Limits at infinity; continuity

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(*) = similar to a HW problem.

Limits at infinity

1. Sketch a single function $f(x)$ that satisfies all of the following:
   - $\lim_{x\to\infty} f(x) = 0$.
   - $\lim_{x\to 0} f(x) = \infty$.
   - $\lim_{x\to -\infty} f(x) = \infty$.
2. If $\lim_{x\to\infty} f(x) = L$, what is $\lim_{x\to\infty} -f(x)$? What about $\lim_{x\to-\infty} f(-x)$? Is this true even when $L = \infty$ or $-\infty$?
3. Evaluate $\lim_{x\to\infty} f(x)$ for the following functions:
   (a) $f(x) = \ln x$.
   (b) $f(x) = -\ln x$.
   (c) $f(x) = \ln \frac{1}{x}$. Hint: use the power rule for logarithms.
   (d) $f(x) = \log_b x$ for any base $b > 1$. Hint: use the change of base rule for logarithms.
   (e) $f(x) = \log_b x$ for any base b with $0 < b < 1$. Why is this different from part d)?
4. Evaluate $\lim_{x\to\infty} f(x)$ for the following functions:
   (a) $f(x) = x^n$ for any positive integer n.
   (b) $f(x) = \frac{1}{x^n} = x^{-n}$ for any positive integer n.
   (c) $f(x) = \sqrt{x} = x^{1/2}$.
   (d) $f(x) = \sqrt[3]{x} = x^{1/3}$.
   (e) $f(x) = x^{1/p}$ for any positive integer p. What do you think is $\lim_{x\to\infty} x^c$ for *any* c in the interval $(0, 1)$ (even when c is really, really small, e.g. $x^{0.0000001}$)?
5. Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for each of the following functions. Are there any horizontal asymptotes?

(a) $f(x) = 3 + \frac{10}{x^2}$.

(b) $f(x) = \frac{3+2x+4x^2}{x^2}$.

(c) $f(x) = \frac{\cos x^5}{\sqrt{x}}$.

(d) $f(x) = 3x^2 - 9x^7$.

(e) $f(x) = 3x^7 + x^2$.

(f) $f(x) = \frac{3x^2-7}{x^2+5x}$.

(g) $f(x) = \frac{3x^3-7x}{x^4-5}$.

(h) $f(x) = \frac{40x^5+x^2}{10x^4-2x}$.

(i) $f(x) = \frac{-x^3+1}{2x+8}$.

(j) $f(x) = \frac{4x^3+1}{2x^3+\sqrt{16x^6+1}}$. (*)

(k) $f(x) = \frac{\sqrt{x^2+1}}{2x+1}$. (*)

(l) $f(x) = \frac{\sqrt[3]{x^6+8}}{4x^2+\sqrt{3x^4+1}}$.

(m) $f(x) = 3x - \sqrt{9x^2+1}$.

(n) $f(x) = -3e^{-3x}$. (*)

(o) $f(x) = \frac{50}{e^x}$.

(p) $f(x) = \sin x$.

(q) $f(x) = x \sin x$.

(r) $f(x) = x + \sin x$.

Based on these examples, when does a function have horizontal asymptotes?

Continuity

1. What does it mean *graphically* for a function $f(x)$ to be continuous at $x = a$? On the interval $[a, b]$? On $(-\infty, \infty)$?

2. State the definition of continuity of a function $f(x)$ at $x = a$.

3. Sketch a function $f(x)$ for which $f(0)$ and $\lim_{x\to 0} f(x)$ are both defined but $f(x)$ is not continuous at $x = 0$.

4. Give an example of a function with infinitely many discontinuities.

5. Give an example of a function with infinitely many discontinuities but *no* vertical asymptotes. Hint: two such functions appeared on HW #1!

6. At which points is $f(x)$ discontinuous? State the intervals of continuity.

   (a) $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ x^2 + 3x & \text{if } x \le 1 \end{cases}$. (*)

   (b) $f(x) = \begin{cases} \frac{x^2-4x+3}{x-3} & \text{if } x \ne 3 \\ 2 & \text{if } x = 3 \end{cases}$.

7. Let
$$f(x) = \begin{cases} x^2 + x & \text{if } x < 1 \\ 4x + b & \text{if } x \ge 1 \end{cases}.$$
   Determine the value of b that will make $f(x)$ continuous at $x = 1$.

8. State the intermediate value theorem in precise mathematical terms. Then explain informally what it is saying.

9. Use the intermediate value theorem to prove that the equation
$$2x^3 + x - 2 = 0$$
   has a solution on the interval $(-1, 1)$.