INTRODUCTION TO RESOLUTION OF SINGULARITIES : BLOW UP

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Abstract

Roughly speaking, singular point is a point on a space that cannot generate a tangent space in normal way. However, nonsingular points which is also called as regular points are able to generate a tangent space in regular way, and this allows us to observe invariants and global properties of the whole space by looking at tangent space at nonsingular points. Therefore, if we have a fact that an arbitrary space is nonsingular space, we can recover not only each local parts of the space but also global invariants of the space. Therefore, many mathematicians tried to build methods to resolve singularities. In this survey, I will suggest definitions related with singularities and show a typical way of resolution which is called 'blow up'.

0. Preliminaries

In order to talk about singularities we need to define tangent spaces. In this section, I will go through definitions related with tangent spaces and singularities, and I will also give several definitions related with resolution of singularities.

Note. In this survey, all spaces are algebraic varieties. Also k is always algebraic closed field, and assume that we are always working on $\mathbb{A}_k^n$ or $\mathbb{P}_k^n$.

Before giving the definition of tangent space, there are many ways to define a tangent space of a space at a point. In *Hitchin's Lecture note*, he defines a tangent space of a manifold M at a point a as $(C^\infty(M)/Z_a)^*$ which is the dual space of cotangent space at a point a, and in *Milnor's book - Characteristic Classes*, he defines tangent space as a space of tangent lines that passes through a (i.e. $\{\gamma'(0)|\gamma : \mathbb{R} \to M, \gamma(0) = a, \gamma \in C^\infty\}$). Moreover, in algebraic geometry, a tangent space of a affine variety X at a point x is defined by $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ where $\mathfrak{m}_x$ is a maximal ideal in a coordinate ring $k[X]$ associated with a point x.

In this survey, I will define a tangent space of a variety X at a point x with the number of intersections between X and a line that passes through x which will be called as *intersection multiplicity*, and in order to be simple, I will assume that an affine variety X contains 0(i.e. $0 \in X$) and usually talk about tangent spaces at $x = 0$. This is because, we can also construct similar definitions for arbitrary points by using translation, and if we work on arbitrary points, then that could just increase the number of meaningless variables.

Definition 0.1. (*Intersection multiplicity of a line with a variety*) Let's say a is a nonzero point on $\mathbb{A}^n$ and X is a affine variety defined as the zero locus of polynomials $F_1, \cdots, F_m \in k[x_1, \cdots, x_n]$. *Intersection multiplicity* of a line $L = \{ta \in \mathbb{A}^n | t \in k\}$ with a variety $X = V(F_1, \cdots, F_m)$ is the multiplicity of $t = 0$ as a root of the polynomial

$$f(t) := lcm(F_1(ta), \cdots, F_m(ta))$$

in one variable t.

Example 0.2. For $X = \{(x, y) \in \mathbb{A}^2 | y = x^2\} = V(y - x^2) \subset \mathbb{A}^2$, an arbitrary line passes through 0 can be expressed as $L_a = \{t(a_1, a_2)|t \in k\}$ where $a = (a_1, a_2) \neq (0, 0)$. In this case $F_1 = y - x^2$, and $F(ta) = ta_2 - t^2a_1^2 = t(a_2 - ta_1^2)$. So, if we think about the intersection multiplicity of a line $L_{(1,1)}$ with a variety X, since the multiplicity of $t = 0$ as a root of $t(1 - t)$ is 1, we can conclude that

$$\text{(Intersection multiplicity of a line } L_{(1,1)} \text{ with a variety } X) = 1$$

Note. $X \cap L = \{(ta_1, ta_2)|F(ta_1, ta_2) = 0\}$ is sometimes denoted as $\{F(ta) = 0\}$.

Definition 0.3. (*Tangent at 0*) A line L is called *tangent to X at 0*, if

$$\text{(Intersection multiplicity of } L \text{ with } X \text{ at } 0) \geqslant 2.$$

Note. If 0 is a singular point, then the minimum of intersection multiplicities of any lines with X at 0 determines whether the singularity 0 is double point or multiple points.

Example 0.4. Recall the above example, $X = V(y - x^2) \subset \mathbb{A}^2$ and $L_a = \{t(a_1, a_2) | t \in k\}$ where $a = (a_1, a_2) \neq (0, 0)$. We observed that $F(ta) = t(a_2 - a_1^2 t)$. If $a_2 = 0$, then $F(at) = -a_1^2 t^2$ and this implies that the multiplicity of $t = 0$ as a root of $F(at)$ is 2. Therefore, $L_{(a_1,0)}$ is tangent to $V(y - x^2)$ at 0, and we can easily think that $L_{(a_1,0)} = L_{(1,0)}$ is a x-axis.

Example 0.5. Let's think about the case $X = V(y(y - x^2)) \subset \mathbb{A}^2$. Geometrically, X will look like union of a line(x-axis) and a parabola that passes through 0. So, one might think that $L_{(1,0)}$ is the only line that is tangent to X at 0, because all the other line is not tangent to a parabola. But this is wrong conclusion with this definition. Since $F(x, y) = y(y - x^2)$, $F(at) = a_2 t^2 (a_2 - a_1^2 t)$ and we can check that the multiplicity of $t = 0$ of $F(at)$ is greater than equal to 2 for any $0 \neq a \in \mathbb{A}^2$. Therefore, all lines passing through 0 is tangent to X. Geometrically, this case can be explained as any line passes through not only intersects with a parabola but also the line in X, and we need to think that there is an overlap at 0 between a parabola and the line in X.

Definition 0.4. (Tangent Space) Let's say that X is a variety and x is a point in X. A *tangent space to X at x* is the geometric locus of points on lines which are tangent to X at x, denoted by Θ_x or $\Theta_{X,x}$.

Note. In some books tangent spaces are denoted as $T_x X$, TX_x or $D_x X$. And although I did not mention about the algebraic structure of tangent space, almost always tangent space is considered as a k-vector space.

Example 0.6. For $X = V(y - x^2) \subset \mathbb{A}^2$, $L_{(1,0)}$ is the only line that is tangent to X at 0. Therefore, in this case, $\Theta_0 = \Theta_{X,0} = \{\text{all points on } L_{(1,0)}\ \} = L_{(1,0)}$.

Example 0.7. For $X = V(y(y - x^2)) \subset \mathbb{A}^2$, we have shown that all lines in $\mathbb{A}^2$ passing through 0 is tangent to X. Therefore, in this case, $\Theta_0 = \Theta_{X,0} = \mathbb{A}^2$.

So far, we defined intersection multiplicity of a line with a variety, and by using this algebraic concept, we defined tangency and tangent plane. From now, I will define singularity with these definitions with using dimensions.

Definition 0.8. (*Singular and Nonsingular points on a irreducible variety*) Let's say X is an irreducible variety and define $s := \min_{x \in X}(\dim \Theta_x)$. For any point $x \in X$ we can define singular and nonsingular points as following :

$$\begin{aligned} x \text{ is a } \textit{singular point} &\iff \dim \Theta_x > s \\ x \text{ is a } \textit{nonsingular point} &\iff \dim \Theta_x = s \end{aligned}$$

Note. The value of $s := \min_{x \in X}(\dim \Theta_x)$ turns out to be $s = \dim X$.

Definition 0.9. (*Resolution of Singularities*) For any variety X, if there exists a smooth variety Y and a regular birational map $\pi : Y \to X$, then a map π is called a *resolution of singularities* of X.

The existence of a resolution of singularities was proved by Heisuke Hironaka for the case of *char* $k = 0$. For the *char* $k = p > 0$ case, it is known for curves and surfaces, but it is still open for general cases.

Definition 0.9.a. (*Rational function and Rational map*) Let X be an variety. A function $f : X \dashrightarrow \mathbb{A}^1$ is called a *rational function*, if there exists polynomials g, h such that $f(x) = \dfrac{g(x)}{h(x)}$ for all $x \in X$. Moreover, if a map $F : X \dashrightarrow \mathbb{A}^m$ is a collection of rational functions, which means if $F = (f_1, \cdots, f_m)$ and if all f_j are rational functions, we call F as a *rational map.*

For rational functions and rational maps, they use $\dashrightarrow$ instead of $\to$. Since $h(x) = 0$ could happen for some $x \in X$, for this case $f(x)$ cannot be well-defined. But if we remind that important property of a function is well-definedness, this trouble should not happen when we are defining a function. Although there are points that make trouble for defining f on X, it is still possible to define f for all $x \in X$ such that $h(x) \neq 0$, and on this open set $\{x \in X | h(x) \neq 0\}$, f is a function. By these reasons, mathematicians usually use $\dashrightarrow$ instead of $\rightarrow$ while defining rational functions or maps.

Definition 0.9.b. (*Regular at a point*) A rational function f is called *regular* at a point x if there is a representation $f = \frac{g}{h}$ with $h(x) \neq 0$. (Roughly speaking, $f(x) \neq \frac{*}{0}$)

1. Resolution on Plane curves

In 1964, Heisuke Hironaka proved that every quasi-projective variety can be desingularized (i.e. every variety is birationally equivalent to smooth projective variety) over the characteristic 0 field. In his proof, he constructed desingularizing process with concrete algorithm which is called *blowing up.* The reason why this method is called as *blowing up* is the result variety looks like a grenade that extracts a line on each base point, and each line has the same direction with a line that passes through a base point and the origin on the base space. In this section, I will explain this well-organized method and I will try to resolve several singularities of plane curves.

Definition 1.1. (*Blowing up a point p in $\mathbb{A}^n$*) Key of this *blow up* is leaving $\mathbb{A}^n \backslash \{p\}$ part in $\mathbb{A}^n$ itself, and replacing p with the set of all lines passing through p in $\mathbb{A}^n$. Therefore, the *blowup of $\mathbb{A}^n$ at a point p* is a set $B_p(\mathbb{A}^n) \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ such that

$$B_p(\mathbb{A}^n) = (\{p\} \times \mathbb{P}^{n-1}) \cup \{(x, l) \in (\mathbb{A}^n \backslash \{p\}) \times \mathbb{P}^{n-1} | x \in l\}$$

Consider $\mathbb{P}^{n-1}$ as a set of all lines starting from p in $\mathbb{A}^n$ (regarding p as a origin in $\mathbb{A}^n$).

In other words, $B_p(\mathbb{A}^n)$ is a set of all pairs $(x, l) \in \mathbb{A}^n \times \mathbb{P}^{n-1}$ such that l is a line starting from p passing through x. This tells us why $B_p(\mathbb{A}^n)$ contains all lines at p more easily, because all lines starting from p must pass the point p.

In fact, for any point $p = (p_1, \cdots, p_n) \in \mathbb{A}^n$, if we say that $\big((x_1, \cdots, x_n), [z_1, \cdots, z_n]\big) \in B_p(\mathbb{A}^n)$, we can think this set as the zero locus of polynomials $\{(X_i - p_i)Z_j - (X_j - p_j)Z_i | 1 \leqslant i, j \leqslant n\}$ in variables of X_i, X_j, Z_i, and Z_j. So we can say that $B_p(\mathbb{A}^n)$ is not just set but also a variety. Moreover, sometimes a variety $B_p(\mathbb{A}^n)$ with its natural projection π :

$$\begin{gathered} \pi : B_p(\mathbb{A}^n) \rightarrow \mathbb{A}^n \\ (x, l) = \big((x_1, \cdots, x_n), [z_1, \cdots, z_n]\big)\big) \mapsto x = (x_1, \cdots, x_n) \end{gathered}$$

is called the *one-point blowup of* $\mathbb{A}^n$.

Note. In *Joe Harris' book*, he defines *blow up* (of $\mathbb{P}^n$ at a point) as a graph $\tilde{\mathbb{P}^n} := \Gamma_\varphi$ of the rational projection map $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$. When the rational projection map $\varphi : X \dashrightarrow \mathbb{P}^n$ is given, the graph of φ is a Zariski closure of $\{(x, y) \in X \times \mathbb{P}^n | \varphi(x) = y\}$ in $X \times \mathbb{P}^n$ denoted by Γ_φ.

However, one more interesting thing for this space $B_p(\mathbb{A}^n) \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ is that we can think this object as a trivial line bundle. If we construct a projection map onto $\mathbb{P}^{n-1}$ (base space : $\mathbb{P}^{n-1}$)

$$\begin{gathered} \pi_2 : B_p(\mathbb{A}^n) \rightarrow \mathbb{P}^{n-1} \\ (x, l) \mapsto l \end{gathered}$$

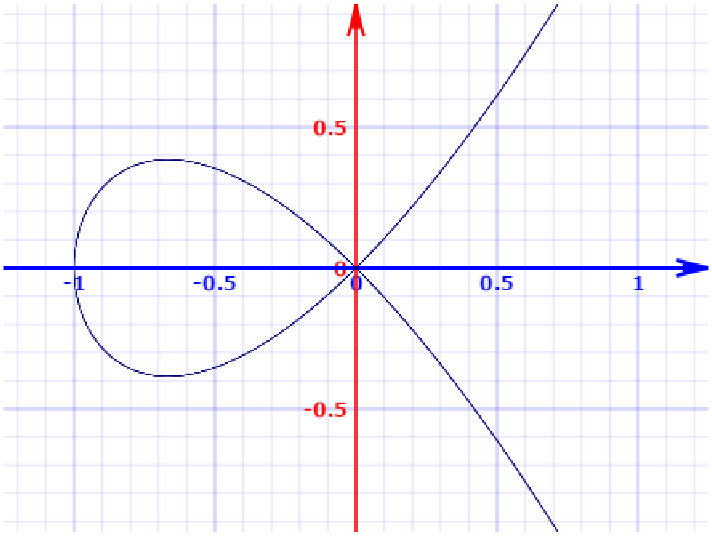
then by regrading π_2 as a projection map, $B_p(\mathbb{A}^n)$ is the *tautological line bundle* or *trivial line bundle* over $\mathbb{P}^{n-1}$.

So far, we talked about one-point blowup of $\mathbb{A}^n$. But this is just the most typical type of blowup. We can generalize $\mathbb{A}^n$ to more general varieties, or we can blowup a larger subvariety of $\mathbb{A}^n$ instead of a point. In general, it is possible to blowup arbitrary closed subvarieties of an arbitrary quasi-projective variety, but in this section it is enough to define the blowup of a point in an arbitrary affine variety.

Definition 1.2. (*Blowup of an affine variety at a point*) Let $V \subset \mathbb{A}^n$ be an affine variety and p be a point in V. The *blowup of V at p*, denoted by $B_p(V)$, is the Zariski closure of $\pi^{-1}(V \backslash \{p\})$ in $\mathbb{B}_p(\mathbb{A}^n)$.

Note. π is a rational natural projection such that $\pi : V \subset \mathbb{A}^n \dashrightarrow \mathbb{P}^n$ defined by $\pi(x_1, \cdots, x_n) = [x_1, \cdots, x_n]$.

Example 1.3. Let X be a curve $\{(x,y) \in \mathbb{A}^2 | y^2 = x^3 + x^2\}$ (i.e. $X = V(y^2 - x^3 - x^2)$). We can easily check that $\min_{x \in X}(\dim \Theta_x) = 1$ and $\dim \Theta_0 = 2$, and this implies that 0 is a singular point on X. If we substitute (x,y) with (t, mt) in $F(x,y) = y^2 - x^3 - x^2$, then this will give us $F(t, mt) = (-1 + m^2)t^2 - t^3$. From here, we can check that minimal intersection multiplicity of a tangent line and X is 2, thus we can determine 0 is a *double point*, and by checking that, $-1 + m^2 = 0$ has two real roots for m, we can also call 0 as *crunode.*



(graph of $y^2 = x^3 + x^2$)

Now let's think about the canonical projection map $\pi : \mathbb{A}^2 \times \mathbb{P}^1 \dashrightarrow \mathbb{A}^2$ defined by $\pi\big((x,y),[x,y]\big) = (x,y)$. Then, we can define $B_0(\mathbb{A}^2) = \big(\{(0,0)\} \times \mathbb{P}^1\big) \cup \big\{\big((x,y),[x,y]\big) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid (x,y) \neq (0,0)\big\}$. In order to construct $B_0(X)$, we first need to think about $\pi^{-1}(X\backslash\{0\})$. The preimage of $X\backslash\{0\}$ is determined by

$$\pi^{-1}(X\backslash\{0\}) = \big\{\big((x,y),[x,y]\big) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid (x,y) \in X\backslash\{0\}\big\}$$

Since $\mathbb{P}^2$ can be thought as union of two affine spaces, namely $\mathbb{P}^2 = U_0 \cup U_1$ where $U_i = \{[Z_0, Z_1] \mid Z_i \neq 0\}$, we can also think $\pi^{-1}(X\backslash\{0\})$ as a union of $\pi^{-1}(X\backslash\{0\}) \cap (\mathbb{A}^2 \times U_0)$ and $\pi^{-1}(X\backslash\{0\}) \cap (\mathbb{A}^2 \times U_1)$.

$$\begin{aligned}
\pi^{-1}(X\backslash\{0\}) &= \big(\pi^{-1}(X\backslash\{0\}) \cap (\mathbb{A}^2 \times U_0)\big) \cup \big(\pi^{-1}(X\backslash\{0\}) \cap (\mathbb{A}^2 \times U_1)\big) \\
\pi^{-1}(X\backslash\{0\}) \cap (\mathbb{A}^2 \times U_0) &= \big\{\big((x,y),[x,y]\big) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid (x,y) \in X, x \neq 0\big\} \\
&= \big\{\big((x,y),[x,y]\big) \mid y^2 = x^3 + x^2,\ x \neq 0\big\} \\
&= \big\{\big((x,mx),[1,m]\big) \mid m^2 = x + 1,\ x \neq 0,\ m \in k\big\} \\
&\cong \big\{(x,m) \in \mathbb{A}^2 \mid m^2 = x + 1,\ x \neq 0,\ m \in k\big\}
\end{aligned}$$

Since Zariski closure of $\big\{(x,m) \in \mathbb{A}^2 \mid m^2 = x + 1,\ x \neq 0,\ m \in k\big\}$ in $\mathbb{A}^2$ is $V(Y^2 - X - 1)$, if we think about corresponding set in $\mathbb{A}^2 \times U_0$, then we can conclude that Zariski closure of $\pi^{-1}(X\backslash\{0\}) \cap (\mathbb{A}^2 \times U_0)$ in $(\mathbb{A}^2 \times U_0)$ is

$$\big(\pi^{-1}(X\backslash\{0\}) \cap (\mathbb{A}^2 \times U_0)\big) \cup \Big\{\big((0,0),[1,1]\big), \big((0,0),[1,-1]\big)\Big\},$$

similarly we can also find that the Zariski closure of $\pi^{-1}(X\backslash\{0\}) \cap (\mathbb{A}^2 \times U_1)$ in $(\mathbb{A}^2 \times U_1)$ is

$$\big(\pi^{-1}(X\backslash\{0\}) \cap (\mathbb{A}^2 \times U_1)\big) \cup \Big\{\big((0,0),[1,1]\big), \big((0,0),[-1,1]\big)\Big\},$$

As a result, the blowup of X at a point 0 is

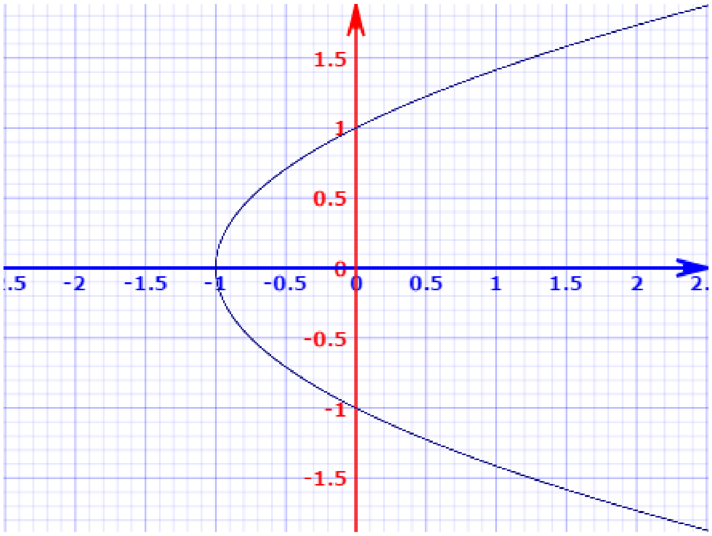
$$\begin{aligned}
B_p(X) &= (\text{Zariski closure of } \pi^{-1}(X\backslash\{0\})) \\
&= (\text{Zariski closure of } \pi^{-1}(X\backslash\{0\}) \cap (\mathbb{A}^2 \times U_0)) \cup (\text{Zariski closure of } \pi^{-1}(X\backslash\{0\}) \cap (\mathbb{A}^2 \times U_1)) \\
&= \pi^{-1}(X\backslash\{0\}) \cup \Big\{\big((0,0),[1,1]\big), \big((0,0),[1,-1]\big)\Big\}
\end{aligned}$$

Now let's talk about why two points $\big((0,0),[1,1]\big)$ and $\big((0,0),[1,-1]\big)$ are smooth on this variety. Let's think about a point $\big((0,0),[1,1]\big)$. We can see this is a point in $\mathbb{A}^2 \times \mathbb{P}^1$, but we can also think this is a point in $\mathbb{A}^2 \times U_0$. Moreover, all points in $\mathbb{A}^2 \times U_0$ can be expressed as $\big((x,y),[1,m]\big)$, and by corresponding a point

$\big((x,y),[1,m]\big) \in \mathbb{A}^2 \times U_0$ with a point $(x,y,m) \in \mathbb{A}^3$, we can consider a neighborhood of $\big((x,y),[1,m]\big)$ is $\mathbb{A}^2 \times U_0$ as a neighborhood of (x,y,m) in $\mathbb{A}^3$. With this correspondence, we can also interpret $B_0(X) \cap (\mathbb{A}^2 \times U_0)$ as $\big\{(x,mx,m) \in \mathbb{A}^3 \mid m^2 = x+1\big\}$, or simply $\big\{(x,m) \in \mathbb{A}^2 \mid m^2 = x+1\big\}$. Since we used the same correspondence while simplifying $\big((x,y),[1,m]\big)$ and $B_0(X) \cap (\mathbb{A}^2 \times U_0)$, we can conclude that

$$\big((0,0),[1,1]\big) \text{ is a smooth point on } B_0(X) \iff (0,0,1) \text{ is a smooth point on } \big\{(x,mx,m) \in \mathbb{A}^3 \mid m^2 = x+1\big\}$$
$$\iff (0,1) \text{ is a smooth point on } \big\{(x,m) \in \mathbb{A}^2 \mid m^2 = x+1\big\}$$

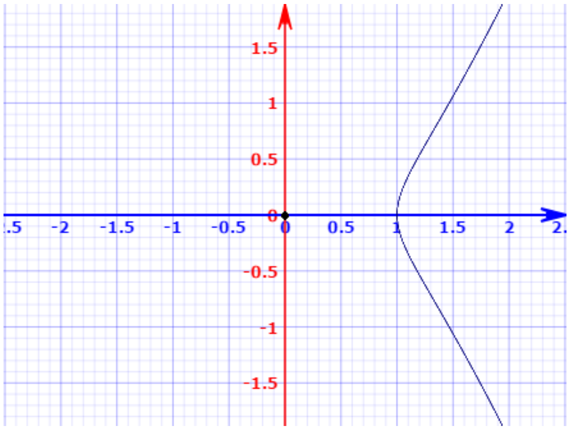
If we think about the variety $V(Y^2 - X - 1)$ in $\mathbb{A}^2$, it will give a nice parabola as following :



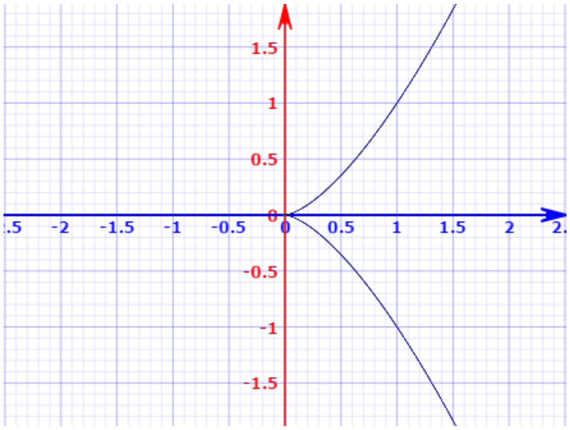
(graph of $y^2 = x + 1$)

and there is no doubt to say that $(0,1)$ is a smooth point on this parabola. Therefore, we can conclude that $\big((0,0),[1,1]\big)$ is a smooth point on $B_0(X)$. Similarly, we can conclude that $\big((0,0),[1,-1]\big)$ is a smooth point on $B_0(X)$, because $(0,-1)$ is a smooth point on the preceding parabola.

Note. Another types of double points : If the coefficient of t^2 was a quadratic polynomial in variable m with no real roots, then it is called *acnode*, and if it had single root with multiplicity, then it is called *cusp*. We can see *acnode* in the graph of $y^2 = x^3 - x^2$, and *cusp* in the graph of $y^2 = x^3$, both at a point 0.



(graph of $y^2 = x^3 - x^2$)



(graph of $y^2 = x^3$)

2. Resolution on Varieties

Blow up can be used not only on curves but also on general algebraic varieties. The definition of the blow up of an arbitrary variety along an ideal or a subvariety, comes from the generalization of the idea of how the blow up of $\mathbb{A}^n$ along a point was defined. If we think about the definition of $B_0(\mathbb{A}^n)$, which is a blow up of $\mathbb{A}^n$ along a point 0, this space can be interpreted as a Zariski closure of the graph of $\pi : \mathbb{A}^n \backslash \{0\} \to \mathbb{P}^{n-1}$ defined by $\pi(x_1, \cdots, x_n) = [x_1, \cdots, x_n]$. Since this morphism π can be though as a rational map $\pi : \mathbb{A}^n \dashrightarrow \mathbb{P}^{n-1}$, we can interpret the blow up of $\mathbb{A}^n$ along a point 0 is equivalent to the (Zariski closure of) graph of the rational map $\pi : \mathbb{A}^n \dashrightarrow \mathbb{P}^{n-1}$. Thus, by using the idea of Zariski closure of graph of a certain rational map, we can define blow up of the varieties along an ideal or a subvariety.

Definition 2.1. (*Blowup of a variety along the an ideal*) Let X be an affine algebraic variety, and $\mathbb{C}[X]$ is a coordinate ring of X. Take finite number of functions $F_1, \cdots, F_r$ in $\mathbb{C}[X]$, and define an ideal I by

$$I = \langle F_1, \cdots, F_r \rangle \subset \mathbb{C}[X].$$

Then, *blow up of the variety X along the ideal I* is the graph B of the rational map $F : X \dashrightarrow \mathbb{P}^{r-1}$, defined by

$$\begin{gathered} F(x) = [F_1(x), \cdots, F_r(x)] \text{ for } x \in X \backslash Y \\ (Y : \text{ the zero locus of } F_1, \cdots, F_r \text{ in } X) \end{gathered}$$

together with the natural projection map

$$\begin{gathered} \pi : B \longrightarrow X \qquad \text{where } B \subset X \times \mathbb{P}^{r-1} \\ \pi(x, F(x)) = x \end{gathered}$$

This graph B is denoted by $B_I(X)$.

If we think about all $x \in X \backslash Y$, $Y = X \cap V(F_1, \cdots, F_r)$ implies that $(F_1(x), \cdots, F_r(x)) \neq (0, \cdots, 0)$. So, for these elements x, $[F_1(x), \cdots, F_r(x)] \in \mathbb{P}^{r-1}$ is well-defined. Moreover, if we restrict $\pi : B_I(X) \longrightarrow X$ to $B_I(X) \backslash \pi^{-1}(Y)$, then π maps $B_I(X) \backslash \pi^{-1}(Y)$ to $X \backslash Y$, and the map $id_X \times F : X \backslash Y \to B_I(X) \backslash \pi^{-1}(Y)$, which is defined by $x \mapsto (x, F(x))$, exactly becomes an inverse morphism of $\pi : B_I(X) \backslash \pi^{-1}(Y) \to X \backslash Y$. Therefore, $\pi : B_I(X) \backslash \pi^{-1}(Y) \to X \backslash Y$ is an isomorphism, and this implies that $B_I(X)$ and X are birationally equivalent through π.

By the way, if we try to look for the above definition of $B_I(X)$, we defined $B_I(X)$ with the generators $F_1, \cdots, F_r$ of I. Because the rational map F, can be defined by using $F_1, \cdots, F_r$. However, in fact, the isomophism class of the blowup $B_I(X)$ only depends on I, which means different choice of generators of I will produce another blowup which is still isomorphic to the previous one. Therefore, isomorphic class of $B_I(X)$ is well-defined over many choice of generators of I.

Definition 2.2. (*Blowup of a variety along its subvariety*) Let's say that X is an affine variety, and Y is a subvariety of Y. Since Y is a subvariety of X, there exists corresponding radical ideal $\mathbb{I}(Y) \subset \mathbb{C}[X]$. The *blowup of X along a subvariety Y* is $B_{\mathbb{I}(Y)}(X)$, and it is denoted by $B_Y(X)$.

Note. *Rees Algebra*(Blowup Algebra) Let R be a commutative ring, and let I be an ideal of R. Then *Rees algebra* of I is defined by

$$\mathrm{Rees}_R(I) := \bigoplus_{j \geqslant 0} I^j = R \oplus I \oplus I^2 \oplus I^3 \oplus \cdots$$

This definition is used in *Aluffi's book* 164 page, and in *Eisenbud's book* 148 page, he introduce this R-algebra as *blowup algebra of I in R*, denoted by $B_I R$. We need to think about for about why this algebra is called as blowup algebra. Because, if we define R as the coordinate k-algebra, for example if $k = \mathbb{C}$ then $R = \mathbb{C}[X]$, and if we say that Y is a subvariety of X, then the radical ideal $\mathbb{I}(Y)$ is an ideal in $\mathbb{C}[X]$. In this case, blowup algebra of $\mathbb{I}(Y)$ in $\mathbb{C}[X]$ is defined by

$$\mathrm{Rees}_{\mathbb{C}[X]}(\mathbb{I}(Y)) = B_{\mathbb{I}(Y)}\mathbb{C}[X] = \mathbb{C}[X] \oplus \mathbb{I}(Y) \oplus \mathbb{I}(Y)^2 \oplus \cdots.$$

And this graded ring is exactly same as the coordinate ring of blowup $B_Y(X)$ of X along a subvariety Y. Therefore, *Rees Algebra* or *Blowup algebra* is a coordinate ring of a blowup variety.

3. Closing

For the field of characteristic 0, resolution of singularity is proved by Hironaka 53 years ago, in 1964. But in the field of positive characteristics, it is not proved. According to *Bulletin of AMS*, which was published in 2011, there are two current streams of work. One stream is finding general proof of resolution in arbitrary dimension, and the other stream is attempt to construct and improve methods of resolution in low dimensions.

For the first stream, although new invariants of singularity have been produced and new geometry and algebra are found, the general problem is not solved yet. However, in the second stream, there were big progresses.

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