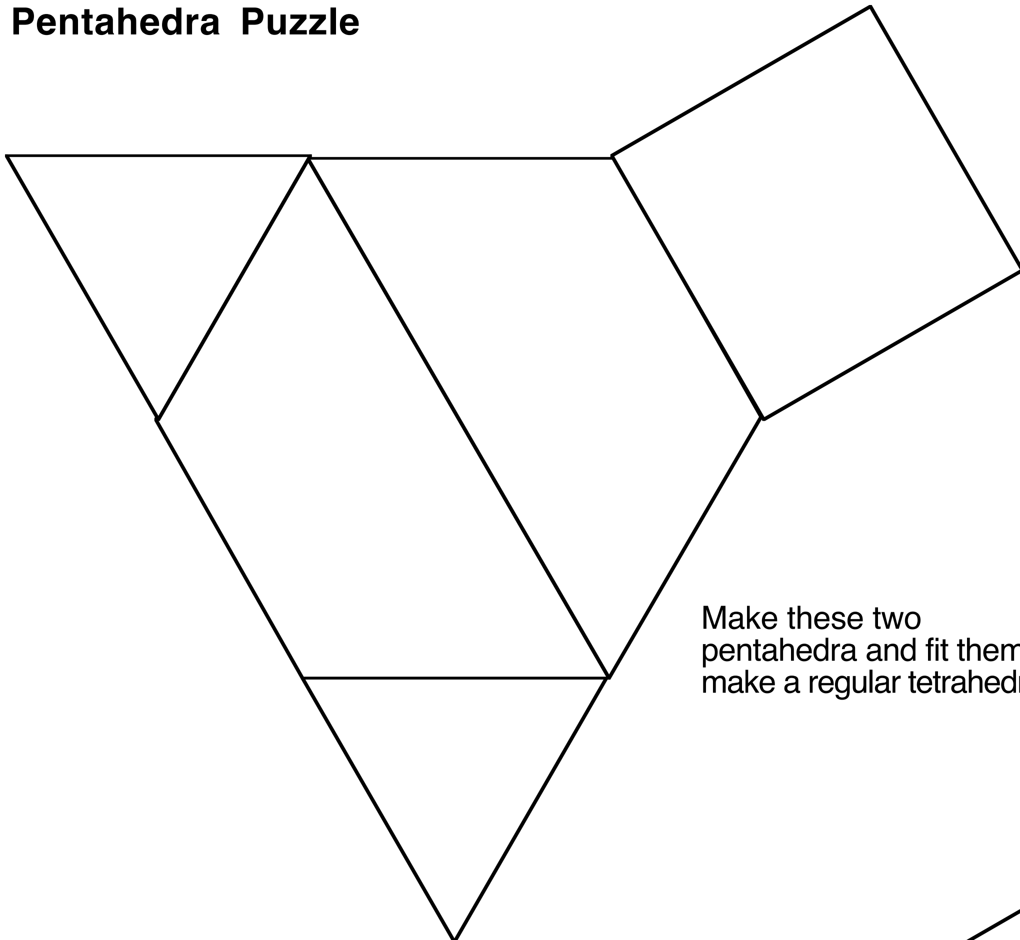


4) Imagine six shapes of solid gold, each one the exact dimensions of red Cuisenaire rod. If they were to be melted down, could the gold be contained in an 1in x 1in x 1in cube? Explain your reasoning.

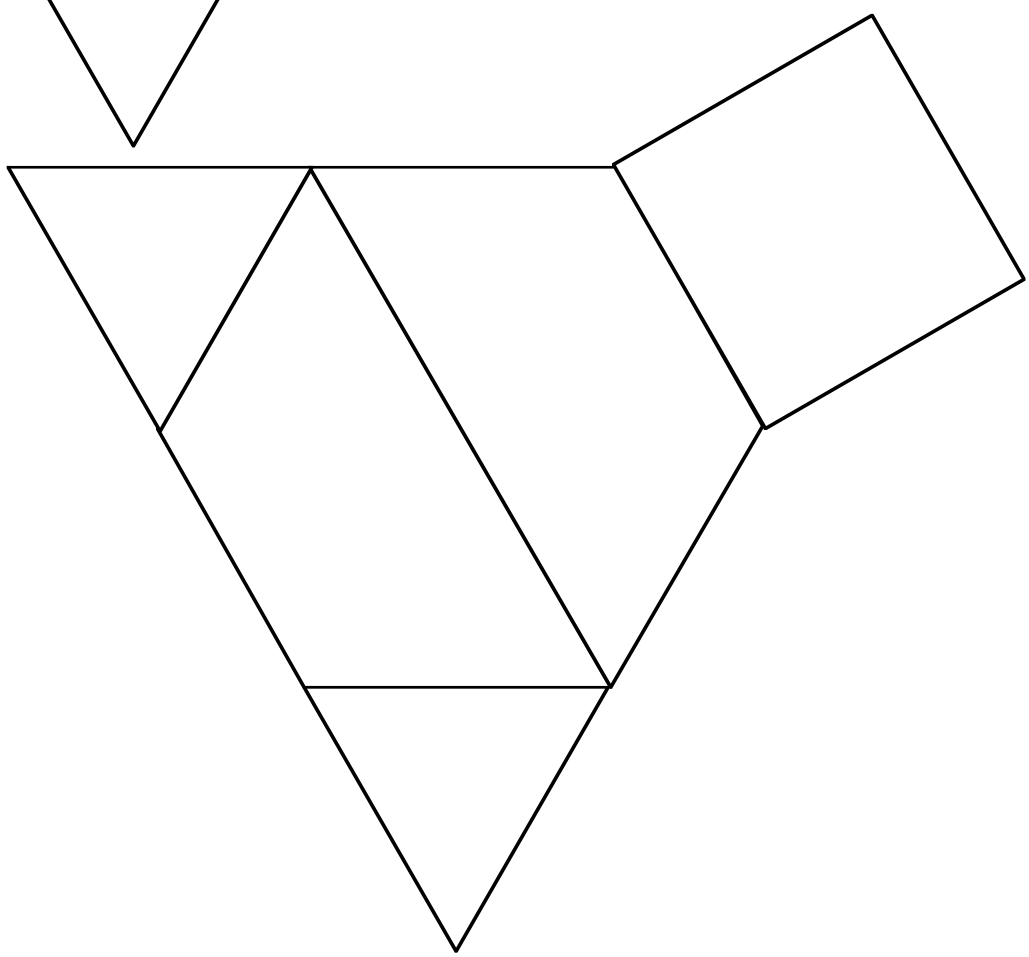




Pentahedra Puzzle

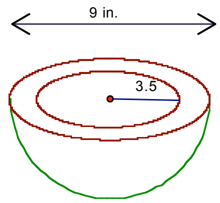


Make these two
pentahedra and fit them together to
make a regular tetrahedron.

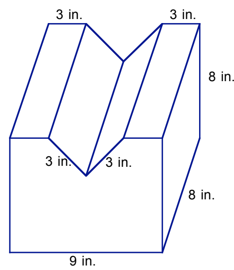


Problem Set #3

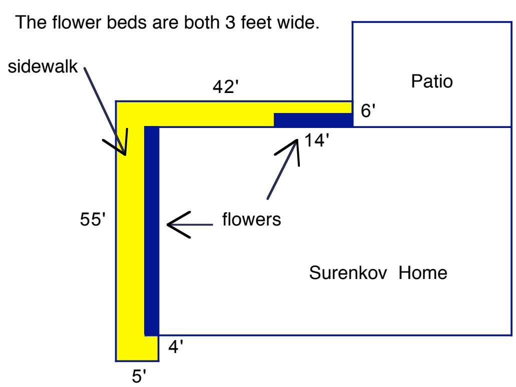
1. Find the surface area and volume:



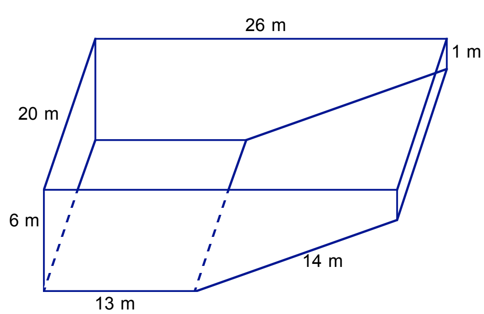
2. Find the surface area and volume:



3. A sketch of the Surenkov home is shown in the following figure, with the recent sidewalk addition shaded. Using the measurement indicated on the figure and the fact that the sidewalk is 4 inches thick with right angles at all corners, determine how many cubic yards of concrete were used.

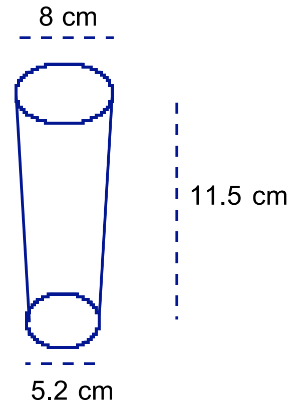


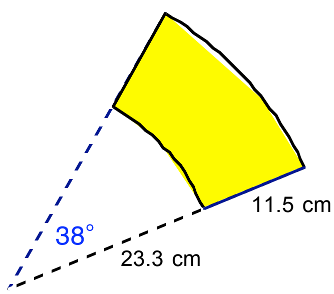
4. How many square meters of tile are needed to cover the inside of the swimming pool illustrated below?



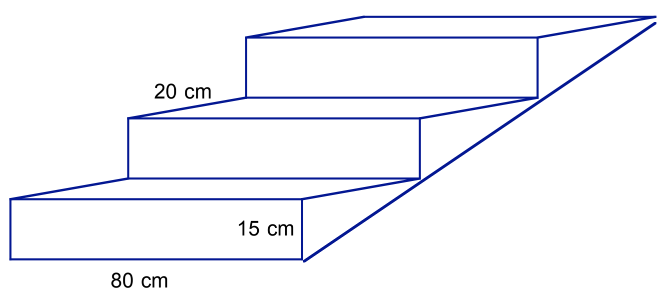
5. How much water does the pool in the previous problem hold?

6. A paper cup has the shape shown in the first drawing (a frustrum). If the cup is sliced open and flattened, the sides of the cup have the second shape (the shaded part of a sector). Use the dimensions given to calculate the number of square meters of paper used in the construction of 10,000 of these cups





7. The first three steps of a 10-step staircase are shown at the right.



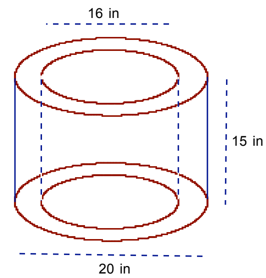
a) Find the amount of concrete needed to make the exposed portion of the 10-step staircase.

b) Find the amount of carpet needed to cover the fronts, tops, and sides of the steps.

8. A soft-drink cup is in the shape of a right circular cone with capacity 250 ml. If the radius of the circular base is 5 cm, how deep is the cup?

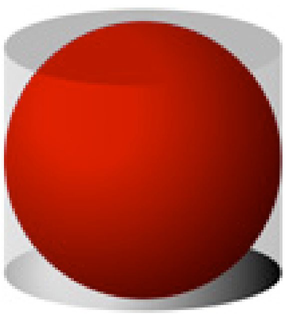
9. A rock placed in an aquarium measuring 2 ½ ft long by 1 ft wide causes the water level to rise ¼ inch. With the rock in place, the water level in the aquarium is ½ inch from the top. The owner wants to add to the aquarium 200 solid marbles, each with a diameter of 1.5 cm. Will the addition of these marbles cause the water in the aquarium to overflow?

10. The bases of a prism are regular hexagons. The other sides are squares. If the perimeter of one of the bases is 72 inches, what is the surface area of the prism? What is the volume of the prism?



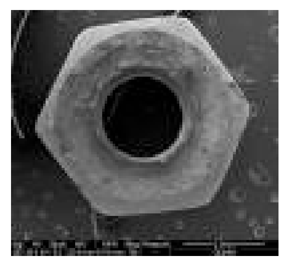
11. Find the surface area of the wooden napkin ring pictured at the right.

12. Find the volume of the wooden napkin ring pictured at the right.



13. Archimedes' tomb bore an engraving of a sphere inscribed in a right circular cylinder to commemorate a discovery of which he was particularly proud. The discovery concerned the ratio of the volume of the sphere to the volume of the cylinder and the ratio of the surface area of the sphere to the total surface area of the cylinder. Find those two ratios in simplest form.

14. Calculate the volume and surface area this bolt. The hole through the nut has diameter 1 cm. Each side of the hexagon is 2 cm. and the depth of the bolt is 0.75.



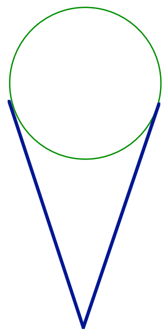
15. Scotty just moved into a new house and the landscaper ordered 1 cubic yard of topsoil for his 15′ by 24′ garden. If the topsoil is spread evenly, about how thick will it be – a light dusting, about 1″, or about 1′? Justify your answer.

16. If one were to double each dimension of a fish tank that is the shape of a rectangular prism, how would the capacity of the tank also double?

17. The Great Pyramid of Khufu (also known as the Cheops) was original 481 feet tall with a square base that was 751 feet on each side. Answer these questions:

a) How many football fields (100 yards by 160 feet) would fit in the same area as the base of this pyramid?
b) This pyramid has the same volume as about how many bedrooms that are 15 feet by 15 feet by 12 feet?

18. If the ice cream in an ice cream cone were to melt, could the cone hold all of the ice cream. Assume that the cone is a perfect cone, with a height of 5 inches and a circular base with a diameter of 2.5 inches, and that the ice cream is a perfect sphere with a diameter equal to the diameter of the cone.

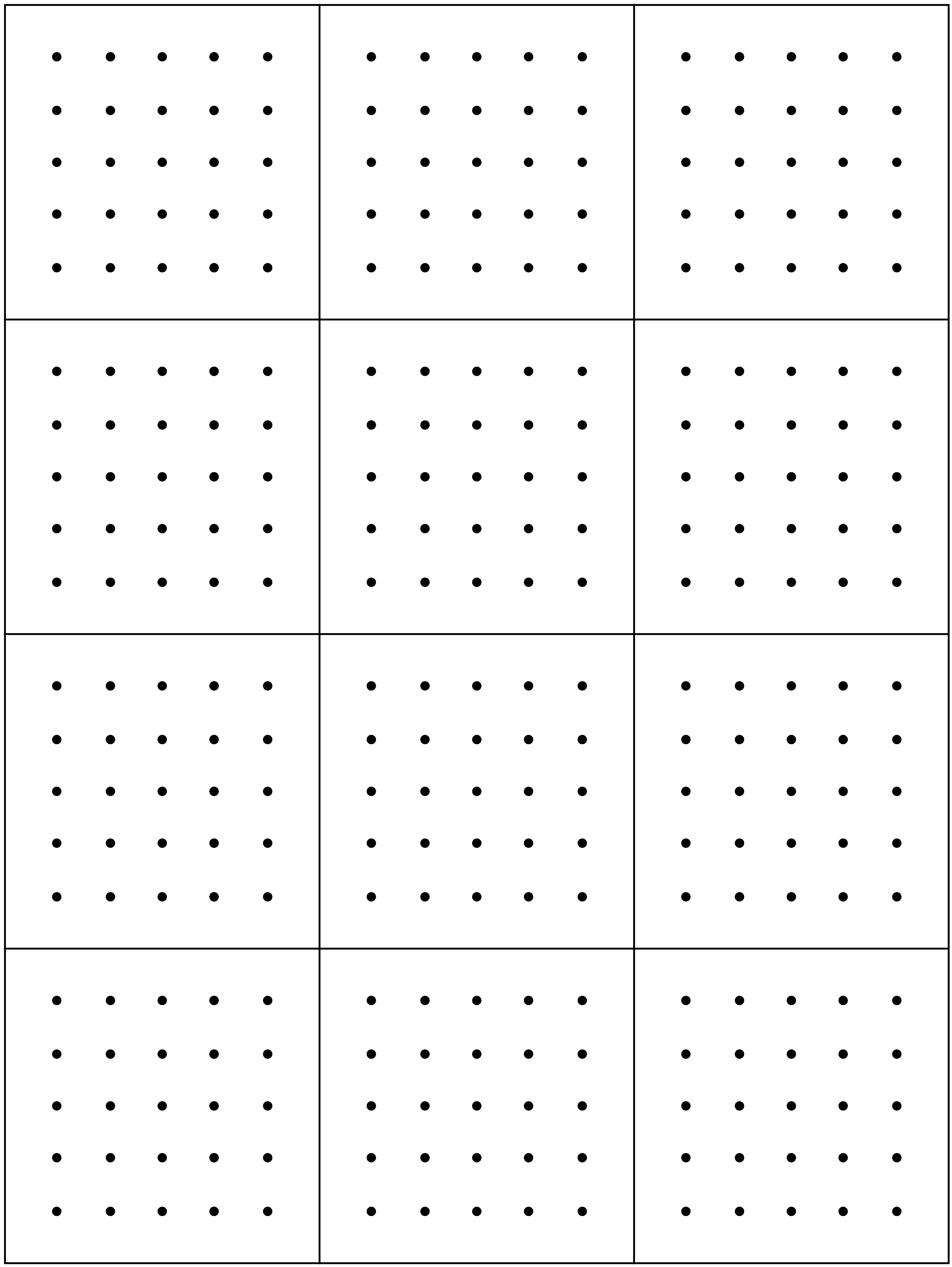


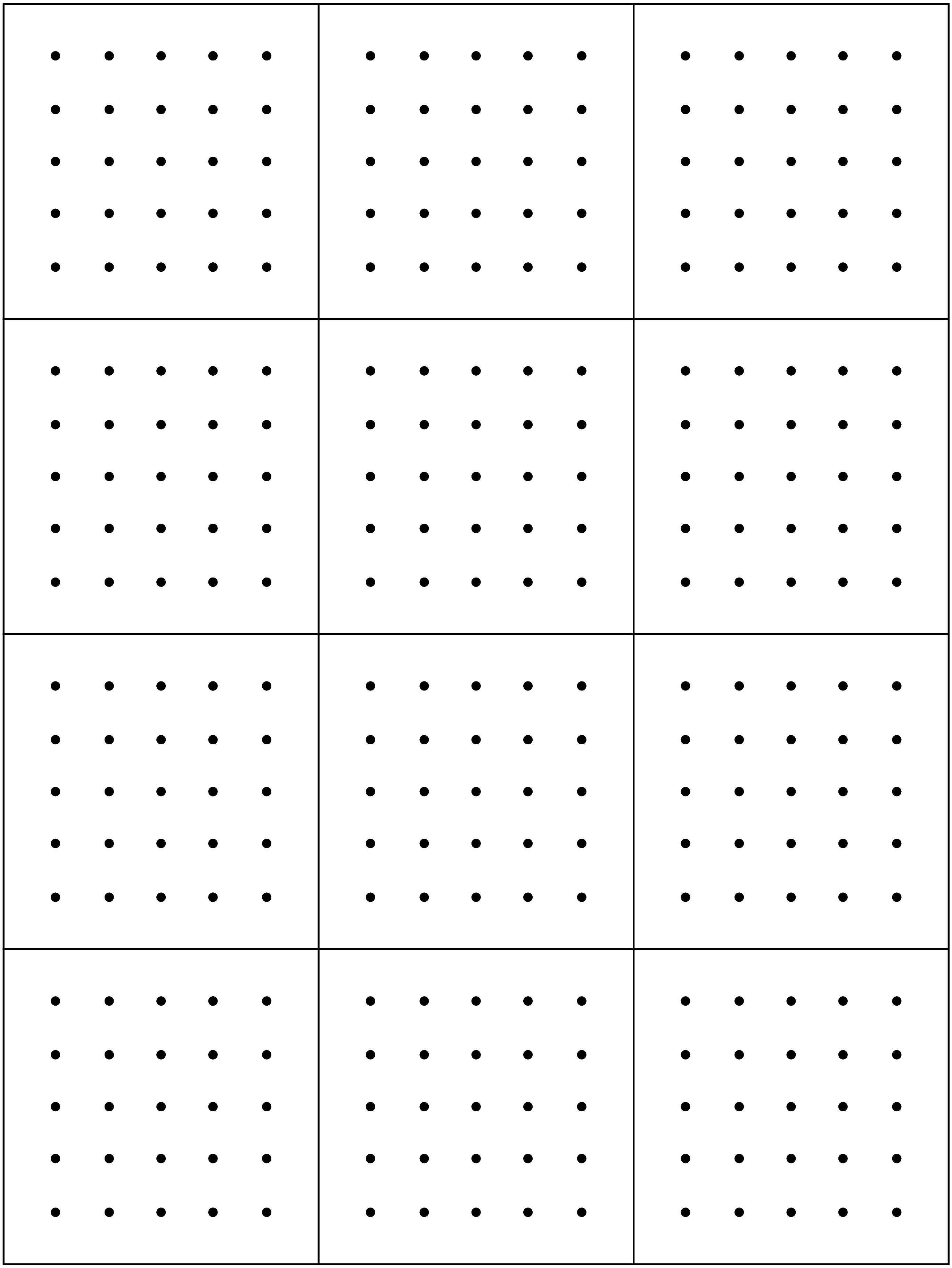
Centimeter graph paper

Centimeter graph paper

Centimeter graph paper

Centimeter graph paper





Centimeter Dot Paper

Centimeter Dot Paper