Non-Parallelizable Real Projective Space and Stiefel-Whitney Classes

See-Hak Seong

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Abstract

Parallelizability allows us to define a collection of continuous vector fields on a manifold which forms a basis on every tangent space. Therefore, many properties of tangent spaces of a manifold can be easily discovered by observing this collection of vector fields. However, mathematicians find out that S^2 is not parallelizable and mathematicians need to decide whether an arbitrary manifold is parallelizable or not. In this paper, I will suggest one method that many mathematicians used in order to conclude non-parallelizability, named Stiefel-Whitney Classes, and I will show you how it works on real projective spaces $\mathbb{R}P^n$.

0. Preliminaries

In this part, I will briefly state all basic definitions of smooth manifolds and smooth functions. In order to define parallelizablity, we need to define vector bundle and tangent bundle which will be handled in the next section. Since smooth manifolds and their tangent manifolds are usual examples of vector bundle and tangent bundle, this part will help you to construct intuition of bundles.

Definition 0.1. A be any index set. $\mathbb{R}$-vector space $\mathbb{R}^A$ is defined by $\mathbb{R}^A := \{x \mid x : A \to \mathbb{R} \text{ is a function}\}$. For any $\alpha \in A$, define *α-th coordinate of* x by the value $x(\alpha)$ and denote it as x_α. For any function $f : Y \to \mathbb{R}^A$, define $f_\alpha(y) :=$ *α-th coordinate of* $f(y)$.

Example 0.2. For $A = \{1, 2, \cdots, n\}$, $\mathbb{R}^A$ is denoted by $\mathbb{R}^n$. And for arbitrary function $f : Y \to \mathbb{R}^n$, f can be interpreted as $f = (f_1, \cdots, f_n)$.

Definition 0.3. (*Smooth functions on* $\mathbb{R}^n$) For an open subset $U \subset \mathbb{R}^n$, a function $f : U \to M \subset \mathbb{R}^A$ is called *smooth* if $f_\alpha : U \to \mathbb{R}$ is C^∞ for all $\alpha \in A$. In this case, $(\dfrac{\partial f}{\partial u_i})_\alpha = \dfrac{\partial f_\alpha}{\partial u_i}$.

Note. In some papers, the word 'map' is used instead of 'continuous function'.

Definition 0.4. (*Smooth manifolds*) A subset $M \subset \mathbb{R}^A$ is called a *smooth manifold* of dimension $n \geqslant 0$ if for any $x \in M$, there exists an open set $U \in \mathbb{R}^n$ and a smooth function $h : U \to \mathbb{R}^A$ which satisfy following properties :
(a) There exists a neighborhood V of x in M such that $h : U \to V$ is a homeomorphism.
(b) For all $u \in U$, the matrix $[\dfrac{\partial h_\alpha(u)}{\partial u_j}]$ has rank n.
In this case, $V = h(U)$ is called as *coordinate neighborhood* in M, and triple (U, V, h) is called as *local parametrization* of M.

Note. In some book(Boothby) a *local parametrization* (U, V, h) is expressed in another way by using its inverse. In this case, the inverse $h^{-1} : V \to U \subset \mathbb{R}^n$ is defined as *local coordinate system* or *chart* for M. However, in this paper I will define *local coordinate system* differently in the next section.

Definition 0.5. (*Tangent vector, space, manifold*) A vector $v \in \mathbb{R}^A$ is *tangent* to M at a point $x \in M$ if v can be defined as the velocity vector of some smooth path in M that passes x. The collection of those tangent vectors is defiend as *tangent space* of M at x and denoted by DM_x. Moreover, the collection of those tangent spaces with their standard points is defined as *tangent manifold* and denoted as DM.

$$DM := \{(x, v) \in M \times \mathbb{R}^A \mid v \in DM_x\}$$

Definition 0.6. (*Smooth functions on a manifold*) $M \subset \mathbb{R}^A$ and $N \subset \mathbb{R}^B$ are smooth manifolds. Let's take a point $x_0 \in M$ with a local parametrization (U, V, h) of M with $x_0 = h(u_0)$. The function $f : M \to N$ is said to be *smooth at* x_0 if $f \circ h : U \to N \subset \mathbb{R}^B$ is smooth on some neighborhood of u_0. If f is smooth at all points in M, then f is called *smooth.* Moreover, f is called a *diffeomorphism* if f is bijective and both f and f^{-1} are smooth.

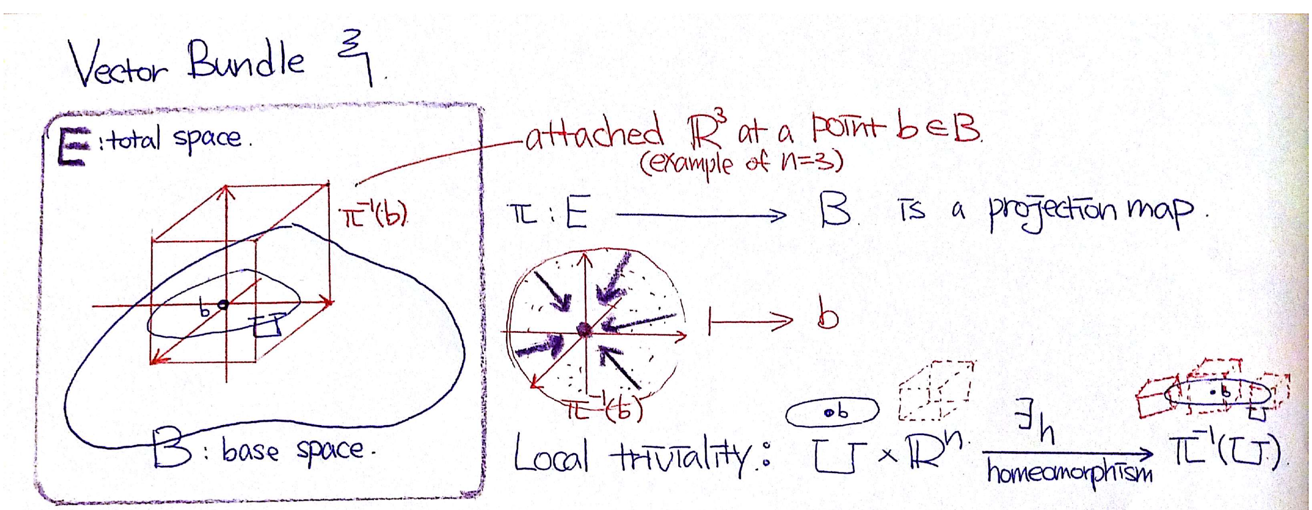
Remark. With all of definitions above, we can construct *the category of smooth manifolds and smooth maps.* Although we need to show several things more, such as whether composition of smooth functions is smooth, since all those things can be proved easily I will just claim that this collection of smooth manifolds and smooth functions satisfies category structure. So if we consider with this category, *diffeomorphism* can be explained as an *isomorphism* between two objects in this category.

1. Vector Bundles

Basic idea of vector bundles comes from attaching arbitrary real vector spaces at each point of a topological space. Although this idea sounds weird, the intuitive example of vector bundle is always suggested in calculus classes in different way, namely tangent space of a sphere. If we recall this example, then we could possibly guess that concept of vector bundle is a generalization of a collection of tangent spaces labelled with each standard points. With this intuition, I will state the definition of vector bundles, tangent bundles, and parallelizability of topological spaces. Moreover, several properties and methods to create new vector bundles will be suggested in this section in order to prepare for the next section.

Note. Among all part of this paper I will fix a topological space B and this will be defined as the *base space.*

Definition 1.1. A *real vector bundle* ξ over B is constructed by following objects :
(a) a topological space $E = E(\xi)$ called a *total space.*
(b) a map $\pi : E \to B$ is called a *projection map*, and for all $b \in B$ a set $\pi^{-1}(b)$ has the $\mathbb{R}$-vector space structure.
And ξ following with E, B and π should satisfy following property :
(c) (*Local triviality*) For every point $b \in B$, there should exist a neighborhood $U \subset B$, an integer $n \geqslant 0$, and a homeomorphism $h : U \times \mathbb{R}^n \to \pi^{-1}(U)$ so that, for each $b \in U$, the correspondence $x \in \mathbb{R}^n \mapsto h(b, x) \in \pi^{-1}(b)$ defines an isomorphism between the vector space $\mathbb{R}^n$ and the vector space $\pi^{-1}(b)$.



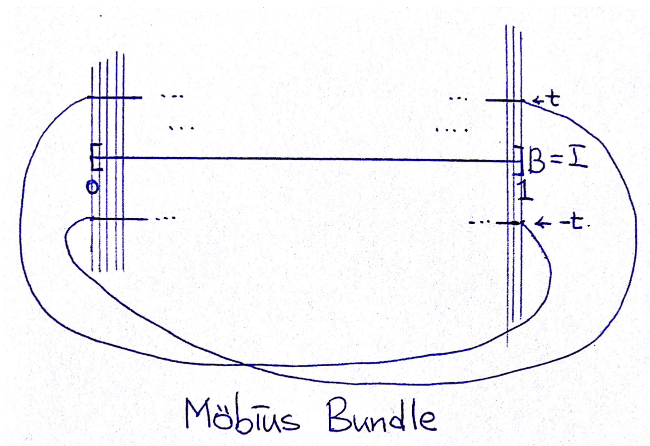
Definition 1.2. The vector space $\pi^{-1}(b)$ is called *fiber over b*. It may be denoted by F_b or $F_b(\xi)$

Definition 1.3. A pair (U, h) which is defined in the definition of vector bundle is called a *local coordinate system for ξ about b*. In special case, if (B, h) becomes a local coordinate system for ξ, then ξ is defined as a *trivial bundle*.

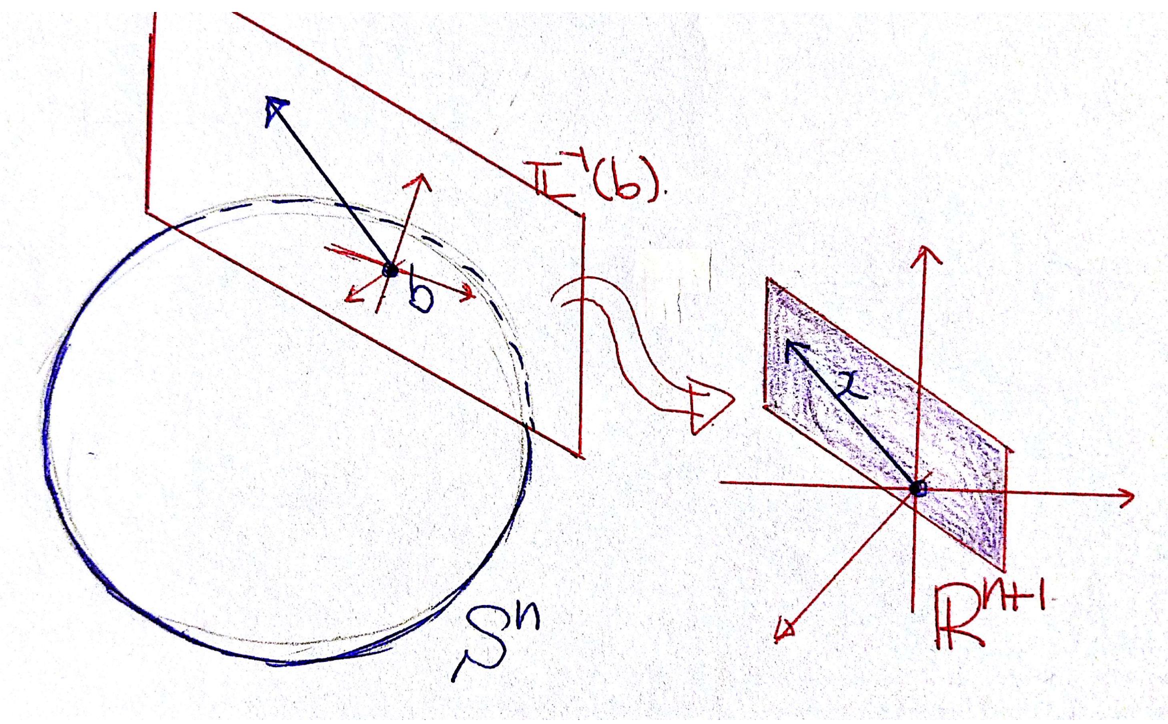
Definition 1.4. A vector bundle ξ is called *n-plane bundle* or *$\mathbb{R}^n$-bundle* if, for all $b \in B$ $F_b = \pi^{-1}(b)$ is a n dimensional $\mathbb{R}$-vector space (i.e. $\dim E(\xi) = n$). In the case of $n = 1$, it is sometimes called as a *line bundle*

Example 1.5. *product* or *trivial bundle* ϵ_B^n : $E = B \times \mathbb{R}^n$ with the projection map $\pi : (b, x) \mapsto b$

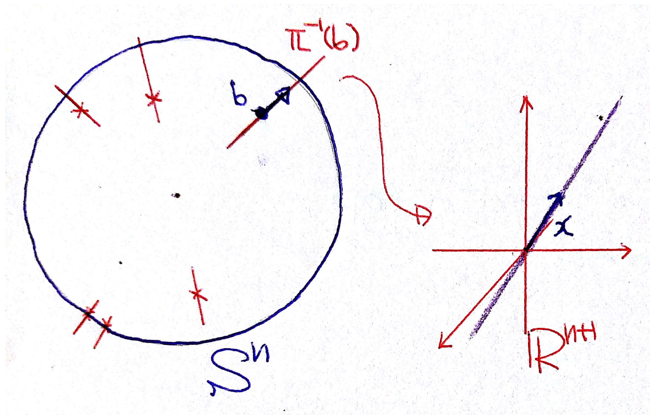
Example 1.6. *Möbius bundle* : $E = I \times \mathbb{R}/(0, t) \sim (1, -t)$, $B = I/\{0\} \sim \{1\}(= S^1)$ with the projection map $\pi : [(i, t)] \mapsto [i]$ (where [ ] means an equivalence class in each spaces)



Example 1.7. *Tangent bundle of unit sphere $S^n \subset \mathbb{R}^{n+1}$* : $E = \{(b, x) \in S^n \times \mathbb{R}^{n+1} \mid b \perp x\}$, $B = S^n$ with the projection map $\pi : (b, x) \mapsto b$.



Example 1.8. *Normal bundle to $S^n \subset \mathbb{R}^{n+1}$* : $E = \{(b, x) \in S^n \times \mathbb{R}^{n+1} \mid DS^n_b \perp x\}$, $B = S^n$ with the projection map $\pi : (b, x) \mapsto b$. *Normal bundle of S^n* is a line bundle.



Definition 1.9. The vector space ξ is called a *smooth vector bundle* if all objects and morphisms are contained in the category of smooth manifolds. To be specific, E, B are smooth manifolds, π is a smooth map, and h should be diffeomorphism.

Example 1.10. *Tangent bundle τ_M of a smooth manifold M : $E = DM = \{(x, v) \in M \times \mathbb{R}^A \mid v \in DM_x\}$,* $B = M$ with the projection map $\pi : (x, v) \mapsto x$. One might guess that $\tau_M = DM$, but since DM has a structure of manifold and τ_M has a structure of vector bundle, it will be better to distinguish them.

After defining the vector bundle, we need to decide what vector bundles are the same. Since an equivalence relation gives the classification scheme, the definition of isomorphism should be constructed at first.

Definition 1.11. Suppose that two vector bundles ξ and η are defined over the same base space B. ξ is *isomorphic* to η, if there exists a homeomorphism $f : E(\xi) \to E(\eta)$ between the total spaces which maps each vector space structured fibers $F_b(\xi)$ isomorphically onto the corresponding vector space structured fibers $F_b(\eta)$. In this case, it is denoted by $\xi \cong \eta$.

To be more precise, an isomorphism $f : E(\xi) \to E(\eta)$ is actually consisted of two maps, one is a homeomorphism f between two topological spaces $E(\xi)$ and $E(\eta)$ and the other is an isomorphic linear map $L_{f,b} : F_b(\xi) \to F_b(\eta)$ which varies over the base point $b \in B = B(\xi) = B(\eta)$. Moreover, in this case, f acts as an identity map on the base space B. With all these description, f maps (b, x) where $x \in F_b(\xi)$ to $(b, L_{f,b}(x))$ where $L_{f,b}(x) \in F_b(\eta)$.

Usually, category lovers defines morphisms before defining isomorphism. So, the order of definition might confuse some people who have read this. However, since we are focusing on classification rather than constructing specific structures, morphism of this category which is named by *bundle map*, will be appeared later and in that case $f\,|_{B(\xi)}$ could not be the identity map.

From now I will suggest the main words of this survey, parallelizable and the real projective space $\mathbb{R}P^n$.

Definition 1.11.(Parallelizable) A manifold M is called *parallelizable* if its tangent bundle τ_M is isomorphic to a trivial bundle.

In Boothby's book, this terminology is defined by using the existence of linearly independent n-vector fields, which is the special case of *cross-section*, where n is the dimension of M. Although both use different words, both indicates the same definition and since the concept of *vector bundle* is more general I will use this definition, and the idea of Boothby will be appeared later in Theorem 1.17.

Definition 1.12. A *cross-section* of a vector bundle ξ with base space B is a continuous function $s : B \to E(\xi)$ such that $s(b) \in F_b(\xi)$ for all $b \in B$. If M is a smooth manifold and $\xi = \tau_M$, cross-section is usually called a *vector field* on M. Moreover, a cross-section called *nowhere zero* if $s(b) \neq 0 \in F_b(\xi)$ for all $b \in B$.

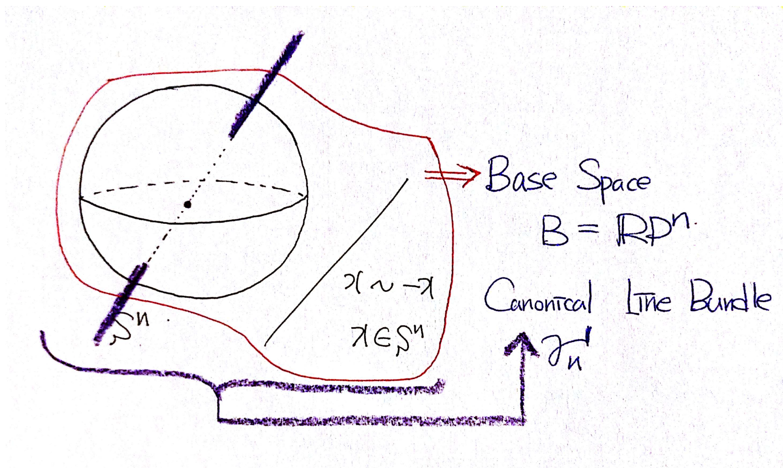
The word *section* comes from $\pi \circ s = id_B$. Since the word *section* is defined as right inverse, the map s can be interpreted as the right inverse of π. And if the reader is friendly with algebraic geometry, then one could guess that the definition of cross-section is similar to the definition of section $s : U \to \coprod_{P \in U} \mathscr{O}_{X,P}$.

Remark 1.13. S^2 is not parallelizable manifold. This can be proved by using the degree of a map $f : S^2 \to S^2$. (See [Hatcher,Algebraic Topology 135pg])

Although I will not suggest the proof of the following statement, but it is well known that $S^1,\ S^3,\ S^7$ are the only spheres that are paralellizable. This facts can be found in [Hatcher,Vector Bundles and K-theory 59pg].

Definition 1.14.($\mathbb{R}P^n$ and γ_n^1) The *real projective space* $\mathbb{R}P^n$ can be defined as the quotient space of $S^n \subset \mathbb{R}^{n+1}$, quotient by antipodal points. All points in $\mathbb{R}P^n$ can be interpreted as $\{\pm x\}$ where $x \in S^n$. Moreover, the canonical line bundle γ_n^1 over $\mathbb{R}P^n$ as following :

$$E(\gamma_n^1) = \{(\{\pm x\}, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \mid \exists c \in \mathbb{R}\ v = cx\}$$
$$\pi : E(\gamma_n^1) \to \mathbb{R}P^n \text{ is defined by } \pi(\{\pm x\}, v) = \{\pm x\}$$



Theorem 1.15. The bundle γ_n^1 is not isomorphic to trivial bundle for $n \geqslant 1$.

Proof. We can always define nowhere zero cross-section on a trivial real line bundle. However, if we define cross-section $s : \mathbb{R}P^n \to E(\gamma_n^1)$ and quotient map $p : S^n \to \mathbb{R}P^n$, then there is a continuous real valued function on S^n such that $s(p(x)) = (\{\pm x\}, t(x)x)$ for all $x \in S^n$. Since $p(x) = p(-x) = \{\pm x\}$, we can say that $t(-x) = -t(x)$. And by using the *Intermediate Value Theorem*, t should have a zero value in somewhere. Thus, we cannot define nowhere zero cross-section on $\mathbb{R}P^n$ and this ends the proof. □

The big game of the above theorem was proving the given bundle is non isomorphic, and key idea was that the trivial bundle admits a nowhere zero cross-section. Nowhere zero could be meaningful in line bundles, but this concept will not be useful for higher dimension. Therefore, we need to find replacement for this idea and the concept of linearly independent will help to generalize this concept.

Definition 1.16. A collection $\{s_1, \cdots, s_n\}$ of cross-sections of a vector bundle ξ is called *nowhere dependent* if, for every $b \in B$, the vectors $s_1(b), \cdots, s_n(b)$ are linearly independent in $F_b(\xi)$.

Nowhere dependent is also can be interpreted as everywhere linearly independent. And Now I will show the theorem, which tells how can we generalize the idea of Theorem 1.15.

Theorem 1.17. An $\mathbb{R}^n$-bundle ξ is trivial iff ξ possesses n cross-sections $s_1, \cdots, s_n$ which are nowhere dependent.

Key for this theorem is the following lemma.

Lemma 1.18. There are two vector bundles ξ and η which share the same base space B. Let $f : E(\xi) \to E(\eta)$ is a continuous map such that $f\mid_B = id_B$ and $L_{f,b} : F_b(\xi) \to F_b(\eta)$ is a vector space isomorphism for all $b \in B$. Then, $f$is a homeomorphism, thus isomorphism between two vector bundles.

This lemma is quite straightforward, because the proof only uses the local triviality of vector bundles and the fact that invertible linear map is homeomorphism. And if we construct $f : B \times \mathbb{R}^n \to E(\xi)$ such that $f(b,x) = x_1 s_1(b) + \cdots + x_n s_n(b)$ where $x = (x_1, \cdots, x_n)$ then this lead us to prove the Theorem 1.17 by using above lemma.

Surprisingly, this theorem can be more refined in specific vector bundles, *Euclidean Vector Bundles*.

Definition 1.19. A *Euclidean vector bundle* is a real vector bundle ξ with a continuous function $\mu : E(\xi) \to \mathbb{R}$ such that for each $b \in B$, $\mu \mid_{F_b(\xi)}$ is a positive definite quadratic function. The μ is called as a *Euclidean metric* on ξ.

One might be curious why such a positive definite quadratic function is called *metric*. This is because, any quadratic function implies a symmetric bilinear map, and positive definite allows this map to agree with the property of inner products. And by defining this map as inner product, this allows us to define norm and orthogonality and this is why such quadratic form is called metric. In special case, if $B = M$ is a smooth manifold and $\xi = \tau_M$, a Euclidean metric $\mu : DM \to \mathbb{R}$ is called *Riemannian metric*, and (M, μ) is called *Riemannian manifold*.

So if I use this metric with the idea of Gram-Schmidt process, the Theorem 1.17 can be refined as following.

Theorem 1.20. $\xi \cong \epsilon_B^n$ iff there exist n cross-sections $s_1, \cdots, s_n$ which $\{s_1(b), \cdots, s_n(b)\}$ forms an orthonormal basis of $F_b(\xi)$ for all $b \in B$. (Specifically, Boothby called these cross-sections as *coordinate frames* for the case of $B = M$ is a smooth manifold and $\xi = \tau_M$.)

If we can find any collection of nowhere dependent n cross-sections, we can show that given bundle is trivial. So these theorems(1.17 and 1.20) are very powerful to prove whether the given bundle is trivial when specific collection of cross-sections is suggested. However, as we have shown in γ_n^1, some manifolds do not admit this condition. Moreover, for arbitrary vector bundle, it will be hard to find a full collection of nowhere dependent cross-sections. Therefore, some mathematicians tried to focus on deciding non-trivial vector bundles and this is the one reason why the idea of *Characteristic class* pop up.

1.A. Basic ways to construct vector bundles

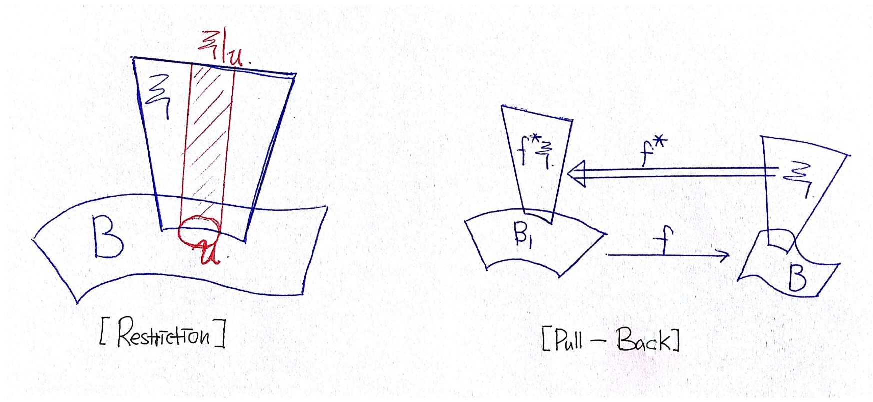
Before getting into the chapter of Stiefel-Whitney class, I will suggest several ways to construct vector bundles. Although I suggest examples of *canonical line bundle* γ_n^1 and tangent bundle τ_M, there are a number of vector bundles and usually they can be constructed by using a number of operators in vector spaces. To be specific, since tensor product, Hom-functor, and direct sum preserves its vector space structure, these operators are often used to construct new bundles from given ones. Moreover, these tools will be useful, because new bundles can be studied by using algebraic properties of these operators.

A.1. Restriction Let ξ be a vector bundle with projection $\pi : E \to B$ and U be a subset of B. By taking $E' = \pi^{-1}(U)$, $\pi \mid_U : E' \to U$, we can get a new vector bundle $\xi \mid_U$, namely *the restriction of* ξ *to* U.

A.2. Pull-Back or Induced Bundle Given a bundle ξ, an arbitrary topological space B_1, and a map $f : B_1 \to B$, one can construct *the induced bundle* $f^*\xi$ *over* B_1 as following :

$$\begin{gathered}\text{Total space } E_1 = \{(b,e) \in B_1 \times E \mid f(b) = \pi(e)\} \\ \text{The projection map } \pi_1 : E_1 \to B_1 \text{ is defined by } \pi_1(b,e) = b \\ \text{For a local coordinate system } (U,h) \text{ for } \xi, \text{ and } U_1 = f^{-1}(U), \\ \text{a local coordinate system } (U_1, h_1) \text{ of } f^*\xi \text{ is defined by} \\ h_1 : U_1 \times \mathbb{R}^n \to \pi_1^{-1}(U_1) \text{ such that } h_1(b,x) = (b, h(f(b),x)).\end{gathered}$$

One big property of pull-back is that it preserves smoothness. And moreover, if one makes induced bundle with a *bundle map*, then induced bundle becomes isomorphic to original one.



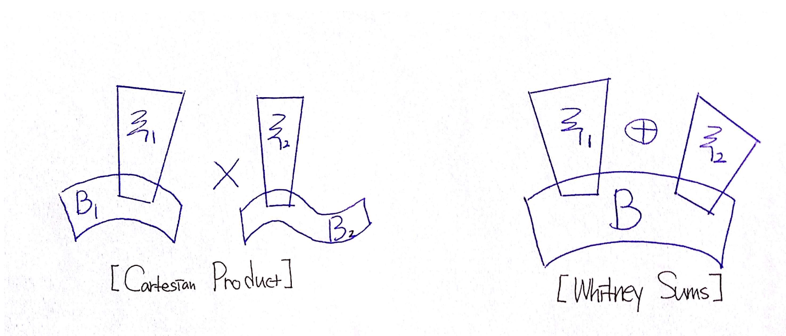
Definition. Let η and ξ be vector bundles(they could have different base spaces). A continuous function $g : E(\eta) \to E(\xi)$ is called a *bundle map* if $L_{g,b} : F_b(\eta) \to F_{g(b)}(\xi)$ is a vector space isomorphism for all $b \in B(\eta)$.

Lemma. If $g : E(\eta) \to E(\xi)$ is a bundle map, then $\eta \cong (g\mid_{B(\eta)})^*\xi$.

A.3. Cartesian products Suppose that there are two vector bundles ξ_1 and ξ_2. The *Cartesian product* $\xi_1 \times \xi_2$ is defined by making Cartesian products for all components.

$$\text{Total space } E(\xi_1 \times \xi_2) = E_1 \times E_2, \text{ Base space } B(\xi_1 \times \xi_2) = B_1 \times B_2,$$
$$\text{Projection map } \pi_1 \times \pi_2 : E_1 \times E_2 \to B_1 \times B_2, \text{ Fiber } (\pi_1 \times \pi_2)^{-1}(b_1, b_2) = F_{b_1}(\xi_1) \times F_{b_2}(\xi_2).$$

A.4. Whitney Sums or Direct Sums Let ξ_1, ξ_2 be vector bundles over the same base space B. Let $d : B \to B \times B$ be a diagonal embedding such that $b \mapsto (b, b)$. Then the bundle $d^*(\xi_1 \times \xi_2)$ over B is the *Whitney sum of* ξ_1 *and* ξ_2, and denoted by $\xi_1 \oplus \xi_2$. In this bundle, each fiber is isomorphic to the direct sum of original ones (i.e. $F_b(\xi_1 \oplus \xi_2) = F_b(\xi_1) \oplus F_b(\xi_2)$).



Definition. A vector bundle ξ is called a *sub-bundle* of η if these two bundles have the same base space B, $F_b(\xi)$ is a sub-vector space of $F_b(\eta)$ for all $b \in B$.

Lemma. Let ξ_1 and ξ_2 are sub-bundles of η with the property that $F_b(\eta) = F_b(\xi_1) \oplus F_b(\xi_2)$ for all $b \in B$. Then this implies that $\eta \cong \xi_1 \oplus \xi_2$.

A.5. Orthgonal complements This method is only available for Euclidean vector bundles. Suppose that η is an Euclidean vector bundle and ξ is sub-bundle of η. By taking all orthogonal parts of ξ in η we can define *the orthogonal complement of* ξ *in* η as following :

$$\text{Total space } E(\xi^\perp) = \{(b, x) \in E(\eta) \mid b \in B, x \in F_b(\eta), x \perp F_b(\xi)\}$$
The projection map is the same as the projection map of η.

In this case, just because of Gram-Schmidt process, $\eta \cong \xi \oplus \xi^\perp$.

A.6. Using other continuous algebraic operations Key property of vector bundle is that all fiber has the vector space structure. And a person who have learned algebra might guess that there are a number of operations that make new vector spaces. For example, for any real vector spaces V and W, $\mathrm{Hom}(V, W)$, $V \otimes W$, $V^* = \mathrm{Hom}(V, \mathbb{R})$, and $\Lambda^k V$ forms new vector spaces. Since all these operations are continuous, it is possible to apply these ideas to vector bundles and this lead us to make a number of new vector bundles.

2. Stiefel-Whitney Classes

Before saying about Stiefel-Whitney Classes, I like to talk about some background of cohomology first. Cohomology can be simply defined as the dual of homology. The idea of homology is quite intuitive and stratightforward, but it is really hard to construct the intuitive view for cohomology. Although cohomology is hard to think intuitively, a lot of mathematicians tried to research this because cohomology allowed them to get a number of invariants of topological spaces. And the Stiefel-Whitney class is just one case of them.

Stiefel-Whitney class is a cohomology class of a vector bundle defined by four axioms. The existence and uniqueness will not be proved in this survey, but since there exists a proof, we will assume these two things in this survey. And basically $H^i(B; G)$ is defined by i-th singular cohomology group of B with coefficients in G, and G will always be $\mathbb{Z}/2\mathbb{Z}$ for the Stiefel-Whitney classes.

Axiom 1. For any vector bundle ξ, there is a corresponding sequence of cohomology classes $\{w_i(\xi)\}$ where $w_i(\xi) \in H^i(B(\xi); \mathbb{Z}/2\mathbb{Z})$ for all $i = 0, 1, \cdots$. This class $\{w_i(\xi)\}$ is called the *Stiefel-Whitney classes of* ξ. Moreover, $w_0(\xi) = 1$ which is the unit element in $H^0(B(\xi); \mathbb{Z}/2\mathbb{Z})$, and $w_i(\xi) = 0$ for all $i > \dim E(\xi)$.

Axiom 2.(Naturality) If $f : B(\xi) \to B(\eta)$ is a bundle map, then $w_i(\xi) = f^* w_i(\eta)$.

Axiom 3.(Whitney Product Theorem) Let ξ and η be vector bundles over the same base.

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \smile w_{k-i}(\eta), \text{ where } \smile \text{ is a cup product.}$$

Usually, cup product will be omitted.

Axiom 4. $w_1(\gamma_1^1) \neq 0$, where γ_1^1 is the canonical line bundle over $\mathbb{R}P^1$.

With all these axioms, following propositions explain basic properties of Stiefel-Whitney classes.

Proposition 2.1. $\xi \cong \eta \implies w_i(\xi) = w_i(\eta)$ for all i

Proposition 2.2. ϵ is a trivial vector bundle $\implies w_i(\epsilon) = 0$ for all $i > 0$ (Usually, ϵ means trivial bundle)

Proposition 2.3. $w_i(\epsilon \oplus \eta) = w_i(\eta)$

Proposition 2.4. If ξ is an Euclidean $\mathbb{R}^n$-bundle, and if admits nowhere linearly dependent k cross-sections, then

$$w_{n-k+1}(\xi) = w_{n-k+2}(\xi) = \cdots = w_n(\xi) = 0 \ (\because \xi \cong \epsilon_B^k \oplus (\epsilon_B^k)^\perp)$$

Stiefel-Whitney classes are defined as the form of $\{w_i(\xi)\}$. However, this sequence form makes people hard to look for its properties. By using the graded ring structure, there is a nice way to show Stiefel-Whitney class without losing any information.

Definition 2.5. $H^{\Pi}(B; \mathbb{Z}/2\mathbb{Z})$ is the graded ring consisting of all formal infinite series $a = a_0 + a_1 + a_2 + \cdots$ such that $a_i \in H^i(B; \mathbb{Z}/2\mathbb{Z})$. The product of two elements in this graded ring is

$$(a_0 + a_1 + a_2 + \cdots)(b_0 + b_1 + b_2 + \cdots) = (a_0 b_0) + (a_1 b_0 + a_0 b_1) + (a_2 b_0 + a_1 b_1 + a_0 b_2) + \cdots$$

The *total Stiefel-Whitney class* of $\mathbb{R}^n$-bundle ξ over B is defined by

$$w(\xi) = 1 + w_1(\xi) + w_2(\xi) + \cdots + w_n(\xi) + 0 + \cdots \in H^{\Pi}(B; \mathbb{Z}/2\mathbb{Z}).$$

With the structure of the graded ring, we can briefly explain that $w(\xi \oplus \eta) = w(\xi) w(\eta)$.

Lemma 2.6. The collection of all infinite series

$$w = 1 + w_1 + w_2 + \cdots \in H^{\prod}(B; \mathbb{Z}/2\mathbb{Z})$$

with leading term 1 forms a field. In this case, the inverse is written by

$$\overline{w} = 1 + \overline{w_1} + \overline{w_2} + \cdots$$

($\because$ Consider that all elements in power series with a unit leading term is a unit)

Lemma 2.7.(Whitney Duality Theorem) Let τ_M be the tangent bundle of a manifold M in Euclidean space. If ν is the normal bundle, then

$$w_i(\nu) = \overline{w_i}(\tau_M)$$

This result immediately comes from the fact $\tau_M \oplus \nu = \epsilon$ and $w(\tau_M \oplus \nu) = w(\tau_M)w(\nu)$.

So far, I just suggested simple properties of *Stiefel-Whitney classes*. From now, I will try to reach to the goal and show what real projective spaces are non-parallelizable with some lemmas. And we will pretend all real projective spaces as Riemannian manifolds.

Lemma 2.8. $H^{\prod}(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) = \dfrac{(\mathbb{Z}/2\mathbb{Z})[a]}{\langle a^{n+1} \rangle}$

More precise proof of this lemma is written in [Hilton and Wylie 151pg]. But the basic idea for this proof is that $\mathbb{R}P^n = e^0 \sqcup e^1 \sqcup \cdots \sqcup e^n$ where e^is are i dimensional cell.

Lemma 2.9. $w(\gamma_n^1) = 1 + a$

Proof. The standard inclusion $j : \mathbb{R}P^1 \to \mathbb{R}P^n$ implies a bundle map from γ_1^1 to γ_n^1. By Axiom 2, $j^* w(\gamma_n^1) = w(\gamma_1^1)$. Because of Axiom 1 and 4 with Lemma 2.8, we can decide that $w(\gamma_1^1) = 1 + a$. Thus, all these information induce the result. $\square$

With this lemma and the Whitney Duality Theorem, if we define $\gamma^\perp$ as the orthogonal complement of γ_n^1 in ϵ^{n+1}, we can easily show that

$$w(\gamma^\perp) = 1 + a + \cdots + a^n$$

Lemma 2.10. Define τ as the tangent bundle of smooth Riemannian manifold $\mathbb{R}P^n$. Then, the bundle τ is isomorphic to a bundle $\mathrm{Hom}(\gamma_n^1, \gamma^\perp)$.

Proof. Let L be a line through the origin in $\mathbb{R}^{n+1}$ with $L \cap S^n = \{x, -x\}$. Also define $L^\perp \subset \mathbb{R}^{n+1}$ as the orthogonal complement of L in $\mathbb{R}^{n+1}$. First of all, an elemnt in the tangent manifold $D\mathbb{R}P^n$ can be defined by $\{(x, v), (-x, -v)\}$ where $(x, v) \in DS^n$. Since $x \cdot x = 1$ and $x \cdot v = 0$, we can say that the set $\{(x, v), (-x, -v)\}$ corresponds to a linear mapping $l : L \to L^\perp$ such that $l(x) = v$. With this idea, we can construct canonical vector space isomorphism from the fiber $F_{\{\pm x\}}(\tau)$ to the vector space $\mathrm{Hom}(L, L^\perp)$ for each $\{\pm x\} \in \mathbb{R}P^n$. This allows us to decide $\tau \cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp)$. $\square$

Theorem 2.11. Let τ be a tangent bundle of $\mathbb{R}P^n$ and ϵ^1 be a trivial line bundle over the same base. Then,

$$\tau \oplus \epsilon^1 \cong \underbrace{\gamma_n^1 \oplus \gamma_n^1 \oplus \cdots \oplus \gamma_n^1}_{(n+1)-times}$$

Therefore, the total Stiefel-Whitney class of the tangent bundle of $\mathbb{R}P^n$ is written by

$$w(\tau) = (1 + a)^{n+1} = 1 + \binom{n+1}{1} a + \binom{n+1}{2} a^2 + \cdots + \binom{n+1}{n} a^n \in H^i(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) = \frac{(\mathbb{Z}/2\mathbb{Z})[a]}{\langle a^{n+1} \rangle}$$

($w(\tau)$ is denoted by $w(\mathbb{R}P^n)$, and called as the total Stiefel-Whitney class of $\mathbb{R}P^n$)

Proof. First of all $\epsilon^1 \cong \mathrm{Hom}(\gamma_n^1, \gamma_n^1)$, because the Hom-bundle admits nowhere zero cross-section and (Theorem 1.17). And following steps will summarize whole steps:

$$\begin{aligned}
\tau \oplus \epsilon^1 &\cong \tau \oplus \mathrm{Hom}(\gamma_n^1, \gamma_n^1) && (\because \text{First step}) \\
&\cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp) \oplus \mathrm{Hom}(\gamma_n^1, \gamma_n^1) && (\because \text{Lemma 2.10.}) \\
&\cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp \oplus \gamma_n^1) && (\because \text{Algebraic property of Hom and direct sum}) \\
&\cong \mathrm{Hom}(\gamma_n^1, \epsilon^{n+1}) && (\because \text{A.5. Orthogonal complements}) \\
&\cong \mathrm{Hom}(\gamma_n^1, \bigoplus_{k=1}^{n+1} \epsilon^1) && \\
&\cong \bigoplus_{k=1}^{n+1} \mathrm{Hom}(\gamma_n^1, \epsilon^1) && (\because \text{Algebraic property of Hom and direct sum}) \\
&\cong \bigoplus_{k=1}^{n+1} \gamma_n^1 && (\because \mathrm{Hom}(\gamma_n^1, \epsilon^1) \cong \gamma_n^1,\ \exists \text{a Euclidean metric on } \gamma_n^1)
\end{aligned}$$

Since $w(\tau) = w(\tau \oplus \epsilon^1)$, we can conclude that $w(\tau) = (w(\gamma_n^1))^{n+1} = (1+a)^{n+1} \in \dfrac{(\mathbb{Z}/2\mathbb{Z})[a]}{\langle a^{n+1} \rangle}$. ☐

Now by using this theorem, we can immediately find what $\mathbb{R}P^n$s are not parallelizable with simple number theoretical computations.

Corollary 2.11.(Stiefel) $w(\mathbb{R}P^n) = 1 \iff n+1$ is a power of 2.
This implies that only $\mathbb{R}P^{2^r-1}$ types of real projective spaces can be parallelizable.

Proof. One big property that helps us to get this result is that Stiefel-Whitney class only plays in the field of $F_2 = \mathbb{Z}/2\mathbb{Z}$. Since the base field has characteristic 2, we can easily show that $(1+a)^{2^r} = 1 + a^{2^r}$. So if $n+1 = 2^r$, $w(\mathbb{R}P^n) = 1 + a^{n+1} = 1$. However, if $n+1 = 2^r m$ with odd number $m > 1$, then by using binomial theorem, $w(\mathbb{R}P^n) = 1 + ma^{2^r} + \cdots \neq 1$.

And if the Stiefel-Whitney classes of the given bundle ξ is not 1, by using contrapositive statement of Proposition 2.1 and 2.2, we can conclude that ξ is not trivial and this lead us to get the goal of this survey. ☐

3. Closing - Other Characteristic Classes

Although I have done this survey, I am still curious what is the geometric intuition of characteristic classes. In Hatcher's book, he says that all characteristic classes measure some how a vector bundle is twisted, or nontrivial. Although it does not give any intuition, but one thing that I guessed is that a lot of mathematicians tried to find the way to distinguish from trivial bundles. The reason of this fact is also still curious for me. Leaving my curiosity behind, since a lot of mathematicians liked to find the way to distinguish from trivial, they created several more types of Characteristic classes. If I list them as possible as I could, following things will be the well-known characteristic classes :

Stiefel-Whitney Classes : $w_i(\xi) \in H^i(B; \mathbb{Z}/2\mathbb{Z})$ for a real vector bundle $E \to B$.

Chern Classes : $c_i(\xi) \in H^{2i}(B; \mathbb{Z})$ for a complex vector bundle $E \to B$.

Pontryagin Classes : $p_i(\xi) \in H^{4i}(B; \mathbb{Z})$ for a real vector bundle $E \to B$.

Euler Classes : $e(\xi) \in H^n(B; \mathbb{Z})$ for an oriented $\mathbb{R}^n$-vector bundle $E \to B$.

Briefly speaking, Chern classes is a complex version of Stiefel-Whitney classes, and Pontryagin classes is a refinement of Stiefel-Whitney classes. Specifically for orientable cases, it can be further refined to Euler Classes. However, it is still hard to construct geometric intuition for all these classes, I will leave it as a task for readers.

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