CALIBRATING DETERMINACY STRENGTH IN LEVELS OF THE BOREL HIERARCHY

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Abstract. We analyze the set-theoretic strength of determinacy for levels of the Borel hierarchy of the form $\mathbf{\Sigma}^0_{1+\alpha+3}$, for $\alpha < \omega_1$. Well-known results of H. Friedman and D.A. Martin have shown this determinacy to require $\alpha+1$ iterations of the Power Set Axiom, but we ask what additional ambient set theory is strictly necessary. To this end, we isolate a family of Π_1-reflection principles, Π_1-RAP_α, whose consistency strength corresponds exactly to that of $\Sigma^0_{1+\alpha+3}$-Determinacy, for $\alpha < \omega_1^{\mathrm{CK}}$. This yields a characterization of the levels of L by or at which winning strategies in these games must be constructed. When $\alpha = 0$, we have the following concise result: the least θ so that all winning strategies in Σ^0_4 games belong to $L_{\theta+1}$ is the least so that $L_\theta \models$ "$\mathcal{P}(\omega)$ exists + all wellfounded trees are ranked".

§1. Introduction. Given a set $A \subseteq \omega^\omega$ of sequences of natural numbers, consider a game, $G(A)$, where two players, I and II, take turns picking elements of a sequence $\langle x_0, x_1, x_2, \dots \rangle$ of naturals. Player I wins the game if the sequence obtained belongs to A; otherwise, II wins. For a collection Γ of subsets of ω^ω, Γ *determinacy*, which we abbreviate Γ-DET, is the statement that for every $A \in \Gamma$, one of the players has a winning strategy in $G(A)$. It is a much-studied phenomenon that Γ-DET has mathematical strength: the bigger the pointclass Γ, the stronger the theory required to prove Γ-DET. Allowing Γ to range over pointclasses in Baire space, we obtain a natural measuring rod for the strength of theories ranging from weak fragments of second order arithmetic, up to the large cardinal axioms of higher set theory.

Our interest in this paper is a fine calibration of the strength of determinacy for levels $\mathbf{\Sigma^0_\alpha}$ of the Borel hierarchy. Results of this kind can be traced along two main trajectories. On the one hand, in reverse mathematics, the strength of determinacy is measured in terms of *provability*: An optimal result would be an isolation of some subsystem of second order arithmetic provably equivalent to Γ-DET over some weak base theory. On the other hand, in set theory, determinacy strength is measured in terms of *consistency strength*: An optimal result would be a characterization of some minimal model whose existence is equivalent (again over some weak base theory) to Γ-DET.

On both fronts, the question of strength for the lowest levels of the Borel hierarchy has been settled. $\mathbf{\Sigma}^0_1$-DET was early on proved by Gale and Stewart [5] and later shown by Steel [15] to be equivalent over ACA_0 to ATR_0; and an

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analysis by Blass [2] implies that Σ^0_1-DET is equivalent to the existence of a wellfounded model of $\mathsf{KP}+$ Infinity, so that winning strategies in Σ^0_1 games are constructed at or before ω_1^{CK} in L. Tanaka [16] refined Wolfe's [19] original proof of Σ^0_2-DET and showed this determinacy to be equivalent to an axiom asserting the stabilization of Σ^1_1-monotone inductive operators; this in turn was inspired by work of Solovay (see [8]) from which follows a characterization of the least level of L witnessing determinacy in terms of the closure ordinal of such operators.

Already at the level of Σ^0_3-DET, a calibration of determinacy strength in terms of reverse mathematics becomes problematic. Welch [17] has closely studied this strength, pushing through Davis's [3] proof under minimal assumptions, and establishing that Σ^0_3-DET is provable from Π^1_3-CA_0, but not from Δ^1_3-CA_0; however, Montalbán and Shore show [12] that no reversal is possible, in the strong sense that Σ^0_3-DET (and indeed, any true Σ^1_4 sentence) cannot prove Δ^1_2-CA_0. However, Welch [18] went on to characterize the least ordinal β so that winning strategies for all such games belong to $L_{\beta+1}$ in terms of certain nonstandard models of $V = L$.

Montalbán and Shore go a bit further up the Borel hierarchy, analyzing levels of the difference hierarchy on Π^0_3, showing that $n{-}\boldsymbol{\Pi}^0_3$-DET lies strictly between Δ^1_{n+2}-CA_0 and Π^1_{n+2}-CA_0; again, no reversals are possible. They establish the limit of determinacy provable in second-order arithmetic as essentially $<\omega{-}\boldsymbol{\Pi}^0_3$-DET, and even this determinacy may fail in nonstandard models of second-order arithmetic.

In this paper, we consider the next level, Σ^0_4, and more generally, levels of the form $\Sigma^0_{1+\alpha+3}$ for $\alpha < \omega_1^{\mathrm{CK}}$. By an early result of Friedman [4], full Borel determinacy requires ω_1 iterations of the Power Set Axiom, and even Σ^0_5-DET is not provable in second order arithmetic nor indeed, full ZFC^- (ZFC minus the Power Set Axiom). Martin later improved this to Σ^0_4-DET and proved the corresponding generalization for higher levels of the hyperarithmetical hierarchy; combining Montalbán-Shore's fine analysis [12] of $(n{-}\boldsymbol{\Pi}^0_3)$-DET with Martin's inductive proof [10] of Borel determinacy, we may summarize the bounds for these levels known prior to our work as follows.

Theorem 1.1 (Martin, Friedman, Montalbán-Shore). *For $\alpha < \omega_1$, $n < \omega$,*

$$\mathsf{Z}^- + \Sigma_1\text{-}\mathit{Replacement} + \mathcal{P}^\alpha(\omega)\ \mathit{exists} \vdash (n{-}\boldsymbol{\Pi}^0_{1+\alpha+2})\text{-DET},\ \mathit{but}$$
$$\mathsf{ZFC}^- + \mathcal{P}^\alpha(\omega)\ \mathit{exists} \nvdash \Sigma^0_{1+\alpha+3}\text{-DET}.$$

Here Z is Zermelo Set Theory without Choice (including Comprehension, but excluding Replacement). Again the superscript "$-$" indicates removal of the Power Set Axiom. Thus, $\alpha+1$ iterations of the Power Set Axiom are necessary to prove $\Sigma^0_{1+\alpha+3}$-DET. However, the question remained as to what additional ambient set theory is strictly necessary. More precisely, can one isolate a natural fragment of "$\mathsf{Z}^- + \Sigma_1$-Replacement $+\ \mathcal{P}^{\alpha+1}(\omega)$ exists" whose consistency strength is precisely that of $\Sigma^0_{1+\alpha+3}$-DET? Furthermore, can one characterize in a meaningful way the least level of L at which winning strategies in these games are constructed?

In this paper, we show this is the case. We introduce a family of natural reflection principles, Π_1-RAP_α, and show in a weak base theory that the existence of a wellfounded model of Π_1-RAP_α is equivalent to $\Sigma^0_{1+\alpha+3}$-DET, for $\alpha < \omega_1^{\mathrm{CK}}$. In particular, we show that the least ordinal θ_α so that winning strategies in all $\Sigma^0_{1+\alpha+3}$ games belong to $L_{\theta_\alpha+1}$ is precisely the least so that $L_{\theta_\alpha} \models \Pi_1$-$\mathsf{RAP}_\alpha$.

To give the reader a sense of the where these principles lie in terms of strength, we state at the outset a chain of nonreversible consistency strength implications that will be proved in the course of the paper.

THEOREM 1.2. *Each of the following theories proves the existence of a wellfounded model of the next.*

1. KP + "$\mathcal{P}(\omega)$ *exists*" + Σ_1-*Comprehension.*
2. $\mathsf{KP} + \boldsymbol{\Sigma}^0_4$-DET.
3. $\mathsf{KPI}_0 + \Sigma^0_4$-DET.
4. Π_1-RAP_0.
5. KP + "$\mathcal{P}(\omega)$ *exists*" + Σ_1-*Comprehension restricted to countable sets.*

It turns out that in the $V = L$ context, Π_1-RAP_α is equivalent to an easily stated axiom concerning the existence of ranking functions for open games, so that the ordinals θ_α can be rather simply described. In particular, letting $\theta = \theta_0$, we have the following: L_θ is the least level of L satisfying "$\mathcal{P}(\omega)$ exists, and all wellfounded trees on $\mathcal{P}(\omega)$ are ranked."

The paper is organized as follows. We begin in Section 2 by reviewing some basic facts about admissibility theory and L. In Section 3, after introducing the abstract principles Π_1-$\mathsf{RAP}(U)$, we focus on Π_1-$\mathsf{RAP}(\omega)$, proving some basic consequences and obtaining useful equivalents in the $V = L$ context. In Section 4, we connect these principles to determinacy, in particular proving Σ^0_4-DET assuming the existence of a wellfounded model of Π_1-$\mathsf{RAP}(\omega)$. In Section 5, we prove our lower bound in the case of Σ^0_4-DET, making heavy use of the results of Section 3. Section 6 carries out the analogous arguments for levels of the hyperarithmetical hierarchy of the form $\Sigma^0_{\alpha+3}$, for $1 < \alpha < \omega_1^{\mathrm{CK}}$. We conclude in Section 7 with some remarks concerning the complexity of winning strategies.

§2. Preliminaries: admissibility, L, and illfounded models. The main results of this paper concern certain weak subsystems of ZFC. We begin by reviewing these and cataloguing those facts that we require in the sequel.

Let Γ be a class of formulas in the language of set theory. Γ-*Comprehension* (often called Γ-Separation) is the axiom scheme containing universal closures of all formulas of the form

$$(\forall a)(\exists z)(\forall u) u \in z \leftrightarrow u \in a \wedge \phi(u, p_1, \ldots, p_k),$$

where ϕ is a $k+1$-ary formula in Γ.

Γ-*Collection* is the scheme consisting of the universal closures of formulas

$$(\forall u \in a)(\exists v)\phi(u, v, p_1, \ldots, p_k) \to (\exists b)(\forall u \in a)(\exists v \in b)\phi(u, v, p_1, \ldots, p_k)$$

for $k+2$-ary formulas in Γ. In this paper, Γ will always be one of Δ_0 or Σ_1.

We take as our background theory BST (Basic Set Theory), which consists of the axioms of Extensionality, Foundation, Pair, Union, Δ_0-Comprehension, and the statement that Cartesian products exist. Unless otherwise stated, all of the models we consider satisfy at least BST (so “transitive model” really means “transitive model of BST”).

Kripke-Platek Set Theory, KP, is BST together with the axiom scheme of Δ_0-Collection; note that all axioms in the schema of Σ_1-Collection and Δ_1-Comprehension are then provable in KP. A transitive set M is called *admissible* if the structure $(M, \in)$ satisfies KP. KPI_0 is the theory BST together with the assertion that every set x belongs to an admissible set; KPI is the union of KP and KPI_0. The standard reference for admissible set theory is Barwise's [1].

The most important feature of admissible sets for our purposes is their ability to correctly identify wellfounded relations.

PROPOSITION 2.1. *Let M be an admissible set, and suppose $T \in M$ is a tree. Then T is wellfounded if and only if there is a* ranking function *$\rho \in M$, that is, a map $\rho : T \to \mathrm{ON}^M$ such that $\rho(s) < \rho(t)$ whenever $s \supsetneq t$ for $s, t \in T$; in particular, $\{s \in T \mid T_s$ is wellfounded$\}$ is Σ_1-definable over M.*

Here $T_s = \{t \in T \mid s \subseteq t \text{ or } t \subseteq s\}$. Note however that $T \in M$ may be illfounded even though no infinite branch through T belongs to M.

Our determinacy strength lower bounds require some basic fine structure theory of L. We therefore regard Gödel's L as stratified into Jensen's levels J_α, and will also make reference to the auxiliary S-hierarchy further stratifying the J-hierarchy: $S_0 = \emptyset$, $S_{\alpha+1}$ is the image of $S_\alpha \cup \{S_\alpha\}$ under a finite list of binary operations generating the rudimentary functions, and we set, for limit ordinals λ, $J_\lambda = S_\lambda = \bigcup_{\alpha<\lambda} S_\alpha$. In particular, we index the J_α by limit ordinals, $\mathrm{ON} \cap J_\alpha = \alpha$ for all limit α, and when $\omega \cdot \alpha = \alpha$, we have $L_\alpha = J_\alpha$.

The following definition/theorem summarizes all of the fine structural facts we require. For details and proofs, see [14].

THEOREM 2.2. *Every J_α is a model of BST. Moreover, there are Σ_1 formulas σ, τ, ϕ, ψ so that for all limit α,*

1. *For all $\beta < \alpha$ and $x \in J_\alpha$, $x = S_\beta$ iff $J_\alpha \models \sigma(x, \beta)$.*
2. *(Σ_1 Skolem functions) τ defines a partial function $h_1^{J_\alpha} : [\alpha]^{<\omega} \rightharpoonup J_\alpha$ that is onto J_α.*
3. *(Σ_1 satisfaction) Letting $\langle \phi_i \rangle_{i\in\omega}$ be a fixed recursive enumeration of all Σ_1 formulas in the language of set theory, we have, for all i, $J_\alpha \models \phi_i(p_1, \ldots, p_k)$ if and only if $J_\alpha \models \phi(i, \langle p_1, \ldots, p_k \rangle)$.*
4. *(The first projectum) Define the Σ_1-*projectum *of J_α to be the least ordinal $\rho_1^{J_\alpha} \leq \alpha$ for which there is a Σ_1 (in parameters) subset of $\rho_1^{J_\alpha}$ which does not belong to J_α. Then there is some finite $p \subset \alpha \setminus \rho_1^{J_\alpha}$ so that the partial map $\xi \mapsto h_1^{J_\alpha}(p \cup \{\xi\})$ is a surjection from a subset of $\rho_1^{J_\alpha}$ onto J_α.*
5. *(Canonical wellorderings) ψ defines a wellorder $<_L^{J_\alpha}$ of J_α.*
6. *(Condensation) Suppose $H \prec_{\Sigma_1} J_\alpha$ with H transitive. Then $H = J_\beta$ for some $\beta \leq \alpha$.*
7. *(Acceptability) If $\rho < \alpha$ and $\mathcal{P}(\rho) \cap J_{\alpha+\omega} \setminus J_\alpha$ is nonempty, then there is a surjection $f : \rho \to \alpha$ in $J_{\alpha+\omega}$.*

Note that by upwards absoluteness of Σ_1 formulas, every Σ_1 statement true of parameters in J_α also is true in all J_β for $\beta > \alpha$; consequently, the canonical wellorder of J_β extends that of J_α, and $h_1^{J_\alpha}(a) = h_1^{J_\beta}(a)$ when the former exists. By a typical abuse of notation, we write $h_1^{J_\alpha}(X) = h_1^{J_\alpha}[\omega \cup X]^{<\omega}$. When $\rho_1^{J_\alpha} = \beta$ we say J_α *projects to* β.

An ordinal α is called *admissible* if $J_\alpha = L_\alpha$ is an admissible set. The least admissible ordinal greater than ω is the Church-Kleene ordinal ω_1^{CK}. Note that $L_{\omega_1^{\mathrm{CK}}}$ projects to ω. More generally, letting ω_1^x be the least non-computable ordinal relative to the real x, we have that ω_1^x is least $> \omega$ so that $L_{\omega_1^x}[x]$ is admissible. Note that whenever there is a real x so that α is the least admissible ordinal with $x \in J_\alpha$, we have $\rho_1^{J_\alpha} = \omega$ (see e.g. Theorem V.7.11 in [1]). The following proposition tells us that there are only two possible values of $\rho_1^{J_\alpha}$ when α is admissible.

Proposition 2.3 (cf. [1], Theorem V.5.9). *Suppose α is admissible, and let κ be the supremum of the cardinals of J_α. Then $\rho_1^{J_\alpha} \in \{\kappa, \alpha\}$.*

Proof. If $\rho_1^{J_\alpha} < \alpha$, then it is a cardinal of J_α. Suppose towards a contradiction that $\rho_1^{J_\alpha} < \kappa$. By (4) of Theorem 2.2, there is a finite set of ordinals p so that $\alpha \subseteq h_1^{J_\alpha}(\rho_1^{J_\alpha} \cup p)$. Let $\tau < \alpha$ be a cardinal of J_α greater than $\rho_1^{J_\alpha}$. For $\beta < \tau$, let $\gamma(\beta)$ be the least limit ordinal so that $\beta \in h_1^{J_{\gamma(\beta)}}(\rho_1^{J_\alpha} \cup p)$; note that the map $\beta \mapsto \gamma(\beta)$ is Σ_1-definable in J_α. By Σ_1-Collection, $\gamma = \sup_{\beta<\tau} \gamma(\beta) < \alpha$. But now there is a map from $\rho_1^{J_\alpha}$ onto τ definable over J_γ, contradicting the fact that τ is a cardinal in J_α. ⊣

Our methods rely on an analysis of *illfounded* models of fragments of set theory. Recall any model $\mathcal{M} = \langle M, \varepsilon \rangle$ in the language of set theory has a unique largest downward ε-closed submodel on which ε is wellfounded, the *wellfounded part of* $\mathcal{M}$, denoted $\mathrm{wfp}(\mathcal{M})$. When ε is extensional on M, we identify $\mathrm{wfp}(\mathcal{M})$ with its transitive isomorph, and denote $\mathrm{wfo}(\mathcal{M}) = \mathrm{wfp}(\mathcal{M}) \cap \mathrm{ON}$. A model $\mathcal{M}$ of BST is an *ω-model* if $\omega \in \mathrm{wfo}(\mathcal{M})$.

Theorem 2.2 concerns transitive sets of the form J_α, but we will need its consequences to hold even for illfounded models of $V = L$. We therefore officially define "$V = L$" to be the theory consisting of the following:

- There is no largest ordinal.
- Every set x belongs to some S_β; in particular, for all limit α, J_α exists.
- Theorem 2.2 holds for the fixed Σ_1 formulas σ, τ, ϕ, ψ; moreover, the same theorem holds with V in place of J_α.

Note $V = L$ is a recursive theory. When working with an illfounded ω-model $\mathcal{M}$ of $V = L$, we refer to $\mathcal{M}$'s versions of the fine structural objects in the obvious way, e.g. $h_1^{\mathcal{M}}, \rho_1^{\mathcal{M}}$, and so on.

We will require a version of the truncation lemma for admissible structures (cf. Corollary II.8.5 in [1]) specialized to models of $V = L$. Proposition 2.4 differs from the usual truncation lemma both in that $\mathcal{M}$ is not itself assumed to be admissible, and in general $L_{\mathrm{wfo}(\mathcal{M})}$ needn't coincide with $\mathrm{wfp}(\mathcal{M})$ (even when $\mathcal{M} \models V = L$; see e.g. Lemma 2.5).

Proposition 2.4. *Working in* KPI, *let $\mathcal{M} = \langle M, \varepsilon \rangle \models V = L$, and suppose $\mathcal{M}$ is illfounded. Then $L_{\mathrm{wfo}(\mathcal{M})}$ is admissible.*

Proof. This is nearly immediate from Lemma 2.14 of [7]: were $\alpha = \mathrm{wfo}(\mathcal{M})$ not admissible, then α would be obtained as the range of a map $f : \gamma \to \alpha$ for some $\gamma < \alpha$, and so that f is Σ_1-definable over J_α; by upwards absoluteness, the same map is defined in $J_s^{\mathcal{M}}$ for any nonstandard ordinal s, so that $\alpha \in \mathrm{wfp}(\mathcal{M})$, a contradiction.

We give a direct argument in the same spirit and greater detail. Given such $\mathcal{M}$, we know (working in KPI) that $\mathrm{wfp}(\mathcal{M})$ exists. Note $\mathrm{wfo}(\mathcal{M}) = \omega \cdot \alpha$ for some unique α. If $\alpha = 1$ then we're done. So suppose $\alpha > 1$ (so in particular, $\mathcal{M}$ is an ω-model). That $J_\alpha \models \mathsf{BST}$ is automatic. We only need to show $J_\alpha \models \Delta_0$-Collection (from which it follows that $L_\alpha = J_\alpha \models \mathsf{KP}$). One can show by induction on ξ that $S_\xi = S_\xi^{\mathcal{M}} \in \mathrm{wfp}(\mathcal{M})$ for all $\xi < \mathrm{wfo}(\mathcal{M})$; that is, $J_\alpha \subseteq \mathrm{wfp}(\mathcal{M})$.

Assume $a, p \in J_\alpha$ and

$$J_\alpha \models (\forall x \in a)(\exists y)\varphi(x, y, p),$$

where φ is Δ_0. Then in $\mathcal{M}$,

$$\mathcal{M} \models (\forall x \in a)(\exists y)\varphi(x, y, p).$$

Let σ be a nonstandard ordinal of $\mathcal{M}$. In $S_\sigma^{\mathcal{M}}$, define

$$\begin{aligned} F(x) = \xi \iff & x, p \in S_{\xi+1} \wedge (\exists y \in S_{\xi+1})\varphi(x, y, p) \\ & \wedge (\forall y \in S_\xi)(\neg\varphi(x, y, p) \vee x \notin S_\xi \vee p \notin S_\xi). \end{aligned}$$

Notice that $J_\alpha \subset S_\sigma^{\mathcal{M}}$, and by absoluteness, $F(x) < \omega \cdot \alpha$ for each $x \in a$. Since $\mathcal{M}$ satisfies BST, we have that the union of the $F(x)$,

$$\tau = \{\eta \in \sigma \mid (\exists x \in a)(\exists \xi \in \sigma)\eta \in \xi \wedge S_\sigma^{\mathcal{M}} \models F(x) = \xi\}$$

is an ordinal in $\mathcal{M}$, and must be contained in $\omega \cdot \alpha$. So $\tau \in \mathrm{wfp}(\mathcal{M})$, hence $\tau < \omega \cdot \alpha$. We have $S_\tau \in J_\alpha$, and

$$(\forall x \in a)(\exists y \in S_\tau)\varphi(x, y, p).$$

This proves the needed instance of Δ_0-Collection, so $J_\alpha \models \mathsf{KP}$. ⊣

If $\mathcal{M}$ is an illfounded model of $V = L$, then $\mathcal{M}$ stratifies into levels $J_a^{\mathcal{M}}$ for $a \in \mathrm{ON}^{\mathcal{M}}$. Proposition 2.4 indicates that such models can be thought of as a standard part $J_{\mathrm{wfo}(\mathcal{M})}$ with nonstandard levels $J_a^{\mathcal{M}}$ stacked on top. An important feature of this picture is *overspill*; it is an immediate consequence of acceptability.

Lemma 2.5. *Suppose $\mathcal{M}$ is an ω-model of $V = L$ with $\beta = \mathrm{wfo}(\mathcal{M})$ and that $\kappa \in L_\beta$ is the largest cardinal of L_β. Say $X \in \mathcal{M}$ is a* nonstandard code *if $X \subseteq \kappa$ codes a linear order of κ so that $\mathcal{M}$ has an isomorphism from X onto some nonstandard ordinal of $\mathcal{M}$. Then*

$$\{X \in \mathcal{M} \setminus L_\beta \mid X \text{ is a nonstandard code}\}$$

is nonempty, and has no $<_L^{\mathcal{M}}$-least element.

At center stage in this paper is the *Power Set Axiom*, which asserts that for every set X there exists a set $\mathcal{P}(X)$ whose elements are precisely the subsets of X. We consider restricted instances of the Power Set Axiom of the form "$\mathcal{P}^\alpha(\omega)$ exists" for fixed $\alpha < \omega_1$ (here $\mathcal{P}^{\alpha+1}(\omega) = \mathcal{P}(\mathcal{P}^\alpha(\omega))$ and $\mathcal{P}^\lambda(\omega) = \bigcup_{\alpha<\lambda} \mathcal{P}^\alpha(\omega)$

for limit λ). Whenever we say $\mathcal{M} \models$ "$\mathcal{P}^\alpha(\omega)$ exists", $\mathcal{M}$ will always be an ω-model so that α is computable relative to some real $x \in \mathcal{M}$. In this situation, $\alpha \in \mathrm{wfo}(\mathcal{M})$, and it makes sense to regard this axiom as first-order (possibly in parameter x) in the language of set theory.

We conclude this section with a remark concerning the base theory. Since the main results of this paper involve models of fragments of set theory (including instances of the Power Set Axiom), it is natural to work in a suitably weak set theory, and the strongest assumption used in the proof of our main result is closure under the next admissible set (see Theorems 5.1 and 6.6). These proofs are therefore carried out in KPI_0. However, calibrations of determinacy strength are traditionally done in second order arithmetic, and our results can be so formulated. Since the consequences of KPI_0 for second order arithmetic are precisely those of Π^1_1-CA_0, our main theorem, rephrased in the language of second order arithmetic, is provable in the latter theory:

THEOREM 2.6 (Π^1_1-CA_0). *Let $x \in \omega^\omega$ and $\alpha < \omega_1^x$. Then $\Sigma^0_{1+\alpha+3}(x)$-DET is equivalent to the existence of a real coding a wellfounded model $(M, \in)$ in the language of set theory so that $x \in M$, and $(M, \in) \models \Pi_1$-RAP_α.*

§3. The Π_1-Reflection to Admissibles Principle.

We now define the main theory of interest in this paper.

DEFINITION 3.1. Let U be a transitive set. The Π_1-*Reflection to Admissibles Principle for U* (denoted Π_1-$\mathsf{RAP}(U)$) is the assertion that $\mathcal{P}(U)$ exists, together with the following axiom scheme, for all Π_1 formulae $\phi(u)$ in the language of set theory: Suppose $Q \subseteq \mathcal{P}(U)$ is a set and $\phi(Q)$ holds. Then there is an admissible set M so that

- $U \in M$.
- $\bar{Q} = Q \cap M \in M$.
- $M \models \phi(\bar{Q})$.

We chose this particular formulation for its simplicity. The sets U we consider are sufficiently well-behaved that Π_1-$\mathsf{RAP}(U)$ gives a bit more.

Say U *admits power tuple coding* if there is a bijective map $c : \mathcal{P}(U)^{<\omega} \to \mathcal{P}(U)$ so that the relations $a \in c(s)$, $a \in c^{-1}(x)_i$, and $c(s) = x$ are all $\Delta_0(\{U\})$ (that is, definable from the parameter U with all quantifiers bounded). Note then that if M is transitive satisfying BST and $U \in M$, then any set $Q \subseteq \mathcal{P}(U)^{<\omega}$ in M can be coded by a set $Q \subseteq \mathcal{P}(U)$ in M.

LEMMA 3.2. *Suppose U is a transitive set that admits power tuple coding, and Π_1-$\mathsf{RAP}(U)$ holds. Let $\phi(u_1, \ldots, u_m, v_1, \ldots, v_n)$ be a Π_1 formula and fix sets $p_i \subseteq U^{<\omega}$, $Q_j \subseteq \mathcal{P}(U)^{<\omega}$ for $1 \le i \le m$, $1 \le j \le n$ so that $\phi(p_1, \ldots, p_m, Q_1, \ldots, Q_n)$ holds. Then there is an admissible set M so that*

- *$U \in M$ and $M \models$ "$\mathcal{P}(U)$ exists".*
- *For $1 \le i \le m$, $p_i \in M$; for $1 \le j \le n$, $\bar{Q}_j = Q_j \cap M \in M$.*
- *$M \models \phi(p_1, \ldots, p_m, \bar{Q}_1, \ldots, \bar{Q}_n)$.*

PROOF. First note that given $Q \subset \mathcal{P}(U)$, the relations $u = U$ and $v = Q$ are both Δ_0-definable from $Q' = Q \cup \{U\}$, and this allows us refer to the coding map

$c : \mathcal{P}(U)^{<\omega} \to \mathcal{P}(U)$ in a $\Delta_0(\{Q'\})$ fashion. So suppose $Q_1 \subseteq \mathcal{P}(U)^{<\omega}$ is given, and $(\forall x)\psi(Q_1, x)$ holds, where ψ is Δ_0. Let $Q = c[Q_1]$; the given Π_1 statement is equivalent to

$$(\forall u)(\forall x) u = c^{-1}[Q] \to \psi(u, x).$$

This can be phrased as a $\Pi_1(Q')$ statement and so can be reflected to an admissible set M where it holds of $\bar{Q}' = Q' \cap M$. Note then by absoluteness of the coding map c, we have $c^{-1}[\bar{Q}] = Q_1 \cap M$, so that $(\forall x)\psi(Q_1 \cap M, x)$ holds in M, as desired.

Similar uses of coding allow us to reflect statements involving finite lists of parameters $p_1, \ldots, p_m, Q_1, \ldots, Q_n$; that $p_i \cap M = p_i$ follows from transitivity of M and the assumption that $U \in M$. Finally, we can ensure $M \models$ "$\mathcal{P}(U)$ exists" by including $\mathcal{P}(U)$ as one of the Q_j; then $\bar{Q}_j = Q_j \cap M = \mathcal{P}(U)^M \in M$. $\dashv$

We will first be concerned mainly with Π_1-$\mathsf{RAP}(\omega)$, which we abbreviate simply as Π_1-RAP. Π_1-RAP does not imply Δ_0-Collection, so cannot prove KP. However, it does prove many Σ_1 consequences of admissibility. For example, from the following lemma, we have Σ_1-Recursion along wellfounded relations on $\mathcal{P}(\omega)$. Recall a relation R is *wellfounded* if every nonempty subset of its domain has an R-minimal element. For binary R and $a \in \mathrm{dom}(R)$, let $\mathrm{pred}_R(a) = \{b \in \mathrm{dom}(R) \mid bRa\}$, and $\mathrm{tc}_R(a)$ denote the downwards R-closure of an element $a \in \mathrm{dom}(R)$,

$$\mathrm{tc}_R(a) := \{b \in \mathrm{dom}(R) \mid (\exists b_0, b_1, \ldots, b_n) b = b_0, b_n = a, b_i R b_{i+1} \text{ for all } i < n\}.$$

LEMMA 3.3. *Work in Π_1-RAP and suppose R is a wellfounded binary relation on $\mathcal{P}(\omega)^{<\omega}$. Suppose further that $\phi(u, v, w)$ is a Σ_1 formula that provably in KP defines the graph of a binary class function G. Then for every $Q \subseteq \mathcal{P}(\omega)^{<\omega}$, there is a function $F : \mathrm{dom}(R) \to V$ so that for all $a \in \mathrm{dom}(R)$,*

$$F(a) = G(Q, F \restriction \mathrm{pred}_R(a)).$$

PROOF. Suppose towards a contradiction that R is wellfounded, but Q is such that no total function $F : \mathrm{dom}(R) \to V$ as in the lemma exists. This is a Π_1 statement in parameters R, Q, so by Π_1-RAP and Lemma 3.2, reflects to an admissible set M satisfying "$\mathcal{P}(\omega)$ exists".

Working in M, define the map F from $\bar{R}, \bar{Q}$ in the usual way:

$$\begin{aligned} F(a) = Y \iff & (\exists \bar{f} : \mathrm{tc}_{\bar{R}}(a) \to V)[Y = G(\bar{Q}, \bar{f}) \\ & \wedge (\forall b \in \mathrm{dom}(\bar{f}))\bar{f}(b) = G(\bar{Q}, \bar{f} \restriction \mathrm{pred}_{\bar{R}}(b))]. \end{aligned}$$

The fact that $M \models \mathsf{KP}$ ensures this definition can be expressed in a Σ_1 fashion over M. Since we reflected a failure of this instance of Σ_1-Recursion to M, there must be some $y \in \mathrm{dom}(\bar{R})$ so that $F(y)$ does not exist. Consider the set

$$D = \{y \in \mathrm{dom}(\bar{R}) \mid M \models \text{"}F(y) \text{ does not exist"}\}.$$

We have D nonempty; note though, that we needn't have $D \in M$. By wellfoundedness of R, let $y_0 \in D$ be R-minimal. It follows that in M, $F(y)$ is defined whenever $y \bar{R} y_0$, so F is defined on $\mathrm{tc}_{\bar{R}}(y_0) = \bigcup_{y \bar{R} y_0} \mathrm{tc}_{\bar{R}}(y)$. By Σ_1-Replacement in M, we have that $\bar{f} = F \restriction \mathrm{tc}_{\bar{R}}(y_0)$ exists in M. But this $\bar{f}$ witnesses the fact that $F(y_0)$ exists, a contradiction. $\dashv$

For a set X, DC_X denotes the *Axiom of Dependent Choices* for X. This weak choice principle states: if R is a binary relation on X so that for every $a \in \mathrm{dom}(R)$, there is some b with bRa, then there is an infinite sequence $\langle a_n \rangle_{n\in\omega}$ so that $a_{n+1} R a_n$ for all n.

COROLLARY 3.4. *Assume* Π_1-RAP *and* $\mathsf{DC}_{\mathbb{R}}$. *Then whenever* T *is a tree on* $\mathcal{P}(\omega)$, *either* T *has an infinite branch, or* T *has a rank function, that is, a map* $\rho : T \to \mathrm{ON}$ *such that* $\rho(s) < \rho(t)$ *whenever* $s \supsetneq t$.

PROOF. Suppose T is a tree on $\mathcal{P}(\omega)$ with no infinite branch. By $\mathsf{DC}_{\mathbb{R}}$, the relation $\supsetneq$ is wellfounded on T. Apply Σ_1-Recursion with the function $G(Q, F) = \sup\{F(s) \mid s \in \mathrm{dom}(F)\}$. ⊣

It turns out that the statement that all wellfounded trees on $\mathcal{P}(\omega)$ are ranked is equivalent to Π_1-RAP in the $V = L$ context. Besides having intrinsic interest, this fact will be useful for our determinacy strength lower bounds.

THEOREM 3.5 (joint with Itay Neeman). *Let* $V = L$ *and assume* ω_1 *exists, and that every tree on* $\mathcal{P}(\omega)$ *is either illfounded or ranked. Then* Π_1-RAP *holds; moreover, every instance of* Π_1-RAP *is witnessed by some* L_α *with* α *countable.*

PROOF. We may assume ω_2 does not exist, since otherwise Π_1-RAP follows immediately. Suppose $Q \subseteq \mathcal{P}(\omega)$ and that $\phi(Q)$ holds for some Π_1 formula ϕ. Let $\tau > \omega_1$ be sufficiently large that $Q \in J_\tau$. Let T be the tree of attempts to build a complete, consistent theory of a model $\mathcal{M}$ so that

- $\mathcal{M}$ is illfounded,
- $\bar{Q} = \mathcal{M} \cap Q \in S_t$ for some $t \in \mathrm{wfo}(\mathcal{M})$,
- $\mathcal{M} \models V = L + \phi(\bar{Q})$ + "ω_1 exists".

In slightly more detail: Let $\mathcal{L}^*$ be the language of set theory together with constants $\{d_n\}_{n\in\omega} \cup \{a_n\}_{n\in\omega} \cup \{t, q\}$. Fix some standard coding $\sigma \mapsto \#\sigma$ of sentences in the language of $\mathcal{L}^*$ so that $\#\sigma > k$ whenever d_k appears in σ. All nodes in T are pairs of the form $\langle f, g \rangle$, where $f : n \to \{0, 1\}$ and $g : n \to \tau \cup \mathcal{P}(\omega)$, and the set $\{\sigma \mid f(\#\sigma) = 1\}$ is a finite theory in $\mathcal{L}^*$, consistent with the following:

- "$d_{n+1} \in d_n$" for each $n \in \omega$.
- "t is an ordinal, q is a set of reals, and $q \in S_t$".
- $V = L + \phi(q)$ + "ω_1 exists and $\omega_1 \in t$".
- $\mu \to \psi(a_{\#\mu})$, for sentences μ of the form $(\exists x)(x \in t \vee x \subset \omega) \wedge \psi(x)$.

The point of the last clause is to have the a_i serve as Henkin constants witnessing statements asserting existence of a real or of an ordinal below t. Finally, the function g is required to assign values in $\mathcal{P}(\omega) \cup \tau$ to the Henkin constants in a way compatible with the theory; in particular, respecting the theory's order for elements of t (so that $f(\#(a_i \in a_j \in t)) = 1$ implies $g(i) < g(j) < \tau$), membership of reals in Q (so that $f(\#(a_i \in q)) = 1$ implies $g(i) \in Q$), and membership of naturals in reals ($f(\#(n \in a_i)) = 1$ implies $n \in g(i)$).

Suppose T is illfounded. A branch through T then yields f giving a complete and consistent theory in $\mathcal{L}^*$ together with assignment of constants g. Let $\mathcal{M}$ be the term model obtained from this theory. By construction, $\mathcal{M}$ is illfounded, $\mathcal{M} \models V = L$, and setting $\bar{Q} = \{g(i) \mid a_i^{\mathcal{M}} \in q^{\mathcal{M}}\}$, we have $\bar{Q} = Q \cap M \in S_t^{\mathcal{M}}$; moreover, by the assignment of elements of τ to terms below t, we have that $\mathcal{M}$

has wellfounded part containing $S_t^{\mathcal{M}}$. By Proposition 2.4, if $\alpha = \mathrm{wfo}(\mathcal{M})$, then $L_\alpha \models \mathsf{KP} + \phi(\bar{Q}) +$ "ω_1 exists", as needed.

Now suppose towards a contradiction that T is not illfounded. Since T is clearly coded by a tree on $\mathcal{P}(\omega)$, we have that T is ranked. Let $\rho : T \to \mathrm{ON}$ be the ranking function. We construct a branch through T using the function ρ; f will be the characteristic function of the complete theory of $J_{\rho(\emptyset)}$, interpreting t by τ, q by Q, and inductively choosing values $g(i)$ in $\tau \cup \mathcal{P}(\omega)$ to be $<_L$-least witnessing existential statements holding in $J_{\rho(\emptyset)}$. All that remains is to decide on interpretations for the constants d_n, corresponding to the descending sequence of ordinals. So let $x_0 = \rho(\emptyset)$, and having chosen the fragment s of the branch up to k, let $x_{k+1} = \rho(s)$. Then interpret d_i in the theory by x_i.

At each finite stage of the above construction, the theory chosen is satisfied under the appropriate interpretation in $J_{\rho(\emptyset)}$, so we may always extend the branch by one step. But then $\{x_k \mid k \in \omega\}$ is an infinite descending sequence of ordinals, a contradiction. ⊣

We remark that with some extra work, the $V = L$ assumption can be replaced with more natural hypotheses. Namely, we have the converse of Lemma 3.3: If $\mathsf{DC}_{\mathbb{R}}$ holds and we have Σ_1-Recursion along wellfounded relations on $\mathbb{P}(\omega)$, then Π_1-RAP holds. The extra assumption of Σ_1-Recursion guarantees the existence of all levels of the hierarchy of sets constructible relative to the parameter set Q; the argument of Theorem 3.5 may then be carried out inside $L(Q)$ to give the desired instance of Π_1-RAP. ($\mathsf{DC}_{\mathbb{R}}$ is important in part to ensure the existence branches through illfounded trees.)

Proposition 3.6. *Suppose M is a transitive model of* Π_1-RAP*. Then* Π_1-RAP *holds in* $L^M = \bigcup\{L_\alpha \mid \alpha \in M \wedge M \models$ "L_α *exists*"$\}$.

Proof. Work in M. By Lemma 3.3, whenever $A \subseteq \mathcal{P}(\omega)$ codes a prewellorder of $\mathcal{P}(\omega)$, there is an ordinal α so that $\alpha = \mathrm{otp}(\mathcal{P}(\omega)/\equiv_A, <_A)$, and L_α exists. Since $\mathcal{P}(\omega)$ exists, there is a Δ_0 prewellorder of $\mathcal{P}(\omega)$ in order type ω_1, so L_{ω_1} exists. It follows that $L \models$ "ω_1 exists". So it's sufficient to show that every tree on $\mathcal{P}(\omega)$ in L is either ranked or illfounded in L.

So let T be a tree on $\mathcal{P}(\omega)$ with $T \in L^M$. If T is not ranked in L^M, reflect this fact to an admissible level L_α. Then $\bar{T} = T \cap L_\alpha \subseteq T$, and $L_\alpha \models$ "$\bar{T}$ is not ranked". Since L_α is admissible, there is a branch through $\bar{T}$ definable over L_α. In particular, $\bar{T} \subseteq T$ is illfounded in L. Hence by Theorem 3.5, Π_1-RAP holds in L $(= L^M)$, as needed. ⊣

Note that assuming KP + "$\mathcal{P}(\omega)$ exists" + Σ_1-Comprehension, this theory reflects from V to L, and if α is least so that $\alpha > \omega_1^L$ and $L_\alpha \prec_{\Sigma_1} L$ (such exists by Σ_1-Comprehension), then by Theorem 3.5, $L_\alpha \models \Pi_1$-RAP. Similarly, by the last sentence of the same theorem combined with Corollary 3.4, Π_1-RAP implies the existence of levels of L satisfying KP + "$\mathcal{P}(\omega)$ exists"; by Proposition 2.3, such levels automatically satisfy $\rho_1 > \omega$. Taken all together, we have established those implications of Theorem 1.2 between (1), (4), and (5).

§4. Proving Determinacy. In this section, we prove a key lemma connecting the principles Π_1-$\mathsf{RAP}(U)$ to determinacy for certain infinite games, and use this lemma to give a proof of Σ_4^0-DET starting from a model of Π_1-RAP. We shall see later that Π_1-RAP does not itself imply Σ^0_4-DET; rather, it only guarantees that for every Σ^0_4 set $A \subseteq \omega^\omega$, either I has a winning strategy in $G(A;\omega^{<\omega})$, or else there is a $\boldsymbol{\Delta}_1$-definable winning strategy for II. Since we aren't assuming Δ_1-Comprehension, it is possible that II's definable strategy will not be a set, and indeed, this scenario must occur in minimal models of Π_1-RAP. Thus the hypothesis that guarantees determinacy of all Σ^0_4 games is the existence of such a model (Note that this situation is in complete analogy with that of Σ^0_1-DET and models of KP; see [2]).

We first recall some basic definitions and terminology.

Fix a set X. By a *tree on* X we mean a set $T \subseteq X^{<\omega}$ closed under initial segments. Let $[T]$ denote the set of infinite branches of T; for $s \in T$, T_s denotes the set $\{t \in T \mid t \subseteq s \text{ or } s \subseteq t\}$, the "subtree with stem s". For a set $A \subseteq [T]$, the *game on* T *with payoff* A, denoted $G(A;T)$, is played by two players, I and II, who alternate choosing elements of X,

$$\begin{array}{c|cccccccc} \mathrm{I} & x_0 & & x_2 & \dots & x_{2n} & & \dots \\ \hline \mathrm{II} & & x_1 & & \dots & & x_{2n+1} & \dots \end{array}$$

with the requirement that $\langle x_0,\dots,x_n\rangle \in T$ for all n. If a terminal node is reached, the last player to have made a move wins; otherwise, we obtain an infinite play $x \in [T]$, and Player I wins the play if $x \in A$; otherwise, Player II wins.

A *strategy for I* in a game on T is a subtree $\sigma \subseteq T$ so that whenever $s \in \sigma$ has even length, there is a unique $x \in X$ so that $s^\frown\langle x\rangle \in T$; and if $s \in \sigma$ has odd length and $s^\frown\langle x\rangle \in T$, then also $s^\frown\langle x\rangle \in \sigma$ (a strategy for I puts no restrictions on moves by II). Note that due to the presence of terminal nodes in the tree, I needn't have a strategy at all. Whenever it is more convenient to do so, we identify a strategy with the induced partial function $\sigma : T \rightharpoonup X$ assigning an even-length position to the unique next move in σ. We let $\mathrm{Strat}_{\mathrm{I}}(T)$ denote the set of strategies for Player I in T, and $\mathrm{Strat}^*_{\mathrm{I}}(T)$ is the set of *partial strategies*,

$$\mathrm{Strat}^*_{\mathrm{I}}(T) = \bigcup_{m\in\omega} \mathrm{Strat}_{\mathrm{I}}(T{\restriction}2m),$$

where here $T{\restriction}k = \{s \in T \mid |s| < k\}$. Analogous definitions are made for Player II, and we let $\mathrm{Strat}^*(T)$ denote the space of all partial strategies (for both players) in T.

We say $x \in [T]$ is *compatible with* (or *according to*) a strategy σ if $x \in [\sigma]$; for $y \in X^{<\omega}$, we let $\sigma * y$ denote the unique play compatible with σ where the opposing player plays the elements of y (if such exists). A strategy σ is *winning* for Player I (Player II) in $G(A;T)$ if every play according to σ belongs to A ($[T]\setminus A$). If Player I has such a winning strategy, we say briefly that *Player I wins* $G(A;T)$. A game $G(A;T)$ is *determined* if one of the players has a winning strategy. For a pointclass Γ, Γ-DET denotes the statement that $G(A;\omega^{<\omega})$ is determined for all $A \subseteq \omega^\omega$ in Γ.

We assume the reader is familiar with Martin's inductive proof of Borel determinacy as presented in e.g. [13]. In particular, we take as fundamental the technical concept of an *unraveling*, reviewing only briefly the relevant definitions and result.

Definition 4.1. Let S be a tree. A *covering* of S is a triple $\langle T, \pi, \psi\rangle$ consisting of a tree T and maps $\pi : T \to S$, $\psi : \mathrm{Strat}^*(T) \to \mathrm{Strat}^*(S)$ such that

(i) The map T is length-preserving and monotone (and so extends naturally to a map $\pi : [T] \to [S]$, setting $\pi(x) = \bigcup_{i\in\omega} \pi(x{\restriction}i)$).

(ii) (Coherence condition.) For all $\sigma \in \mathrm{Strat}^*_{\mathrm{I}}(T)$ and $m \in \omega$, we have $\psi(\sigma) \in \mathrm{Strat}^*_{\mathrm{I}}(S)$ and

$$\psi(\sigma{\restriction}2m) = \psi(\sigma){\restriction}2m;$$

similarly for $\tau \in \mathrm{Strat}^*_{\mathrm{II}}(T)$ (with $\psi(\tau{\restriction}2m{+}1) = \psi(\tau){\restriction}2m{+}1$ for all m).

(iii) (Lifting property.) For $\sigma \in \mathrm{Strat}^*(T)$, if $s \in \psi(\sigma)$ then there is $t \in \sigma$ with $\pi(t) = s$; and whenever $x \in [S]$, if x is compatible with $\psi(\sigma)$, then there is some $y \in [T]$ compatible with σ so that $\pi(y) = x$.

We say a covering *unravels* a set $A \subseteq [S]$ if $\pi^{-1}(A)$ is clopen.

The coherence condition (ii) allows us to regard ψ as a continuous function $\psi : \mathrm{Strat}(T) \to \mathrm{Strat}(S)$ on the space of full strategies. Note by the lifting property (iii) that if $\sigma \in \mathrm{Strat}(T)$ is a winning strategy in $G(\pi^{-1}(A); T)$, then $\psi(\sigma)$ is winning in $G(A; S)$ (for the same player). In particular, if A can be unraveled then $G(A; S)$ is determined.

Martin showed that the members of any countable collection of closed sets can be *simultaneously* unraveled. So if A is a $\mathbf{\Sigma}^0_{1+\alpha}$ subset of $[S]$ and $\langle T, \pi, \psi\rangle$ is the simultaneous unraveling of the countably many closed sets involved in the construction of A, then $\pi^{-1}(A)$ is $\mathbf{\Sigma}^0_\alpha$ as a subset of $[T]$. Thus the unraveling allows us to reduce the determinacy of $G(A; S)$ to that of the topologically simpler game $G(\pi^{-1}(A); T)$.

Our sights set on Σ^0_4-DET, the way forward is clear: apply the unraveling to all closed sets involved in the construction of a $\mathbf{\Sigma}^0_4$ set A, and prove that the $\mathbf{\Sigma}^0_3$ game $G(\pi^{-1}(A); T)$ is determined, roughly imitating Davis's proof of $\mathbf{\Sigma}^0_3$-DET. Happily, the unraveling of a countable sequence of closed subsets of $[S]$ can be carried out in rudimentarily closed models of "$H(|S|^+)$ exists and is well-ordered". However, the tree T obtained is on a set of higher type than S, and in the setting in which we work, $\mathrm{Strat}^*(T)$ will be a proper class. This presents two related difficulties: First, the proof of determinacy of the unraveled game on T becomes rather more delicate; and second, the winning strategy obtained may be a proper class, and we will nonetheless want to apply the unraveling map ψ to it. We will see that both difficulties are taken care of by certain features of the unraveling of closed sets.

We begin by isolating a locality property of Martin's unraveling. Suppose $\tau \in \mathrm{Strat}^*_{\mathrm{II}}(T)$; for $q \in T$ with $|q|$ even, set

$$\tau^q = \tau_q{\restriction}(|q| + 2) = \{r \in \tau \mid r \subseteq q \text{ or } q \subseteq r, \text{ and } |r| < |q| + 2\}.$$

That is, τ^q is just the fragment recording replies by τ to legal moves by Player I at q. Let us say ψ is *local* if whenever $p \in \psi(\tau)$ with $|p|$ odd, there is some finite sequence $q_0, \dots, q_{n-1}$ of positions in T so that

- $|q_i|$ is even for all $i < n$.
- $\pi(q_i) \subseteq p$ for all $i < n$.
- If $\tau, \bar\tau \in \mathrm{Strat}^*_{\mathrm{II}}(T)$ satisfy $\tau^{q_i} = \bar\tau^{q_i}$ for all $i < n$, then $\psi(\tau)(p) = \psi(\bar\tau)(p)$.

(And analogously for $\sigma \in \mathrm{Strat}^*_{\mathrm{I}}(T)$.) Locality may be regarded as a strengthening of the coherence condition (ii); it states that $\psi(\tau)(p)$ depends not on all of $\tau\restriction(|p|+2)$, but only on its replies to one-step extensions of a certain finite list of positions in T which project to initial segments of p.

Definition 4.2. Let M be a set and S a tree. We say a triple $\langle T, \pi, \psi\rangle$ is an *M-covering* if ψ is local, and $\langle T, \pi, \psi\rangle$ satisfies the conditions of Definition 4.1, except the domain of ψ is only required to consist of those strategies σ with $\sigma^q \in M$ for all $q \in \sigma$ of the appropriate length.

An M-covering is an *M-unraveling* of $A \subseteq [S]$ if $\pi^{-1}(A)$ is clopen in $[T]$.

The notions of locality and M-unraveling allow us to cope with the second difficulty alluded to above: Later, we work in a model M where we need to apply ψ to a strategy τ that is only a definable class in M. Nonetheless, we will have that each of the fragments τ^q with $q \in \tau$ belongs to M. If $\langle T, \pi, \psi\rangle$ is an M-covering, then by locality of ψ, the image $\psi(\tau)$ is well-defined. (We remark that we have not relaxed the lifting property (iii) of Definition 4.1; in particular, it holds true of an M-covering for all plays $x \in [\psi(\tau)]$, regardless of whether x belongs to M.)

For the first mentioned difficulty, it will be crucial that the unraveling tree T is "one-sided," in the sense that only Player I's moves in the game on T are of higher type than those in the game on S. This one-sidedness is central to our arguments and, it seems, has not previously been isolated for uses in the literature, though it is obtained in the constructions of [6], [11], [13]. Let us say a tree T in which I plays moves in X and II plays moves in Y is a *tree on X, Y*. The following can be gleaned from a careful reading of the proof of Borel determinacy in [13].

Theorem 4.3 (Martin). *Suppose X, S, M are sets with $X, S \in M$, S a tree on X, and M a transitive model of "$H(|X|^+)$ exists and is well-ordered"; suppose further that $\{S_n\}_{n\in\omega}$ is a sequence in M of subtrees of S. Then there is a triple $\langle T, \pi, \psi\rangle$ so that*

1. *$T, \pi \in M$ and $\langle T, \pi, \psi\rangle$ is an M-unraveling of $[S_n]$, for all $n \in \omega$.*
2. *T is a tree on $X \cup (X \times \mathcal{P}(S)^M), X \cup (2 \times S)$.*
3. *ψ is definable over $H(|X|^+)^M$; in particular, ψ is a $\Delta_0(M)$-definable subset of* $\mathrm{Strat}^*(T) \times \mathrm{Strat}^*(S)$.

In the situation of this section, $X = \omega$ and $S = \omega^{<\omega}$, so the unraveling tree T is (isomorphic to) a tree on $\mathcal{P}(\omega), \omega$. Since II's moves are in ω, I's strategies are *countable* objects. The upshot is that if $H(\omega_1)$ exists, then $\mathrm{Strat}_{\mathrm{I}}(T)$ is a set, and we can bound quantification over strategies for I, thus keeping the complexity of formulae low.

The following is an abstract form of Davis's Lemma towards Σ^0_3-Determinacy, stated for trees on $\mathcal{P}(X), X$. It includes some technical assumptions on X since we do not in general assume the Axiom of Choice. Here a *quasistrategy* W for Player II is defined similarly to a strategy but dropping the uniqueness condition on moves by II; so if $s \in W$ has odd length, then there is some (not necessarily unique) $x \in X$ with $s^\frown\langle x\rangle \in W$.

Lemma 4.4. *Let X be a transitive set so that Π_1-$\mathsf{RAP}(X)$ holds, and suppose $H(|X|^+)$ exists. Assume X can be wellordered and that $\mathsf{DC}_{\mathcal{P}(X)}$ holds; further assume that there is a uniform Δ_0-definable coding of $X^{<\omega}$ by elements of X, as well as elements of $\mathcal{P}(X)^X$ by $\mathcal{P}(X)$. Let T be a tree on $\mathcal{P}(X), X$ with $B \subseteq A \subseteq [T]$, where B is Π^0_2 and A is Borel. Then for any $p \in T$, either*

1. *Player I has a winning strategy in $G(A; T_p)$, or*
2. *There is $W \subseteq T_p$ a quasistrategy for Player II so that*
    - *$[W] \cap B = \emptyset$;*
    - *Player I does not win $G(A; W)$.*

Proof. In the event (2) holds, we say p is *good* (relative to A, B, T). Notice that goodness of p is Σ_1 in parameters; for this, it is important that moves for II come from X, so that strategies for Player I, who plays in $\mathcal{P}(X)$, can be coded by subsets of X. We assume (2) fails for some p and show (1) must hold. So suppose $p \in T$ is not good; using our canonical coding we can reflect this Π_1 statement to an admissible set N with $X \in N$; since ω-sequences in $\mathcal{P}(X)$ are coded by elements of $\mathcal{P}(X)$, $\mathsf{DC}_{\mathcal{P}(X)}$ is Π_1 in parameters, so we can further assume this choice principle holds in N. This move to an admissible set is our use of the main strength assumption of the lemma, namely Π_1-$\mathsf{RAP}(X)$.

So work in N. By an abuse of notation we prefer to refer to N's versions of the relevant objects A, B, T by the same names. Since B is Π^0_2 in $[T]$, we have $B = \bigcap_{n \in \omega} D_n$, where each D_n is open in $[T]$; that is, there are sets $U_n \subseteq T$ so that for all n, $D_n = \{x \in T \mid (\exists i) x{\restriction}i \in U_n\}$. Adjusting the sets U_n as necessary, we may assume that each U_n contains only nodes with odd length $\geq n$.

We now define an operator $\Gamma : \mathcal{P}(T) \to \mathcal{P}(T)$ as follows. Fix $Y \subseteq T$. For $n \in \omega$, let T^Y_n be the tree $\{s \in T \mid (\forall i < |s|) s{\restriction}i \notin U_n \setminus Y\}$. Then let

$$\Gamma(Y) = \{q \in T \mid (\exists n) \text{ Player I does not win } G(A; T^Y_n) \text{ from } q\}.$$

The reason we truncate the tree following minimal odd-length nodes in $U_n \setminus Y$ is that we will be considering an auxiliary game where Player I is trying to enter the open set D_n while avoiding Y; the auxiliary game on T^Y_n ends when a node in $U_n \setminus X$ is reached, and in this case, Player I is the winner, because the length of the node reached is odd by the way we defined U_n.

Note that the definition of Γ is such that "$x \in \Gamma(Y)$" is Δ_0 in parameters; this is because $H(|X|^+)$, and hence $\mathrm{Strat}_{\mathrm{I}}(T)$, is a set (this is where we use "one-sidedness": T is a tree on $\mathcal{P}(X), X$, so strategies for Player I are essentially functions $f : X^{<\omega} \to \mathcal{P}(X)$, and so are in $H(|X|^+)$). Now define recursively sets $\mathcal{G}_\alpha \subseteq T$ for all $\alpha \in \mathrm{ON}$, by

- $\mathcal{G}_0 = \emptyset$;
- $\mathcal{G}_{\alpha+1} = \Gamma(\mathcal{G}_\alpha)$;
- $\mathcal{G}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{G}_\lambda$ for λ limit.

Note KP gives us enough transfinite recursion to ensure that the sets $\mathcal{G}_\alpha$ exist for all ordinals α; for this it is important that the operator Γ is Δ_0. Note also that Γ is a monotone operator (i.e., if $Y \subseteq Y'$ then $\Gamma(Y) \subseteq \Gamma(Y')$), and therefore the sets $\mathcal{G}_\alpha$ are increasing.

Claim 4.5. *If $q \in T$ belongs to $\mathcal{G}_\alpha$ for some α, then q is good.*

PROOF. The proof is by induction on α. So suppose q, α are such that α is least with $q \in \mathcal{G}_{\alpha+1}$. Let n_0 be the least witness to this, so that I doesn't win $G(A; T_{n_0}^{\mathcal{G}_\alpha})$ from q. We describe a quasistrategy W for II at q as follows: play according to II's non-losing quasistrategy in $G(A; T_{n_0}^{\mathcal{G}_\alpha})$. If at any point we reach a position $r \in U_{n_0}$, then because we are inside II's non-losing quasistrategy, we must have $r \in \mathcal{G}_\alpha$. By inductive hypothesis, r is good; so switch to W^r witnessing goodness of r. Note that W exists by admissibility (the assertion "W^r witnesses goodness of r in T" is Δ_0 in parameters).

We claim this W witnesses goodness of q. For let $x \in [W]$; if $x \restriction i \notin U_{n_0}$ for all $i \in \omega$, then by openness of D_{n_0} we have $x \notin D_{n_0}$, so $x \notin B$. If $x \restriction i \in U_{n_0}$ for some i, then we must have switched to $W^{x \restriction i}$ at the least such i; since $[W^{x \restriction i}] \cap B = \emptyset$, we have $x \notin B$.

We need to show I doesn't win $G(A; W)$. So suppose σ is a strategy for I in W. If all positions r compatible with σ satisfy $r \notin U_{n_0}$, then σ must stay inside II's non-losing quasistrategy for $G(A; T_{n_0}^{\mathcal{G}_\alpha})$. This implies that σ can be extended to a winning strategy for I in $G(A; T_{n_0}^{\mathcal{G}_\alpha})$ from q: Play by σ so long as II stays inside W, and if II strays to r outside W switch to a winning strategy for I in $G(A; T_r)$, which exists by definition of W. (Note that the existence of this extension of σ requires a use of $\mathsf{DC}_{\mathcal{P}(X)}$, to choose the strategies Player I switches to.) But this contradicts the assumptions on q and n_0. If on the other hand $r \in U_{n_0}$ for some r compatible with σ, then $r \in \mathcal{G}_\alpha$, so by inductive hypothesis and definition of the quasistrategy W, W_r witnesses the goodness of r. But that means I doesn't have a winning strategy in $G(A; W_r)$; again, σ cannot be winning for I. ⊣

Since we have (in N) that p is not good, we get $p \notin \mathcal{G}_\infty := \bigcup_{\alpha \in \mathrm{ON}} \mathcal{G}_\alpha$. Note that since we do not have Σ_1-Comprehension in N, $\mathcal{G}_\infty$ may not be a set; indeed, if $\mathcal{G}_\alpha \neq \mathcal{G}_{\alpha+1}$ for all α, then $\mathcal{G}_\infty$ must be a proper class, by admissibility.

CLAIM 4.6. $\Gamma(\mathcal{G}_\infty) = \mathcal{G}_\infty$.

PROOF. Since $\mathcal{G}_\infty$ needn't be a set, we should say what we mean by $\Gamma(\mathcal{G}_\infty)$: This is the class of $q \in T$ so that for some n, I doesn't win $G(A; T_n^{\mathcal{G}_\infty})$ from q. Expanding our definition, this is the same as

$$(\exists n \in \omega)(\forall \sigma \in \mathrm{Strat_I}(T_q))(\exists x \in [\sigma]) x \notin A, \text{ and } (\forall i)(x \restriction i \in \mathcal{G}_\infty \cup (T \setminus U_n)).$$

This statement is Σ_1, because $x \restriction i \in \mathcal{G}_\infty$ is Σ_1 (it is the statement $(\exists \alpha) x \restriction i \in \mathcal{G}_\alpha$, and the relation $s \in \mathcal{G}_\alpha$ is $\boldsymbol{\Delta}_1$ as a relation on $T \times \mathrm{ON}$), and all other quantifiers are bounded (since $\mathrm{Strat_I}(T_q)$ and $[\sigma]$ are sets from the point of view of N).

Suppose $q \in \Gamma(\mathcal{G}_\infty)$. Then by Σ_1-Collection applied inside the admissible N, there is a bound α on ordinals witnessing the "$x \restriction i \in \mathcal{G}_\infty$" clause in the above statement, for various σ, x. That is, $q \in \Gamma(\mathcal{G}_\alpha) \subseteq \mathcal{G}_\infty$; so $\Gamma(\mathcal{G}_\infty) \subseteq \mathcal{G}_\infty$. The reverse inclusion is trivial. ⊣

Using this stabilization of the operator Γ we describe a winning strategy for I in $G(A; T_p)$. Since $p \notin \mathcal{G}_\infty = \Gamma(\mathcal{G}_\infty)$, we have for all n that I wins $G(A; T_n^{\mathcal{G}_\infty})$ at p. Let σ_0 witness this for $n_0 = 0$. Have I play according to σ_0 from the initial position. Now suppose strategies σ_i, n_i have been defined; if at any point in the strategy σ_i I reaches a position p_i with $p_i \in U_{n_i} \setminus \mathcal{G}_\infty$, then I wins $G(A; T_n^{\mathcal{G}_\infty})$ from p_i for every n, since $p_0 \notin \mathcal{G}_\infty$. Let n_{i+1} be least so that no initial segment of p_i is in $U_{n_{i+1}}$ (such exists because of the way we defined the U_n). Let σ_{i+1}

be winning for I in $G(A;T^{\mathcal{G}_\infty}_{n_{i+1}})$, and have I continue according to this strategy. Any infinite play against the strategy σ so defined must either enter the sets $U_0, U_1, U_2, \ldots$ one by one, thus belonging to $B = \bigcap_{n\in\omega} D_n$, or else the play avoids $U_n \setminus \mathcal{G}_\infty$ for some least n; but then the play is compatible with σ_n, so must belong to A.

Some remarks regarding definability are in order. The inductive construction of Player II's goodness-witnessing quasistrategies was uniform, so no choice was required. The same cannot be said of Player I's winning strategy; at the very least, $\mathsf{DC}_{\mathcal{P}(X)}$ is needed to select the various σ_n, and even if N possesses a Δ_1-definable wellordering of $H(|X|^+)$, the definition of σ will (in general) be Σ_2, so the strategy needn't be a set in N. However, we are able to define the strategy over N from the point of view of our model of Π_1-$\mathsf{RAP}(X)$, using $\mathsf{DC}_{\mathcal{P}(X)}$.

We claim that the strategy σ we have described is winning (in V) in $G(A;T_p)$ (where now T is the full tree, rather than its restriction to N). Note first that σ really is a strategy in the true T, since II's moves are in X, and so N's version of T is closed under moves by II.

Suppose $x \in [T_p]$ is according to σ. Then $x \restriction i \in N$ for all i. If for some n the play never enters $U_n \setminus \mathcal{G}_\infty$, then x must be according to some $\sigma_n \in N$. We have N is admissible, A is Borel, and σ_n is winning in $G(A;T^{\mathcal{G}_\infty}_n)$ in N. So by absoluteness, $x \in A$.

So suppose x enters $U_n \setminus \mathcal{G}_\infty$ for every n. Then $x \in \bigcap_{n\in\omega} D_n = B \subseteq A$. Either way, we have any play x compatible with σ is in A, so we have that σ is winning in $G(A;T_p)$. That is, case (1) of the lemma holds. $\dashv$

THEOREM 4.7. *Suppose M is a transitive model of Π_1-RAP, and that M has a wellordering of its reals. Let $A \subseteq \omega^\omega$ be Σ^0_4. Then either I wins $G(A;\omega^{<\omega})$ with a strategy in M, or II has a winning strategy in $G(A;\omega^{<\omega})$ that is $\boldsymbol{\Delta}_1$-definable over M.*

PROOF. Note that Π_1-RAP implies the existence of $H(\omega_1)$, and by assumption, there is a wellorder of $\mathcal{P}(\omega)^M$, and hence of $H(\omega_1)^M$, in M. Working inside M, let $A \subseteq \omega^\omega$ be Σ^0_4, and using Theorem 4.3, let $\langle T, \pi, \psi\rangle$ be the simultaneous M-unraveling of all Π^0_1 sets. By a standard coding, T may be regarded as a tree on $\mathcal{P}(\omega), \omega$. Let $\bar{A} = \pi^{-1}(A)$. Then $\bar{A}$ is Σ^0_3, and we have $\bar{A} = \bigcup_{k\in\omega} B_k$ for some family $\{B_k \mid k \in \omega\}$ of Π^0_2 subsets of $[T]$.

If Player I wins $G(\bar{A};T)$, say with σ, then $\psi(\sigma)$ is easily seen to be a winning strategy in M for I in $G(A;\omega^{<\omega})$ that continues to be winning in V. So suppose I does not win $G(\bar{A};T)$. By Lemma 4.4, there is a quasistrategy $W^\emptyset_0$ for II witnessing the goodness of $\emptyset$ relative to $B_0, \bar{A}$ in T. We may assume this $W^\emptyset_0$ is obtained from the uniform construction of goodness-witnessing quasistrategies for II, as in the proof of Lemma 4.4. We may furthermore assume that $W^\emptyset_0$ is non-losing for II; that is, I doesn't win $G(\bar{A};(W^\emptyset_0)_p)$ for any $p \in W^\emptyset_0$.

Suppose now that we have some fixed quasistrategy $W^p_k \subseteq T$ for II in T_p, with $p \in W^p_k$ a position of length $2k$, and that I doesn't win $G(\bar{A};W^p_k)$. For any $q \in W^p_k$ of length $2k+2$, let W^q_{k+1} be the (uniformly constructed) goodness-witnessing quasistrategy at q guaranteed by applying Lemma 4.4 (to $\bar{A}, B_{k+1}, W^p_k$).

We then define a quasistrategy for II in $G(\bar{A};T)$ by inductively taking the common refinement of the W^p_k. That is, at positions p in T of length $2k$, if

$p^\frown\langle a\rangle \in W^p_k$, then those moves b for II at $p^\frown\langle a\rangle$ are permitted exactly when $q^\frown\langle a,b\rangle \in W^p_k$. Let W be the quasistrategy for II so obtained.

Notice that W is Δ_1-definable over M. Furthermore, if $x \in [W]$, then $x \in [W_k^{x\restriction 2k}]$ for all k; since each $W_k^{x\restriction k}$ witnesses goodness of $x\restriction k$ relative to B_k, we have $x \notin B_k$ for all k, hence $x \notin A$.

A strategy τ for II is easily obtained by refining W, choosing a single successor node at each position of odd length (recall II's moves are in ω). Notice that the strategy τ is $\mathbf{\Delta}_1$-definable in M, but may not be an element of M. However, we have for each $q \in \tau$ with $|q| = 2k$ that $\tau^q \in M$, since τ^q is just a refinement of W^q_k. We may thus use the fact that $\langle T, \pi, \psi\rangle$ is an M-covering to conclude that the strategy $\tau' = \psi(\tau)$ is well-defined; what is more, this strategy τ' is $\mathbf{\Delta}_1$-definable in M by $\mathbf{\Delta}_1$-definability of τ combined with the final clause of Theorem 4.3.

We claim τ' wins $G(A;\omega^{<\omega})$ for II in V. Suppose towards a contradiction that $x \in A$ is a play in ω^ω compatible with τ'. Then using the lifting property of the M-unraveling, we have a play $y \in [T]$ (though possibly $\notin M$) so that y is compatible with τ, and $\pi(y) = x$ (in particular, $y \in \pi^{-1}(A) = \bar{A}$). Then $y \in B_k$ for some k. Now τ on $T_{y\restriction 2k}$ is a refinement of $W_k^{y\restriction 2k}$, so we must have $y \in [W_k^{y\restriction 2k}]$. But inside M, we have $[W_k^{y\restriction 2k}] \cap B_k = \emptyset$; in particular, M thinks the tree of attempts to find a branch through $W_k^{y\restriction 2k}$ in B_k is wellfounded, hence ranked in M, by Claim 3.4. By absoluteness, this contradicts $y \in B_k$. ⊣

We would like to remove the uses of Choice in the previous theorems. One way is to proceed as in Hurkens' [6], proving "quasi-determinacy" of Borel games, which implies full determinacy for games on ω; alternately we may work in L so that Choice is available, and then use Shoenfield absoluteness to show the strategies of L are winning in V (both of these approaches are detailed in Chapter 7 of [13]). By Proposition 3.6, we may adopt the latter approach. We obtain

THEOREM 4.8. *Let $A \subseteq \omega^\omega$ be Σ^0_4, and suppose θ is the least ordinal such that $L_\theta \models \Pi_1$-RAP. Then either I wins $G(A;\omega^{<\omega})$ as witnessed by a strategy $\sigma \in L_\theta$, or else II has a winning strategy τ that is Δ_1-definable over L_θ.*

In particular, it is provable in KP *that the existence of a transitive model of* Π_1-RAP *implies* Σ^0_4-DET.

§5. The Lower Bound. This section is devoted to proving the reversal of Theorem 4.8:

THEOREM 5.1 (KPI_0). Σ^0_4-DET *implies the existence of a transitive model of* Π_1-RAP.

Applying Theorem 5.1 in $L_{\omega_1^{L_\theta}}$, we have failure of Σ^0_4-DET in L_θ. Thus Theorem 4.8 is sharp: θ is the least ordinal so that every Σ^0_4 game is determined as witnessed by a strategy belonging to $J_{\theta+\omega}$.

As a warm-up to help orient the reader and introduce the structure of our lower bound arguments, we first present a proof that $\mathsf{ZF}^- \nvdash \Sigma^0_4$-DET. The argument is a refinement due to Martin of Friedman's [4]; see [11].

Definition 5.2. Let T be a collection of sentences in the language of set theory. We say a structure $\mathcal{M}$ in the language of set theory is *T-small* if it satisfies

$$V{=}L + \mathsf{KP} + \text{ Infinity } + \text{ all sets are countable } + \rho_1 = \omega + (\forall\alpha)L_\alpha \not\models T.$$

Suppose Σ is (the set of codes for) a complete consistent theory extending the displayed theory above. There is a unique (up to isomorphism) minimal model M of Σ, defined as follows. Let

$$M = \{i \in \omega \mid \ulcorner(\exists! u)\phi_i(u)\urcorner \in \Sigma\},$$

where ϕ_i is some standard enumeration of formulae with one free variable in the language of set theory, and $\ulcorner\sigma\urcorner$ denotes the Gödel number of a sentence σ. Let $i \equiv j$ if $i, j \in M$ and $\ulcorner(\forall u)\phi_i(u) \leftrightarrow \phi_j(u)\urcorner \in \Sigma$. Then $\equiv$ is an equivalence relation; we define a relation ε on the equivalence classes by setting $[i]_\equiv \,\varepsilon\, [j]_\equiv$ if $\ulcorner(\exists u)(\exists v)\phi_i(u) \wedge \phi_j(v) \wedge u \in v\urcorner \in \Sigma$. Then the *term model determined by* Σ is

$$\mathcal{M} = (M/{\equiv}, \varepsilon).$$

It is easy to check this is well-defined. Note that $\mathcal{M}$ has standard ω if and only if

$$\begin{aligned}(\forall i)\ulcorner(\exists u)\phi_i(u) \wedge u \text{ is a natural number}\urcorner \in \Sigma \\ \to (\exists n)\ulcorner(\forall u)\phi_i(u) \to u = n\urcorner \in \Sigma.\end{aligned}$$

This is a Π^0_2 property of the theory Σ. Thus the statement "Σ determines a T-small ω-model" is a $\Pi^0_2(T)$ property of Σ.

Let β_0 be the least ordinal so that $L_{\beta_0} \models \mathsf{ZF}^-$. We show that Σ^0_4-DET fails in L_{β_0} by defining a Π^0_4 game G_0 where the players compete to produce ZF^--small models with longest possible wellfounded part. ZF^--smallness of the models produced ensures this comparison can be done in a Π^0_4 manner.

Players I and II play reals $f_{\mathrm{I}}, f_{\mathrm{II}}$, respectively. If f_{I} is not the characteristic function of a complete, consistent theory determining a ZF^--small ω-model, then Player I loses. Otherwise, Player II loses unless f_{II} satisfies the same condition. If neither player has lost, let $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$ denote the models determined by $f_{\mathrm{I}}, f_{\mathrm{II}}$, respectively. (Note that the winning condition of G_0 up to this point is a Boolean combination of Π^0_2 conditions.)

In order to make the rest of the winning condition easier to parse, we will typically quantify over sets such as $\mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}}$ rather than ω, and will frequently compress the Π^0_1 condition "i codes a real $x \in \mathcal{M}_{\mathrm{I}}$ and j codes a real $y \in \mathcal{M}_{\mathrm{II}}$ so that $x = y$" as simply "$x = y$", with the hope that this will make the intended meaning clearer. For example, we write

$$(\forall x \in \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}})(\exists y \in \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}})x = y$$

regarding this as an abbreviation for

$$\begin{aligned}(\forall i)(\exists j)(\forall n)[f_{\mathrm{II}}(\ulcorner\exists! u\varphi_i(u) \wedge u \subseteq \omega\urcorner) = 1 \to (f_{\mathrm{I}}(\ulcorner\exists! u\varphi_j(u) \wedge u \subseteq \omega\urcorner) = 1 \wedge \\ (f_{\mathrm{II}}(\ulcorner\forall u\varphi_i(u) \to n \in u\urcorner) = 1 \leftrightarrow f_{\mathrm{I}}(\ulcorner\forall u\varphi_j(u) \to n \in u\urcorner) = 1))].\end{aligned}$$

We condense this further as "$\mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}} \subseteq \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}}$", which we see is a Π^0_3 condition of $f_{\mathrm{I}}, f_{\mathrm{II}}$. In what follows, we typically omit mention of $f_{\mathrm{I}}, f_{\mathrm{II}}$, which are allowed parameters in all our complexity calculations, and simply say the relation is Π^0_3.

Claim 5.3. *Suppose $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$ are ω-models, and let $x \in \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}}$. Then*

$$\text{“}(\exists y \in \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}})(y \notin \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}} \wedge (y <_L x)^{\mathcal{M}_{\mathrm{II}}})\text{”}$$

is Σ^0_3.

Proof. The statement that there exists a real in $\mathcal{M}_{\mathrm{II}} \setminus \mathcal{M}_{\mathrm{I}}$ is Σ^0_3, as the previous discussion shows. Once a code for such a y has been fixed, the question of whether $(y <_L x)^{\mathcal{M}_{\mathrm{II}}}$ holds is decided by the theory of $\mathcal{M}_{\mathrm{II}}$, hence recursive in f_{II}. ⊣

We may now finish giving our winning condition. If both $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$ are ZF^--small ω-models, then Player I wins G_0 if any of the following hold:

1. $\mathcal{M}_{\mathrm{I}} = \mathcal{M}_{\mathrm{II}}$ or $\mathrm{Th}(\mathcal{M}_{\mathrm{II}}) \in \mathcal{M}_{\mathrm{I}}$.
2. There is a real in $\mathcal{M}_{\mathrm{I}} \cap \mathcal{M}_{\mathrm{II}}$ that codes a wellorder in $\mathcal{M}_{\mathrm{II}}$, but codes an illfounded linear order in $\mathcal{M}_{\mathrm{I}}$.
3. $\mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}} \setminus \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}}$ is nonempty and has no $<_L^{\mathcal{M}_{\mathrm{II}}}$-least element.

Note that if either (2) or (3) hold, then $\mathcal{M}_{\mathrm{II}}$ is illfounded. By the remarks above, (1) and (2) are both Σ^0_2; and it follows easily from Claim 5.3 that (3) is Π^0_4, so the statement “Player I wins G_0” is Π^0_4 in the play.

Claim 5.4. *Neither player has a winning strategy for G_0 in L_{β_0}.*

Proof. First, Player I can have no winning strategy in L_{β_0}. For suppose σ were such a strategy. Let α be the least admissible ordinal so that $\sigma \in L_\alpha$. Since L_α is a ZF^--small ω-model (N.B. the remarks preceding Proposition 2.3), have Player II play against σ with $f_{\mathrm{II}} = \mathrm{Th}(L_\alpha)$. Let $\mathcal{M}_{\mathrm{I}}$ be the model produced by σ. Since $\mathcal{M}_{\mathrm{II}}$ is wellfounded, neither (2) nor (3) can hold. So (1) must hold. If $\mathcal{M}_{\mathrm{I}} = \mathcal{M}_{\mathrm{II}} = L_\alpha$, then II simply copied the theory produced by σ, in which case $\mathrm{Th}(L_\alpha)$ is computable from σ and so belongs to L_α, contradicting the fact that $\rho_1^{L_\alpha} = \omega$. Similarly, if $f_{\mathrm{II}} = \mathrm{Th}(\mathcal{M}_{\mathrm{II}}) \in \mathcal{M}_{\mathrm{I}}$, then $\mathrm{Th}(\mathcal{M}_{\mathrm{I}}) = f_{\mathrm{I}} = \sigma * f_{\mathrm{II}}$ belongs to $\mathcal{M}_{\mathrm{I}}$, contradicting ZF^--smallness of $\mathcal{M}_{\mathrm{I}}$ (specifically, the requirement that $\rho_1^{\mathcal{M}_{\mathrm{I}}} = \omega$). So σ cannot be winning for Player I.

Next, Player II has no winning strategy in L_{β_0}. For if τ were such a strategy, we again set α minimal so that $\tau \in L_\alpha$ and α is admissible, and have Player I play $f_{\mathrm{I}} = \mathrm{Th}(L_\alpha)$. Let $\mathcal{M}_{\mathrm{II}}$ be the model τ responds with. We claim $\mathcal{M}_{\mathrm{II}}$ is illfounded. For if $\mathcal{M}_{\mathrm{II}}$ is wellfounded, then $\mathcal{M}_{\mathrm{II}} = L_\beta$ for some β. If $\beta \leq \alpha$, then Player I wins by condition (1); and if $\beta > \alpha$, then τ, f_{I} both belong to L_β, in which case L_β can compute its own theory, f_{II}, again contradicting ZF^--smallness of $\mathcal{M}_{\mathrm{II}}$. So we have $\mathcal{M}_{\mathrm{II}}$ illfounded, and the same argument shows $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}}) \leq \alpha$.

Since we assumed τ was winning for Player II, there can be no codes for nonstandard ordinals of $\mathcal{M}_{\mathrm{II}}$ that belong to L_α. But then by overspill (Lemma 2.5), the set of nonstandard codes not in $\mathcal{M}_{\mathrm{I}}$ is nonempty with no $<_L^{\mathcal{M}_{\mathrm{II}}}$-least element; that is, (3) holds, and I wins the play, a contradiction. ⊣

We have shown L_{β_0} does not satisfy Σ^0_4-DET, so that $\mathsf{ZF}^- \nvdash \Sigma^0_4$-DET. Note that the only real use of ZF^--smallness of the models above was the fact that if α is the wellfounded ordinal of a ZF^--small ω-model, then new reals are constructed cofinally in L_α, and we can apply overspill. We can carry this a bit further.

Let $T = \mathsf{KP} +$ "$\mathcal{P}(\omega)$ exists", and consider the game G_1 with the winning condition unchanged, except we require both players to play T-small ω-models. Let γ_0 be least so that $L_{\gamma_0} \models T$. Then G_1 is non-determined in L_{γ_0}: for if $\mathcal{M}$ is an illfounded model of $V = L$ coded by a real in L_{γ_0}, then $L_{\mathrm{wfo}(\mathcal{M})}$ is admissible, and must satisfy "all sets are countable." We therefore will have new reals constructed cofinally in $L_{\mathrm{wfo}(\mathcal{M})}$, and the overspill argument in the last paragraph of the proof of Claim 5.4 applies. So we have established $\mathsf{KP} +$ "$\mathcal{P}(\omega)$ exists"$\not\vdash \Sigma^0_4$-DET.

Of course, at the level of Π_1-RAP, there are many admissible models of "$\mathcal{P}(\omega)$ exists", so our winning condition will have to be more elaborate. Our ability to identify overspill in $\mathcal{M}_{\mathrm{II}}$ in a Π^0_4 manner relied on the fact that asserting the existence of a real in $\mathcal{M}_{\mathrm{II}} \setminus \mathcal{M}_{\mathrm{I}}$ was Σ^0_3; what we therefore require is a Σ^0_3 condition that will identify nonstandard ordinals in a similar fashion when $\mathcal{M}_{\mathrm{II}}$ is only assumed to be Π_1-RAP-small. The desired condition is given by Lemma 5.6; the next definition is the first step towards this lemma.

Let (T) denote the sentence

$$\mathcal{P}(\omega) \text{ exists, and every tree } T \text{ on } \mathcal{P}(\omega) \text{ is either ranked or illfounded.}$$

Recall θ is the least ordinal so that $L_\theta \models \Pi_1$-RAP; by Corollary 3.4 and Theorem 3.5, it is also least so that $L_\theta \models (\mathsf{T})$. In order to simplify slightly the remaining arguments, we work with (T) instead of Π_1-RAP.

Note that $\omega \cdot \theta = \theta$, so that $L_\theta = J_\theta$; also, $L_\theta \models$"ω_1 is the largest cardinal," and $\rho_1^{J_\theta} = \omega$.

Proof of Theorem 5.1. Work in KPI_0. Assume Σ^0_4-DET and, towards a contradiction, than there is no transitive model of Π_1-RAP; in particular, θ does not exist. Since KPI_0 holds in V, it holds also in L. Now if ω_1^L exists and α is the least admissible greater than ω_1^L, then L_α satisfies $\mathsf{KP} +$ "$\mathcal{P}(\omega)$ exists", and since every sequence of reals in L belongs to L_α, we have $L_\alpha \models (\mathsf{T})$, so that θ exists. We may therefore assume that all ordinals are countable in L; in particular, there are unboundedly many α so that L_α is admissible and projects to ω (N.B. the remarks following Theorem 2.2).

We define a game G with Π^0_4 winning condition, and argue that G cannot be determined. The game proceeds as follows: players I and II play reals $f_{\mathrm{I}}, f_{\mathrm{II}}$, respectively, coding the theories of (T)-small ω-models $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$; if f_{I} does not code such a model, Player I loses, and similarly for f_{II} and Player II.

Definition 5.5. Recall for $x \in L$ that $\mathrm{rank}_L(x)$ is defined as the least ρ so that $x \in J_{\rho+\omega}$. Working in $\mathcal{M}_{\mathrm{II}}$, suppose $x \in \mathcal{P}(\omega)$, and inductively define

$$\delta(0,x) = \begin{cases} \mathrm{rank}_L(x) & \text{if } J_{\mathrm{rank}_L(x)} \models \text{“}\omega_1 \text{ exists”;} \\ \text{undefined} & \text{otherwise;} \end{cases}$$

$$\delta(k+1,x) = \begin{cases} \begin{array}{l} \delta \text{ least s.t. } J_{\delta(k,x)} \models \text{“}\omega_1 \text{ exists and} \\ (\exists T \in J_{\delta+\omega})\, T \text{ is a tree on } \mathcal{P}(\omega) \text{ that} \\ \text{is neither ranked nor illfounded,”} \end{array} & \text{if such exists;} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We stress that this definition is internal to $\mathcal{M}_{\mathrm{II}}$. Thus if $\mathcal{M}_{\mathrm{II}}$ has standard ω and $\delta(0,x)$ exists, then $\langle \delta(k,x) \rangle$ is a strictly descending sequence of ordinals, so is finite. The fact that (T) fails in every level of $\mathcal{M}_{\mathrm{II}}$ implies that the smallest element of $\langle\delta(k,x)\rangle$ is $\omega_1^{J_{\delta(0,x)}}$.

Notice that if $\mathcal{M}_{\mathrm{II}}$ is illfounded with $\omega_1^{J_{\delta(0,x)}} < \mathrm{wfo}(\mathcal{M}_{\mathrm{II}}) \subseteq \delta(0,x)$, then there is some unique k so that $\delta(k+1,x)$ is wellfounded but $\delta(k,x)$ is nonstandard. By the defining property of $\delta(k+1,x)$, there is some tree $T \in J_{\delta(k+1,x)+\omega} \subset L_{\mathrm{wfo}(\mathcal{M}_{\mathrm{II}})}$ that is neither ranked nor illfounded in $J_{\delta(k,x)}$, hence neither ranked nor illfounded in $L_{\mathrm{wfo}(\mathcal{M}_{\mathrm{II}})}$. The latter set is admissible, so T is in fact illfounded, and a branch through T is definable over $L_{\mathrm{wfo}(\mathcal{M}_{\mathrm{II}})}$. We exploit this fact in the following lemma.

LEMMA 5.6. *There is a Σ^0_3 relation $R(k,\gamma,x)$ such that if $f_{\mathrm{I}}, f_{\mathrm{II}}$ are theories determining (T)-small ω-models $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$, respectively, so that $\mathcal{M}_{\mathrm{I}}$ is wellfounded, and x is the $<_L^{\mathcal{M}_{\mathrm{II}}}$-least element of $\mathcal{P}(\omega) \cap (\mathcal{M}_{\mathrm{II}} \setminus \mathcal{M}_{\mathrm{I}})$, then we have the following:*

(A) $(\forall k \in \omega)(\forall \gamma \in \mathrm{ON}^{\mathcal{M}_{\mathrm{II}}}) R(k,\gamma,x) \rightarrow \delta(k+1,x)$ *is standard;*

(B) $(\forall k \in \omega)$ *if $\delta(k,x)$ is nonstandard and $\delta(k+1,x)$ is wellfounded, then* $(\forall \gamma \in \mathrm{ON}^{\mathcal{M}_{\mathrm{II}}})[R(k,\gamma,x) \leftrightarrow (\gamma < \delta(k,x))^{\mathcal{M}_{\mathrm{II}}} \wedge \gamma$ *is nonstandard*$]$.

Note that when we assert "R is a Σ^0_3 relation (on $\omega \times {\mathcal{M}_{\mathrm{II}}}^2$)" this should be understood to mean that the corresponding relation $\tilde{R} \subseteq \omega^3 \times (\omega^\omega)^2$ (on the codes) is Σ^0_3 (as a relation on $i, j, k, f_{\mathrm{I}}, f_{\mathrm{II}}$).

We shall give the proof of Lemma 5.6 shortly. For now, we use the lemma to finish defining the game G, and to prove Theorem 5.1. Suppose I, II play consistent theories $f_{\mathrm{I}}, f_{\mathrm{II}}$ determining (T)-small ω-models $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$, respectively. Player I wins if

$$(\exists x \in \mathcal{P}(\omega) \cap \mathcal{M}_{\mathrm{I}} \cap \mathcal{M}_{\mathrm{II}}) x \text{ codes a wellorder of } \omega \text{ in } \mathcal{M}_{\mathrm{II}}, \text{ but not in } \mathcal{M}_{\mathrm{I}}.$$

(As before, this implies illfoundedness of $\mathcal{M}_{\mathrm{II}}$.) Otherwise, I wins just in case the following holds:

1. $(\forall x \in \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}})$ if $x \notin \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}}$, then
   (a) $(\exists y \in \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}})(y \notin \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}} \wedge (y <_L x)^{\mathcal{M}_{\mathrm{II}}})$, or
   (b) $(\exists k,\gamma) R(k,\gamma,x)$
   $\wedge (\forall k,\gamma)[R(k,\gamma,x) \rightarrow (\exists k',\gamma') R(k',\gamma',x) \wedge \langle k',\gamma'\rangle <_{\mathrm{Lex}} \langle k,\gamma\rangle]$,
   and
2. $\mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}} \subseteq \mathcal{M}_{\mathrm{I}}$ implies
   (a) $\mathrm{Th}(\mathcal{M}_{\mathrm{II}}) \in \mathcal{M}_{\mathrm{I}}$, or
   (b) $\mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}} \subseteq \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}}$.

Here $<_{\mathrm{Lex}}$ is the lexicographic order on the product $(\omega, \in) \times (\mathrm{ON}^{\mathcal{M}_{\mathrm{II}}}, \in^{\mathcal{M}_{\mathrm{II}}})$. Condition (1) states that if $\mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}} \setminus \mathcal{M}_{\mathrm{I}}$ is nonempty, then either it has no $<_L^{\mathcal{M}_{\mathrm{II}}}$-least element, or taking x to be $<_L^{\mathcal{M}_{\mathrm{II}}}$-minimal, the set of $\langle k,\gamma\rangle$ such that $R(k,\gamma,x)$ holds is nonempty and has no $<_{\mathrm{Lex}}$-least element. In particular, if (1) holds and $\mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}} \setminus \mathcal{M}_{\mathrm{I}}$ is nonempty, then $\mathcal{M}_{\mathrm{II}}$ is illfounded.

CLAIM 5.7. *Assuming θ does not exist, I does not win G.*

PROOF. Suppose towards a contradiction that σ is a winning strategy for I in G. Applying Shoenfield absoluteness (which is provable in KPI_0), we may assume $\sigma \in L$. Let α be the least admissible ordinal so that $\sigma \in L_\alpha$. Then L_α projects to ω and satisfies "all sets are countable". Since θ does not exist, we also have L_α satisfies $(\forall\xi)L_\xi \not\models (\mathsf{T})$. Let f_{II} be the theory of L_α, so that $\mathcal{M}_{\mathrm{II}} = L_\alpha$. Let $\mathcal{M}_{\mathrm{I}}$ be the model that σ replies with.

Since $\mathcal{M}_{\mathrm{II}}$ is wellfounded, there cannot be any real in $\mathcal{M}_{\mathrm{II}} \setminus \mathcal{M}_{\mathrm{I}}$, since the $<_L$-least such would be a witness to failure of (1). So $\mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}} \subseteq \mathcal{M}_{\mathrm{I}}$. In particular, $\sigma \in \mathcal{M}_{\mathrm{I}}$, and we can't have $\mathrm{Th}(\mathcal{M}_{\mathrm{II}}) \in \mathcal{M}_{\mathrm{I}}$, since then $\mathrm{Th}(\mathcal{M}_{\mathrm{I}}) = \sigma * f_{\mathrm{II}} \in \mathcal{M}_{\mathrm{I}}$, contradicting the fact $\mathcal{M}_{\mathrm{I}}$ projects to ω. So (2a) fails, and (2b) must hold; in particular, $\mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}} = \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}}$. This implies $\mathcal{M}_{\mathrm{I}} = \mathcal{M}_{\mathrm{II}}$, since both models satisfy "all sets are countable", and we again have the contradiction $\mathrm{Th}(\mathcal{M}_{\mathrm{I}}) \in \mathcal{M}_{\mathrm{I}}$, since in this case f_{II} is just copying the play by σ. We have that $(1) \wedge (2)$ must fail, so σ cannot be a winning strategy for I. ⊣

Claim 5.8. *Assuming θ does not exist, II does not win G.*

Proof. As before, assume for a contradiction that $\tau \in L$ is a winning strategy for II in G. Let α be admissible with $\tau \in L_\alpha$ and L_α projecting to ω; again, $L_\xi \not\models (\mathsf{T})$ for all $\xi \in \alpha$, since θ does not exist. Let f_{I} be $\mathrm{Th}(L_\alpha)$, so $\mathcal{M}_{\mathrm{I}} = L_\alpha$, and suppose τ replies with model $\mathcal{M}_{\mathrm{II}}$.

We claim $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}}) \leq \alpha$. For otherwise, we would have $\tau \in \mathcal{M}_{\mathrm{I}} \in \mathcal{M}_{\mathrm{II}}$, so that $\mathrm{Th}(\mathcal{M}_{\mathrm{II}}) = \tau * \mathrm{Th}(\mathcal{M}_{\mathrm{I}}) \in \mathcal{M}_{\mathrm{II}}$, a contradiction to the fact that $\mathcal{M}_{\mathrm{II}}$ projects to ω.

Suppose $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}}) = \alpha$. If $\mathcal{M}_{\mathrm{II}}$ is wellfounded, then $\mathcal{M}_{\mathrm{I}} = \mathcal{M}_{\mathrm{II}}$, so that (1) holds vacuously and (2) holds via (2b), a contradiction to τ being winning for II. So $\mathcal{M}_{\mathrm{II}}$ is illfounded. By overspill, there are countable codes for nonstandard ordinals in $\mathcal{M}_{\mathrm{II}}$, and there is no $<_L^{\mathcal{M}_{\mathrm{II}}}$ least such; since II wins the play, none of these codes can belong to $\mathcal{M}_{\mathrm{I}}$. But then (1) holds via (1a), and (2) holds vacuously, again a contradiction.

So we must have $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}}) < \alpha$. Again $\mathcal{M}_{\mathrm{II}}$ cannot be wellfounded, for then (1) holds vacuously and (2) holds via (2a). Since $\mathcal{M}_{\mathrm{II}}$ is illfounded, there is some $x \in \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}} \setminus \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}}$. Since (1a) fails, we can let x be the $<_L^{\mathcal{M}_{\mathrm{II}}}$-least such. We must have $L^{\mathcal{M}_{\mathrm{II}}}_{\mathrm{rank}_L(x)} \models$ "ω_1 exists", and by minimality of x, this ω_1 is standard and contained in $\mathcal{M}_{\mathrm{I}}$. It follows that $\delta(0,x)$ exists, and there is a unique k so that $\delta(k,x)$ is nonstandard and $\delta(k+1)$ is wellfounded. In particular, $R(k,\gamma,x)$ holds for any nonstandard $\gamma < \delta(k,x)$ by (B) of Lemma 5.6. And by (A) of the same Lemma, $R(k',\gamma',x)$ cannot hold for any k',γ' with $k' < k$. But then (1b) holds, contradicting that II wins the play. ⊣

It is easy to check by computations similar to those we have given that G has a Π^0_4 winning condition. Since G is non-determined when θ doesn't exist, this completes the proof of Theorem 5.1, modulo the proof of Lemma 5.6. ⊣

Proof of Lemma 5.6. Recall from Theorem 2.2(2) the Σ_1-Skolem function h_1^M, for ω-models M of $V{=}L$. If such M also satisfies "ω_1 exists"$\wedge(\forall\alpha)L_\alpha \not\models (\mathsf{T})$, then we have $h_1^M(\omega_1^M \cup \{\omega_1^M\}) = M$. To see this, suppose for a contradiction that $H = h_1(\omega_1^M \cup \{\omega_1^M\})$ is a proper subset of M. Failure of (T) in initial segments of M implies ω_1^M is the largest cardinal of M. Then in M, H must be a transitive set, since every set has cardinality at most ω_1^M and $\omega_1^M \subseteq H$. So $H = L_\alpha^M$ for some α an ordinal of M.

Let $T \in H$ be a tree. If T is ranked in M, then the same holds in H since $H \prec_1 M$. Otherwise $\{s \in T \mid T_s \text{ is not ranked in } H\}$ is a subtree of T with no terminal nodes, and this belongs to M by Δ_0-Comprehension. It follows that there is a branch through T in M, so such must belong to H, again by Σ_1-elementarity. But then $H = L_\alpha^M$ is a model of (T), contradicting our assumption on M.

Thus in the models we work with, we can talk about uncountable objects by taking images of countable ordinals by h_1.

We define the Σ_3^0 relation $R(k, \gamma, x)$ to be the conjunction of the following:

1. $\mathcal{M}_{\mathrm{II}} \models$ "$\delta(0, x)$ exists and $(\delta(k+1, x) < \gamma < \delta(k, x))$";
2. $(\exists \beta \in \mathrm{ON}^{\mathcal{M}_{\mathrm{I}}})$
   (a) $(J_\beta \models \mathsf{KP} +$ "ω_1 exists"$)^{\mathcal{M}_{\mathrm{I}}}$
   (b) $(\forall z \in \mathcal{P}(\omega) \cap \mathcal{M}_{\mathrm{I}} \cap \mathcal{M}_{\mathrm{II}})(z \in J_\gamma)^{\mathcal{M}_{\mathrm{II}}} \to (z \in J_\beta)^{\mathcal{M}_{\mathrm{I}}}$
   (c) $(\forall z \in \mathcal{P}(\omega) \cap \mathcal{M}_{\mathrm{I}} \cap \mathcal{M}_{\mathrm{II}})$
   If $\mathcal{M}_{\mathrm{I}} \models$ "z codes $\vec{\xi}, \vec{\eta} \in (\omega_1^{J_\beta})^{<\omega}$ such that $h_1^{J_\beta}(\vec{\xi}, \omega_1^{J_\beta}), h_1^{J_\beta}(\vec{\eta}, \omega_1^{J_\beta})$ exist", then
   $\mathcal{M}_{\mathrm{II}} \models$ "z codes $\vec{\xi}', \vec{\eta}' \in (\omega_1^{J_\gamma})^{<\omega}$ such that $h_1^{J_\gamma}(\vec{\xi}', \omega_1^{J_\gamma}), h_1^{J_\gamma}(\vec{\eta}', \omega_1^{J_\gamma})$ exist", and
   $(h_1^{J_\beta}(\vec{\xi}, \omega_1^{J_\beta}) \in h_1^{J_\beta}(\vec{\eta}, \omega_1^{J_\beta}))^{\mathcal{M}_{\mathrm{I}}}$ iff $(h_1^{J_\gamma}(\vec{\xi}', \omega_1^{J_\gamma}) \in h_1^{J_\gamma}(\vec{\eta}', \omega_1^{J_\gamma}))^{\mathcal{M}_{\mathrm{II}}}$;
   (d) $(\forall z \in \mathcal{P}(\omega) \cap \mathcal{M}_{\mathrm{I}} \cap \mathcal{M}_{\mathrm{II}})$
   If $\mathcal{M}_{\mathrm{I}} \models$ "z codes $\vec{\xi}, \vec{\eta} \in \omega_1^{J_\beta}$ such that $h_1^{J_\beta}(\vec{\eta}, \omega_1^{J_\beta})$ exists"
   and $\mathcal{M}_{\mathrm{II}} \models$ "z codes $\vec{\xi}', \vec{\eta}' \in \omega_1^{J_\gamma}$ such that $h_1^{J_{\rho+\omega}}(\vec{\xi}', \omega_1^{J_\gamma})$ exists, where $\rho = \max\{\omega_1^{J_\gamma}, \mathrm{rank}_L(h_1^{J_\gamma}(\vec{\eta}', \omega_1^{J_\gamma}))\}$", then $\mathcal{M}_{\mathrm{I}} \models$ "$h_1^{J_\beta}(\vec{\xi}, \omega_1^{J_\beta})$ exists";
   (e) $(\exists t \in \mathcal{P}(\omega) \cap \mathcal{M}_{\mathrm{I}} \cap \mathcal{M}_{\mathrm{II}})$
      (i) $\mathcal{M}_{\mathrm{I}} \models$ "t codes $\vec{\tau} \in (\omega_1^{J_\beta})^{<\omega}$ with $h_1^{J_\beta}(\vec{\tau}, \omega_1^{J_\beta})$ a tree on $P(\omega)^{J_\beta}$, that is neither ranked nor illfounded in J_β";
      (ii) $\mathcal{M}_{\mathrm{II}} \models$ "t codes $\vec{\tau}' \in (\omega_1^{J_\gamma})^{<\omega}$ with $h_1^{J_\gamma}(\vec{\tau}', \omega_1^{J_\gamma})$ a tree on $\mathcal{P}(\omega)^{J_\gamma}$ that witnesses the defining property of $\delta(k+1, x)$";
      (iii) $(\forall s \in \mathcal{P}(\omega)^{<\omega} \cap \mathcal{M}_{\mathrm{I}} \cap \mathcal{M}_{\mathrm{II}})$
      $(s \in h_1^{J_\beta}(\vec{\tau}, \omega_1^{J_\beta}))^{\mathcal{M}_{\mathrm{I}}} \leftrightarrow (s \in h_1^{J_\gamma}(\vec{\tau}', \omega_1^{J_\gamma}))^{\mathcal{M}_{\mathrm{II}}}$.

Before proceeding with the proof, we feel obligated to provide some discussion motivating our definition of the relation $R(k, \gamma, x)$. Suppose, as in the assumptions of the lemma, that x is $<_L^{\mathcal{M}_{\mathrm{II}}}$-least in $\mathcal{P}(\omega) \cap (\mathcal{M}_{\mathrm{II}} \setminus \mathcal{M}_{\mathrm{I}})$, and that $\mathcal{M}_{\mathrm{I}}$ is wellfounded. Note that then $J_{\mathrm{wfo}(\mathcal{M}_{\mathrm{II}})}$ is a model of $\mathsf{KP} +$ "ω_1 exists".

To define a relation $R_0(\gamma, x)$ that holds precisely of those γ that are nonstandard ordinals of $\mathcal{M}_{\mathrm{II}}$, a natural first attempt is to state that $R_0(\gamma, x)$ holds if $\gamma < \mathrm{rank}_L(x)^{\mathcal{M}_{\mathrm{II}}}$, and there is an ordinal β of $\mathcal{M}_{\mathrm{I}}$ so that $J_\beta^{\mathcal{M}_{\mathrm{I}}}$ and $J_{\mathrm{rank}_L(x)}^{\mathcal{M}_{\mathrm{II}}}$ have the same reals, $J_\beta^{\mathcal{M}_{\mathrm{I}}}$ is a model of $\mathsf{KP} +$ "ω_1 exists", and $J_\beta^{\mathcal{M}_{\mathrm{I}}}$ is isomorphic to a proper initial segment of $J_\gamma^{\mathcal{M}_{\mathrm{II}}}$ (the intent being that $\beta = \mathrm{wfo}(\mathcal{M}_{\mathrm{II}})$). However, the statement that $\mathcal{P}(\omega) \cap J_\beta^{\mathcal{M}_{\mathrm{I}}} = \mathcal{P}(\omega) \cap J_{\mathrm{rank}_L(x)}^{\mathcal{M}_{\mathrm{II}}}$ is Π_3^0, and so $R_0(\gamma, x)$ would be at least Σ_4^0, too complicated for our purposes.

We therefore compromise: Let X be the set of ordinals of $\mathcal{M}_{\mathrm{I}}$ coded by reals in $\mathcal{M}_{\mathrm{I}} \cap \mathcal{M}_{\mathrm{II}}$. Instead of asserting that all of $J_\beta^{\mathcal{M}_{\mathrm{I}}}$ embeds into $J_\gamma^{\mathcal{M}_{\mathrm{II}}}$, we only require that the Σ_1-hull of X in $J_\beta^{\mathcal{M}_{\mathrm{I}}}$ so embeds, thus saving ourselves a quantifier.

$R(k,\gamma,x)$ then holds just in case there is an admissible level J_β of $\mathcal{M}_{\mathrm{I}}$ satisfying "ω_1 exists" (2a) so that any real in $\mathcal{M}_{\mathrm{I}} \cap J_\gamma^{\mathcal{M}_{\mathrm{II}}}$ belongs to J_β (2b), and such that the Σ_1-hull of $X \cup \{\omega_1^{J_\beta}\}$ is $\in$-embeddable (2c) onto an initial segment of $J_\gamma^{\mathcal{M}_{\mathrm{II}}}$ (2d). (Note that by (2b), minimality of x ensures $X \subseteq \omega_1^{J_\beta}$.)

When the witness β is equal to $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}})$, we have that X is all of $\omega_1^{J_\beta}$, and so γ is illfounded; but it can happen that β is strictly larger than $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}})$, in which case the hull of X in J_β could collapse to a smaller ordinal than $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}})$, so that γ may be standard. This is where the trees come in: if γ is truly nonstandard, then there will be an illfounded tree $T \in J_\beta^{\mathcal{M}_{\mathrm{I}}} \cap J_\gamma^{\mathcal{M}_{\mathrm{II}}}$ which $J_\gamma^{\mathcal{M}_{\mathrm{II}}}$ believes is wellfounded, but which is neither ranked nor illfounded in J_β. By admissibility of J_β, then, there is a branch through T in $J_{\beta+\omega} \subseteq \mathcal{M}_{\mathrm{I}}$. The idea is to use this tree T to help us identify the ordinal $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}})$. The existence of T is asserted by parts (i) and (ii) of condition (2e).

However, we again do not have enough quantifiers to enforce full equality trees $T \in \mathcal{M}_{\mathrm{I}}$ and $T' \in \mathcal{M}_{\mathrm{II}}$, and can only insist that they agree on $J_\beta \cap J_\gamma$ as in part (iii) of (2e). Again minimality of x ensures $T' \subseteq T$, but possibly $T \not\subseteq T'$, and the branch through T in $\mathcal{M}_{\mathrm{I}}$ may not, in fact, be a branch through T'. This is the point of defining the ordinals $\delta(k,x)$: If $R(k,\gamma,x)$ and the assumptions of the lemma hold, then $J_{\delta(k+1,x)+\omega}$ is contained in the image of J_β under the embedding of part (c); and if k is minimal so that $R(k,\gamma,x)$ holds, we must have $\delta(k,x)$ nonstandard. Under our assumptions on $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$ and x, the only $\beta \in \mathrm{ON}^{\mathcal{M}_{\mathrm{I}}}$ that could witness $R(k,\gamma,x)$ for this k is $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}})$, since a larger ordinal would already contain a branch through (any tree in $\mathcal{M}_{\mathrm{I}}$ extending) T', where T' witnesses the defining property of $\delta(k+1,x)$ in $\mathcal{M}_{\mathrm{II}}$. Since J_β embeds onto an initial segment of $J_\gamma^{\mathcal{M}_{\mathrm{II}}}$, we have that γ is nonstandard.

Let us now note that the relation $R(k,\gamma,x)$ is Σ^0_3. The main thing is, we repeatedly used expressions of the form

$$(\forall z \in \mathcal{P}(\omega) \cap \mathcal{M}_{\mathrm{I}} \cap \mathcal{M}_{\mathrm{II}})(\text{Boolean comb. of statements internal to } \mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}});$$

c.f. (2b,c,d) and (iii) in (2e). These should be regarded as abbreviations for

$$(\forall z \in \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}})(\forall z' \in \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}})(z' = z \to (\text{Boolean comb...}))$$

which is clearly Π^0_2 (recall "$z' = z$" is Π^0_1 and internal statements are recursive in the codes). Re-envisioning the statement of $R(k,\gamma,x)$ appropriately, it is now easy to check that it is Σ^0_3.

We now prove that $R(k,\gamma,x)$ is as desired. So let $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$ and $x \in \mathcal{M}_{\mathrm{II}}$ satisfy the hypotheses of Lemma 5.6, namely, that $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$ are ω-models projecting to ω and satisfying KP, $V = L$, "all sets are countable" and "θ does not exist"; that $\mathcal{M}_{\mathrm{I}}$ is wellfounded, and that x is minimal in $\mathcal{P}(\omega) \cap (\mathcal{M}_{\mathrm{II}} \setminus \mathcal{M}_{\mathrm{I}})$. To prove (A), suppose $k \in \omega$ and $\gamma \in \mathrm{ON}^{\mathcal{M}_{\mathrm{II}}}$ are such that $R(k,\gamma,x)$ holds. We need to show $\delta(k+1,x)$ is wellfounded.

Let $\beta \in \mathrm{ON}^{\mathcal{M}_{\mathrm{I}}}$ witness (2) in the definition of R, with $t \in \mathcal{P}(\omega) \cap J_\beta^{\mathcal{M}_{\mathrm{I}}} \cap J_\gamma^{\mathcal{M}_{\mathrm{II}}}$ a witness to (2e). Then let $T \in J_\beta^{\mathcal{M}_{\mathrm{I}}}$, $T' \in J_\gamma^{\mathcal{M}_{\mathrm{II}}}$ be the trees whose existence is asserted in clauses (i),(ii) of (2e). There is a real y computable from t so that y codes tuples $\vec\eta$ in $\mathcal{M}_{\mathrm{I}}$ and $\vec\eta' \in \mathcal{M}_{\mathrm{II}}$ with $h_1^{J_\beta}(\vec\eta, \omega_1^{J_\beta}) = \mathrm{rank}_L(T)$ in $\mathcal{M}_{\mathrm{I}}$, and $h_1^{J_\gamma}(\vec\eta', \omega_1^{J_\gamma}) = \mathrm{rank}_L(T') = \delta(k+1,x)$ in $\mathcal{M}_{\mathrm{II}}$.

Suppose towards a contradiction that $\mathcal{M}_{\rm II}$ is illfounded below $\delta(k+1,x)$. Fix a sequence $\langle \alpha_n \mid n\in\omega\rangle$ of ordinals α_n of $\mathcal{M}_{\rm II}$ so that $\alpha_0=\delta(k+1,x)$ and $(\alpha_{n+1}\in\alpha_n)^{\mathcal{M}_{\rm II}}$ for all n. Inductively, fix tuples $\vec{\xi}'_n\in\omega_1^{J_\gamma}$ in $\mathcal{M}_{\rm II}$, as follows: $\vec{\xi}'_0=\vec{\eta}'$. If $\vec{\xi}'_n$ is fixed so that $h_1^{J_\gamma}(\vec{\xi}'_n,\omega_1^{J_\gamma})=\alpha_n$ in $\mathcal{M}_{\rm II}$, let $\rho_n=\max\{\omega_1^{J_\gamma},\alpha_n\}$ and fix $\vec{\xi}'_{n+1}$ so that $h_1^{J_{\rho_n+\omega}}(\vec{\xi}'_{n+1},\omega_1^{J_\gamma})=\alpha_{n+1}$. Such $\vec{\xi}'_{n+1}$ is guaranteed to exist by the fact that $J^{\mathcal{M}_{\rm II}}_{\rho_n+\omega}$ satisfies "ω_1 exists" $\wedge(\forall\alpha)L_\alpha\not\models(\mathsf{T})$.

Now each $\vec{\xi}'_n$ is coded by some real $y_n\in J^{\mathcal{M}_{\rm II}}_\gamma$, and $\mathcal{P}(\omega)^{J^{\mathcal{M}_{\rm II}}_\gamma}\subset\mathcal{M}_{\rm I}$ by the minimality assumption on x. So $y_0\in\mathcal{M}_{\rm I}$, and we have in $\mathcal{M}_{\rm I}$ that y_0 codes $\vec{\xi}_0\in J_\beta$ so that $h_1^{J_\beta}(\vec{\xi}_0,\omega_1^{J_\beta})$ exists. By inductively applying condition (2d), we can pull back the tuples $\vec{\xi}'_n$ of $\mathcal{M}_{\rm II}$ to tuples $\vec{\xi}_n$ of $\mathcal{M}_{\rm I}$ so that for all n, $h_1^{J_\beta}(\vec{\xi}_n,\omega_1^{J_\beta})$ exists. But then by (2c), $\langle h_1^{J_\beta}(\vec{\xi}_n,\omega_1^{J_\beta})\mid n\in\omega\rangle$ is an infinite $\in^{\mathcal{M}_{\rm I}}$-descending sequence. This contradicts wellfoundedness of $\mathcal{M}_{\rm I}$.

Now let us prove (B) of the Lemma. For the rest of the proof, we let $\mathcal{M}_{\rm I},\mathcal{M}_{\rm II},x$ be as above, and suppose further that k is (unique) such that $\delta(k,x)$ is nonstandard and $\delta(k+1,x)$ is wellfounded. Let $\gamma\in\mathrm{ON}^{\mathcal{M}_{\rm II}}$.

Suppose first that $(\gamma<\delta(k,x))^{\mathcal{M}_{\rm II}}$ and that γ is nonstandard. Clearly (1) in the definition of R holds. Let $\beta=\mathrm{wfo}(\mathcal{M}_{\rm II})$. Then $\delta(k+1,x)<\beta\subset\delta(k,x)$, so $J_\beta\models\mathsf{KP}+$ "ω_1 exists." Our minimality assumption on x ensures $\mathcal{P}(\omega)^{J^{\mathcal{M}_{\rm II}}_\gamma}\subset\mathcal{M}_{\rm I}$, so that in particular, the ω_1 of $J^{\mathcal{M}_{\rm II}}_\gamma$ is a subset of the ordinals of $\mathcal{M}_{\rm I}$. Indeed, it must be a proper subset, as $\mathcal{M}_{\rm I}$ projects to ω. It follows that $\beta\in\mathcal{M}_{\rm I}$, so is a witness to (2a); and the fact that $\mathcal{P}(\omega)^{J_\beta}=\mathcal{P}(\omega)^{J^{\mathcal{M}_{\rm II}}_\gamma}$ implies (2b).

Now, J_β is an initial segment of $\mathcal{M}_{\rm II}$, with $\beta\subseteq\gamma$, and $\omega_1^{J_\beta}=(\omega_1^{J_\gamma})^{\mathcal{M}_{\rm II}}$. It follows that the map

$$h_1^{J_\beta}(\vec{\xi},\omega_1^{J_\beta})\mapsto(h_1^{J_\gamma}(\vec{\xi},\omega_1^{J_\gamma}))^{\mathcal{M}_{\rm II}}$$

is $\in$-preserving (so (2c) holds) and is onto the initial segment of $\mathcal{M}_{\rm II}$ corresponding to J_β, by upward absoluteness of the Σ_1-Skolem function h_1 (so (2d) holds).

Finally, by definition there is some tree $T\in J^{\mathcal{M}_{\rm II}}_{\delta(k+1,x)+\omega}$ that is neither ranked nor illfounded in $(J_{\delta(k,x)})^{\mathcal{M}_{\rm II}}$. Since $\delta(k+1,x)$ is a true ordinal and $\delta(k+1,x)<\beta$, we have $T\in J_\beta$ and T is neither ranked nor illfounded in J_β. If we let $t\in J_\beta$ be any real coding $\vec{\tau}$ so that $h_1^{J_\beta}(\vec{\tau},\omega_1^{J_\beta})=T$, then t is a witness to (2e). Thus $R(k,\gamma,x)$ is satisfied as needed.

Conversely, suppose γ is such that $R(k,\gamma,x)$ holds. Let this be witnessed by $\beta\in\mathcal{M}_{\rm I}$ and $t\in\mathcal{P}(\omega)\cap\mathcal{M}_{\rm I}\cap\mathcal{M}_{\rm II}$. We immediately have $\gamma<\delta(k,x)$, by (1); all that's left is to show γ is nonstandard.

First consider the case that $\omega_1^{J_\beta}=\omega_1^{J^{\mathcal{M}_{\rm II}}_\gamma}$. Then $P(\omega)^{J_\beta}\subseteq\mathcal{M}_{\rm II}$, so by (2c,d), J_β is isomorphic to an initial segment of J_γ. By (2a), J_β is admissible. If we had $\beta\in\mathrm{ON}^{\mathcal{M}_{\rm II}}$, then by failure of (T) in $J^{\mathcal{M}_{\rm II}}_\beta$, we must have that $\omega_1^{J_\beta}$ is countable in $J^{\mathcal{M}_{\rm II}}_{\beta+\omega}$. But $\beta\le\gamma<\delta(0,x)$, while $\omega_1^{J_\beta}=\omega_1^{J_\gamma}=\omega_1^{J_{\delta(0,x)}}$ (the latter computed in J_γ), a contradiction. So $\beta\notin\mathrm{ON}^{\mathcal{M}_{\rm II}}$, even though $\beta\subseteq\gamma$. It follows that γ is nonstandard.

Now suppose $\omega_1^{J_\gamma^{\mathcal{M}_{\mathrm{II}}}} < \omega_1^{J_\beta}$ (the reverse inequality is impossible by our minimality assumption on x). Let $\alpha = \mathrm{wfo}(\mathcal{M}_{\mathrm{II}})$ and let $T \in \mathcal{M}_{\mathrm{I}}$, $T' \in \mathcal{M}_{\mathrm{II}}$ be the trees witnessing (i) and (ii), respectively, in (2e). T' is a tree on $\mathcal{P}(\omega)^{J_\gamma} \subseteq \mathcal{M}_{\mathrm{I}}$, so by (iii) of (2e), we have $T \cap (\mathcal{P}(\omega)^{J_\gamma})^{<\omega} = T'$. Now, since $J_{\delta(k,x)}^{\mathcal{M}_{\mathrm{II}}} \models$ "T' is neither ranked nor illfounded" and $\alpha \subseteq \delta(k,x)$, we must have that $T' \in J_\alpha$ and $J_\alpha \models$ "T' is neither ranked nor illfounded" (since being either ranked or illfounded is Σ_1 and would reflect from J_α to $J_{\delta(k,x)}^{\mathcal{M}_{\mathrm{II}}}$). But J_α is admissible, so there is a branch through T', hence through T, definable over J_α. Since $\alpha < \omega_1^{J_\beta}$, we must have that T is illfounded in J_β. But this contradicts (i) of (2e). This contradiction completes the proof of Lemma 5.6. ⊣

Although we have only referred to the lightface Σ_4^0 sets, it is easy to see that our proofs and results relativize to $\Sigma_4^0(x)$ with real parameters x. Letting $\theta(x)$ be the least ordinal α so that $L_\alpha[x] \models (\mathsf{T})$, we have

Theorem 5.9 (KPI_0). *For all reals x, $\Sigma_4^0(x)$-DET if and only if $\theta(x)$ exists.*

Since (boldface) $\boldsymbol{\Sigma}_1^0 \wedge \boldsymbol{\Pi}_1^0$-DET implies closure under the next admissible, the boldface result goes through in the weaker theory KP:

Theorem 5.10 (KP). *$\boldsymbol{\Sigma}_4^0$-DET if and only if $\theta(x)$ exists for every $x \subseteq \omega$.*

Note also that working in KP + "$\mathcal{P}(\omega)$ exists" + Σ_1-Comprehension, we have that every real is contained in a countable transitive model of Π_1-RAP; hence in this theory we have L_{ω_1} is a model of boldface $\boldsymbol{\Sigma}_4^0$ determinacy. This proves the implications between (1), (2), (3), and (4) of Theorem 1.2; together with the remarks at the end of Section 2, this completes the proof of that theorem.

§6. Generalizing to $\Sigma_{\alpha+3}^0$-DET, for $\alpha > 1$. The generalization of the results about Σ_4^0 from the last two sections to all pointclasses of the form $\Sigma_{\alpha+3}^0$ is obtained in a manner similar to that in the inductive proof of Theorem 1.1 (see [11]). The most significant modification to those arguments involves the identification of the correct higher analogues of Π_1-RAP and (T).

Definition 6.1 (Π_1-RAP_α). Let $\alpha < \omega_1^{\mathrm{CK}}$. Π_1-RAP_α denotes the theory consisting of "$\mathcal{P}^\alpha(\omega)$ exists" together with the axioms of the schema Π_1-$\mathsf{RAP}(\mathcal{P}^\alpha(\omega))$.

In particular, Π_1-RAP_α entails the existence of $\mathcal{P}^{\alpha+1}(\omega)$, and any Π_1 statement in parameters from $\mathcal{P}^{\alpha+2}(\omega)$ can be reflected to an admissible set M with $\mathcal{P}^\alpha(\omega) \subset M$. Note that Π_1-RAP is the same as Π_1-RAP_0. The following is the general form of Theorem 4.7.

Theorem 6.2. *Suppose M is a transitive model of "$H(|\mathcal{P}^\alpha(\omega)|^+)$ exists" + Π_1-RAP_α, and that M has a wellordering of $\mathcal{P}^{\alpha+1}(\omega)$. Let $A \subseteq \omega^\omega$ be $\Sigma_{1+\alpha+3}^0$. Then either I wins $G(A;\omega^{<\omega})$ with a strategy in M, or II has a winning strategy in $G(A;\omega^{<\omega})$ that is $\boldsymbol{\Delta}_1$-definable over M.*

Proof. As Martin [10] shows, the unraveling of closed sets can be iterated into the transfinite, taking inverse limits of the unraveling trees at limit stages. Precisely, assuming in M $\mathcal{P}^{\alpha+1}(\omega)$ exists and can be wellordered, there is an M-covering $\langle T, \pi, \psi\rangle$ that simultaneously unravels all $\Pi_{1+\alpha}^0$ sets, and so that T

is a tree on $\mathcal{P}^{\alpha+1}(\omega), \mathcal{P}^{\alpha}(\omega)$, and this cover is definable over $H(|\mathcal{P}^{\alpha}(\omega)|^{+})$ in M (c.f. Definition 4.1 and Theorem 4.3).

So, work in M as in the hypothesis of the Theorem. Let A be a $\Sigma^0_{1+\alpha+3}$ subset of ω^ω; and let $\langle T, \pi, \psi\rangle$ be the simultaneous unraveling of all $\Pi^0_{1+\alpha}$ sets. Then $\bar{A} = \pi^{-1}(A)$ is a Σ^0_3 subset of $[T]$. Applying Lemma 4.4 with $X = \mathcal{P}^{\alpha}(\omega)$, we have that for any position $p \in T$, either I wins $G(\bar{A}; T)$ or, for any Π^0_2 set $B \subseteq \bar{A} \subseteq [T]$, p is good for II relative to $\bar{A}, B, T$.

The remainder of the proof then is exactly like that of Theorem 4.7. If I doesn't win $G(\bar{A}; T)$, then we can take W for II to be the common refinement at stage k of goodness-witnessing quasistrategies relative to the various B_k (where $\bar{A} = \bigcup_{k\in\omega} B_k$). The desired strategy is $\psi(\tau)$, where τ is any strategy for II refining W; as before, M is sufficiently closed under local definitions involving τ to ensure $\psi(\tau)$ is truly winning for I in $G(A; T)$. ⊣

For the lower bound argument, it will again be helpful to have a natural principle involving trees on $\mathcal{P}^{\alpha+1}(\omega)$ which is equivalent in models of $V = L$ to Π_1-RAP_α. Consider a game tree T. The Gale-Stewart theorem applied to the game $G([T]; T)$ tells us that either I has a strategy in T, or the game tree T is *ranked for Player II*, in the sense that there is a partial map $\rho : T \rightharpoonup \mathrm{ON}$ so that for every s of even length in the domain of ρ and every a with $s^\frown\langle a\rangle \in T$, there is some b so that $\rho(s^\frown\langle a, b\rangle) < \rho(s)$. We let $(\mathsf{T})_\alpha$ denote the following special case of this fact, for $\alpha < \omega_1^{\mathrm{CK}}$.

Definition 6.3 ($(\mathsf{T})_\alpha$). Suppose T is a tree on $\mathcal{P}^{\alpha+1}(\omega), \mathcal{P}^{\alpha}(\omega)$. Then either

- Player I has a strategy in T, or
- The game tree T is ranked for Player II.

Although $(\mathsf{T})_0$ clearly implies (T), and it follows from what we've shown that in L, (T) implies Π_1-RAP and hence $(\mathsf{T})_0$, it's less clear that $(\mathsf{T})_0$ and (T) are equivalent in general (under, say, the weak base theory BST); we conjecture they are but have not been able to show it.

Lemma 6.4. *Suppose $V = L$ and that $\mathcal{P}^{\alpha+1}(\omega)$ exists. Then Π_1-RAP_α holds if and only if $(\mathsf{T})_\alpha$ holds.*

Proof. Assume Π_1-RAP_α. Given a tree T on $\mathcal{P}^{\alpha+1}(\omega), \mathcal{P}^{\alpha}(\omega)$, suppose T is not ranked for II. Then this can be reflected to an admissible set M; in an admissible structure, every closed game is either won by Player II (the open Player), or there is a definable winning strategy for Player I (the closed Player). Since if II won $G([T]; T)$ in M this would easily furnish a rank function ρ for II, we must have a strategy for I that is definable over M, hence belongs to V, and is winning for I in the restriction of the game tree to M, hence (since $\mathcal{P}^{\alpha}(\omega) \subset M$) winning for I in V.

Conversely, suppose $(\mathsf{T})_\alpha$ holds; clearly, using the uniform definable bijection in L of $\mathcal{P}^{\alpha}(\omega)$ with ω_α, it is sufficient to show the version of Π_1-RAP_α holds involving parameters $Q \subseteq \mathcal{P}(\omega_\alpha)$. So let $\phi(Q)$ be Π_1 and true in V, with $Q \subseteq \mathcal{P}(\omega_\alpha)$. Let $\tau > \omega_{\alpha+1}$ be large enough that $Q \in J_\tau$. Consider a game tree T defined as follows: Player II plays ordinals $\xi_n < \omega_\alpha$. The moves of Player I are fragments $\langle f, g\rangle$ much like the nodes of the tree T used in the proof of Theorem 3.5; f is the characteristic function of a consistent theory in the language of set theory plus constant symbols t, q and a_n, c_n, d_n for $n \in \omega$; g assigns elements of $\tau \cup \mathcal{P}(\omega_\alpha)$ to certain of the constants a_n.

The theory played by I is subject to the following rules: it must extend $V = L$ + "$\mathcal{P}^{\alpha+1}(\omega)$ exists", and assert that q is a subset of $\mathcal{P}(\omega_\alpha)$ belonging to its J_t (t an ordinal), and $\phi(q)$ must hold; the a_n must act as Henkin constants for statements asserting the existence of elements of $t \cup \mathcal{P}(\omega_\alpha)$; the $\in$-ordering of the constants c_n must agree with that of the ordinals ξ_n played by II; and $d_{n+1} < d_n$ for all n. Moreover, the assignment of the Henkin constants a_n must respect the order on t as asserted by the theory, as well as membership of the ξ_n in subsets of ω_α (so that $f(\#(c_i \in a_j)) = 1$ iff $\xi_i \in g(j)$) and of the elements of $\mathcal{P}(\omega_\alpha)$ in Q (so that $f(\#(a_j \in q)) = 1$ iff $g(j) \in Q$).

Since GCH holds, the tree T is evidently equivalent to one on $\mathcal{P}^{\alpha+1}(\omega), \mathcal{P}^{\alpha}(\omega)$. We claim Player I has a strategy in T. Otherwise, by $(\mathsf{T})_\alpha$, T is ranked for Player II. Let ρ be the rank function. Consider a play of the game where I plays the theory of $L_{\rho(\emptyset)+1}$; I interprets q by Q, the constants a_n are interpreted by witnesses to the appropriate existential statements, the c_n are interpreted as the ξ_n played by II, and the d_n are interpreted by ordinals furnished by the rank function (when the time comes to interpret the constant d_n, we must be at a position p of length at least $2n$, so interpret d_n by $\rho(p \restriction 2n)$).

We have described how to obtain an infinite play for I; but this gives an infinite descending sequence of ordinals, a contradiction.

So Player I has a strategy σ in T; then $\sigma : \mathcal{P}^{\alpha}(\omega)^{<\omega} \rightharpoonup \mathcal{P}^{\alpha+1}(\omega)$ is an element of $H(|\mathcal{P}^{\alpha+1}(\omega)|) = L_{\omega_{\alpha+1}}$. Let G be a $L_{\omega_{\alpha+1}}$-generic filter for the poset $\mathrm{Col}(\omega, \omega_\alpha)$ to collapse ω_α to ω. (Note this makes sense since $L_{\omega_{\alpha+1}} \models \mathsf{ZFC}^-$.) Have II play against I's strategy σ with G, so that II plays an enumeration of ω_α in order-type ω.

Now in $L_{\omega_{\alpha+1}}[G]$, $\sigma * G$ yields a complete theory of an illfounded model $\mathcal{M}$ of $V = L$ + "$\mathcal{P}^{\alpha+1}(\omega)$ exists" + $\phi(\bar{Q})$; by the rules of the game, $\mathcal{M}$ is wellfounded up to $\tau > \omega^M_{\alpha+1}$; and since II plays all ordinals below ω_α, we have $\omega^{\mathcal{M}}_\alpha = \omega^V_\alpha$, so that $\mathcal{P}^\alpha(\omega) \subseteq \mathrm{wfp}(\mathcal{M})$. By Proposition 2.4, $L_{\mathrm{wfo}(\mathcal{M})}$ is admissible, and satisfies $\phi(\bar{Q})$, thus witnessing the desired instance of Π_1-RAP_α. $\dashv$

By a similar argument to that given in Proposition 3.6, we can use the equivalence of Π_1-RAP_α with $(\mathsf{T})_\alpha$ to show that for transitive models M, Π_1-RAP_α reflects from M to L^M; thus we can eliminate the need for the Axiom of Choice in Theorem 6.2:

Theorem 6.5. *For all $\alpha < \omega_1^{\mathrm{CK}}$, if there is a transitive model of Π_1-RAP_α, then $\Sigma^0_{\alpha+3}$-DET holds.*

Let θ_α be least so that $L_{\theta_\alpha} \models$ "$\mathcal{P}^{\alpha+1}(\omega)$ exists" + $(\mathsf{T})_\alpha$.

Theorem 6.6 (KPI_0). *$\Sigma^0_{1+\alpha+3}$-DET implies θ_α exists.*

Proof. As before, we define a Friedman-style game G with a $\Pi^0_{1+\alpha+3}$ winning condition. Player I and II play reals $f_{\mathrm{I}}, f_{\mathrm{II}}$ coding the characteristic functions of complete, consistent theories determining $(\mathsf{T})_\alpha$-small ω-models $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$, respectively. If this rule is broken, the winner is decided appropriately.

We need a lemma concerning the complexity of comparing elements of $\mathcal{M}_{\mathrm{I}}$ to those of $\mathcal{M}_{\mathrm{II}}$. Essentially, it states that increasing the type of the elements by 1 increases the Borel rank of the equality relation by 1. The main complication is that sensibly comparing elements of $\mathcal{P}^{\beta+1}(\omega)$ requires equality of $\mathcal{P}^{\beta}(\omega)$ between the levels in $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$ where these elements are constructed.

LEMMA 6.7. *Let $\beta < \omega_1^{\mathrm{CK}}$. Let $\mu, \nu \in \mathrm{ON} \cup \{\mathrm{ON}\}$ of $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$, respectively. Then*

- *The relation "$\mathcal{P}^{\beta}(\omega)^{L_\mu^{\mathcal{M}_{\mathrm{I}}}} = \mathcal{P}^{\beta}(\omega)^{L_\nu^{\mathcal{M}_{\mathrm{II}}}}$" is $\Pi^0_{1+\beta+1}$;*
- *Suppose $x, y \in \mathcal{P}^{\beta+1}(\omega)$ of $L_\mu^{\mathcal{M}_{\mathrm{I}}}$ and $L_\nu^{\mathcal{M}_{\mathrm{II}}}$, respectively; and that the clause above holds. Then the relation "$x = y$" is $\Pi^0_{1+\beta}$.*

As usual, we mean that the relations in $f_{\mathrm{I}}, f_{\mathrm{II}}$ and the codes for μ, ν, x, y have the stated complexity.

PROOF. By induction on β. For $\beta = 0$, we regard the statement that "$\omega^{\mathcal{M}_{\mathrm{I}}} = \omega^{\mathcal{M}_{\mathrm{II}}}$" as asserting that both models have standard ω, which is Π^0_2; and we have already seen that if this is the case, then equality of reals x, y is Π^0_1 in the codes.

If $\beta = \gamma + 1$, then the relation "$\mathcal{P}^{\beta}(\omega)^{L_\mu^{\mathcal{M}_{\mathrm{I}}}} = \mathcal{P}^{\beta}(\omega)^{L_\nu^{\mathcal{M}_{\mathrm{II}}}}$" is captured by

$$\mathcal{P}^{\gamma}(\omega)^{L_\mu^{\mathcal{M}_{\mathrm{I}}}} = \mathcal{P}^{\gamma}(\omega)^{L_\nu^{\mathcal{M}_{\mathrm{II}}}},\ (\forall x \in \mathcal{P}^{\gamma+1}(\omega)^{L_\mu^{\mathcal{M}_{\mathrm{I}}}})(\exists y \in \mathcal{P}^{\gamma+1}(\omega)^{L_\nu^{\mathcal{M}_{\mathrm{II}}}})(x = y),$$
$$\text{and } (\forall x \in \mathcal{P}^{\gamma+1}(\omega)^{L_\nu^{\mathcal{M}_{\mathrm{II}}}})(\exists y \in \mathcal{P}^{\gamma+1}(\omega)^{L_\mu^{\mathcal{M}_{\mathrm{I}}}})(y = x).$$

By inductive hypothesis, the first clause is $\Pi^0_{1+\gamma+1}$, and "$x = y$" (and $y = x$) here has complexity $\Pi^0_{1+\gamma}$. So the whole expression is $\Pi^0_{1+\gamma+2}$, that is, $\Pi^0_{1+\beta+1}$.

For the second item, let $x, y \in \mathcal{P}^{\beta+1}(\omega)$ of $L_\mu^{\mathcal{M}_{\mathrm{I}}}, L_\nu^{\mathcal{M}_{\mathrm{II}}}$, respectively. Then $x = y$ iff

$$(\forall u \in \mathcal{P}^{\beta}(\mathcal{M}_{\mathrm{I}}))(\forall v \in \mathcal{P}^{\beta}(\mathcal{M}_{\mathrm{I}}))(u = v \to ((u \in x)^{\mathcal{M}_{\mathrm{I}}} \leftrightarrow (v \in y)^{\mathcal{M}_{\mathrm{II}}})).$$

By inductive hypothesis, "$u = v$" is $\Pi^0_{1+\gamma}$. So the displayed line is $\Pi^0_{1+\beta}$, as claimed.

The proof at limits is similar, and in fact, since equality of $\mathcal{P}^{\lambda}(\omega)$ between the models is equivalent for limit λ to equality of $\mathcal{P}^{\xi}(\omega)$ for all $\xi < \lambda$, both relations in this case turn out to be Π^0_λ. (Note the importance of the fact that λ is assumed to be recursive, and the relations above are uniform in the codes.) $\dashv$

We seek to describe the level of least disagreement of $\mathcal{M}_{\mathrm{II}}$ with $\mathcal{M}_{\mathrm{I}}$. Previously, this was witnessed by the least constructed real of $\mathcal{M}_{\mathrm{II}}$ not belonging to $\mathcal{M}_{\mathrm{I}}$; in the present situation, we look for sets witnessing least disagreement of type $\beta \leq \alpha$, in the following sense:

DEFINITION 6.8. Suppose $\beta < \omega_1^{\mathrm{CK}}$ and that $x \in \mathcal{P}^{\beta+1}(\omega)^{\mathcal{M}_{\mathrm{II}}}$. We say x *witnesses disagreement at* β if for some $\mu \in \mathrm{ON}^{\mathcal{M}_{\mathrm{I}}}$, $x \subseteq \mathcal{P}^{\beta}(\omega)^{L^{\mathcal{M}_{\mathrm{II}}}_{\mathrm{rank}_L(x)}} = \mathcal{P}^{\beta}(\omega)^{L_\mu^{\mathcal{M}_{\mathrm{I}}}}$ (in particular, both models believe $\mathcal{P}^{\beta}(\omega)$ exists), and for every $z \in \mathcal{P}^{\beta+1}(\omega)^{\mathcal{M}_{\mathrm{I}}}$ there is some u belonging to this common $\mathcal{P}^{\beta}(\omega)$ that is in the symmetric difference of x and z.

Arguing as in the proof of Lemma 6.7, the relation "x witnesses disagreement at β" is $\Sigma^0_{1+\beta+2}$ in the codes.

Just as before, we require a means of identifying the height of $\mathrm{wfp}(\mathcal{M}_{\mathrm{II}})$ in the event that $L_{\mathrm{wfo}(\mathcal{M}_{\mathrm{II}})}$ satisfies "$\omega_{\alpha+1}$ exists". The device is again a function that steps down incrementally from an ordinal to its $\omega_{\alpha+1}$, using failures of $(\mathsf{T})_\alpha$.

DEFINITION 6.9. Inside $\mathcal{M}_{\mathrm{II}}$, suppose $x \in \mathcal{P}^{\alpha+1}(\omega)$. Put

$$\delta_\alpha(0,x) = \begin{cases} \mathrm{rank}_L(x) & \text{if } J_{\mathrm{rank}_L(x)} \models \text{“}\omega_{\alpha+1} \text{ exists”;} \\ \text{undefined} & \text{otherwise;} \end{cases}$$

$$\delta_\alpha(k+1,x) = \begin{cases} \begin{array}{l} \delta \text{ least s.t.} J_{\delta_\alpha(k,x)} \models \text{“}\omega_{\alpha+1} \text{ exists and} \\ (\exists T \in J_{\delta+\omega}) T \text{ is a tree on } \mathcal{P}^{\alpha+1}(\omega), \mathcal{P}^{\alpha}(\omega) \\ \text{ witnessing failure of } (\mathsf{T})_\alpha\text{”} \end{array} & \text{if such exists;} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We have the following analogue of Lemma 5.6.

LEMMA 6.10. *There is a $\Sigma^0_{1+\alpha+2}$ relation $R_\alpha(k,\gamma,x)$ such that if $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$ are $(\mathsf{T})_\alpha$-small ω-models, $\mathcal{M}_{\mathrm{I}}$ is wellfounded, $L^{\mathcal{M}_{\mathrm{II}}}_{\mathrm{rank}_L(x)} \models$ “$\omega_{\alpha+1}$ exists”, and x is the $<_L^{\mathcal{M}_{\mathrm{II}}}$-least element of $\mathcal{P}^{\alpha+1}(\omega)$ witnessing disagreement at α, then:*

(A) $(\forall k \in \omega)(\forall \gamma \in \mathrm{ON}^{\mathcal{M}_{\mathrm{II}}}) R_\alpha(k,\gamma,x) \to \delta_\alpha(k+1,x)$ *is standard;*

(B) $(\forall k \in \omega)$ *if* $\delta_\alpha(k,x)$ *is nonstandard and* $\delta_\alpha(k+1,x)$ *is wellfounded, then* $(\forall \gamma \in \mathrm{ON}^{\mathcal{M}_{\mathrm{II}}})(R_\alpha(k,\gamma,x) \leftrightarrow (\gamma < \delta_\alpha(k,x))^{\mathcal{M}_{\mathrm{II}}} \wedge \gamma$ *is nonstandard).*

The definition of R_α and the proof of the Lemma closely resemble those in Lemma 5.6, so we omit them; note though that in addition to the obvious modifications, we require of any $\beta \in \mathrm{ON}^{\mathcal{M}_{\mathrm{I}}}$ witnessing $R_\alpha(k,\gamma,x)$ that $\mathcal{P}^\alpha(\omega)^{L^{\mathcal{M}_{\mathrm{I}}}_\beta} = \mathcal{P}^\alpha(\omega)^{L^{\mathcal{M}_{\mathrm{II}}}_{\mathrm{rank}_L(x)}}$ (which is $\Pi^0_{1+\alpha+1}$) so that comparing codes for elements of $\omega_{\alpha+1}$ makes sense. Observe now the engine making the lemma go is the fact that if T is a game tree in an admissible structure which does not have a ranking function for II, then there is a strategy for I (the closed player) defined over the admissible set. The role before played by the newly defined branch now belongs to this strategy.

We may now give the winning condition. Suppose a play $f_{\mathrm{I}}, f_{\mathrm{II}}$ with term models $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$, respectively, is such that no rules have so far been broken. I wins the game if there are $\beta \le \alpha$ and sets z, z' in $\mathcal{P}^{\beta+1}(\omega)$ of $L^{\mathcal{M}_{\mathrm{I}}}_{\mathrm{rank}_L(z)+1}, L^{\mathcal{M}_{\mathrm{II}}}_{\mathrm{rank}_L(z')+1}$, respectively, so that

- $\mathcal{P}^\beta(\omega)^{L^{\mathcal{M}_{\mathrm{I}}}_{\mathrm{rank}_L(z)}} = \mathcal{P}^\beta(\omega)^{L^{\mathcal{M}_{\mathrm{II}}}_{\mathrm{rank}_L(z')}}$,
- $z = z'$,
- z' codes an ordinal in $\mathcal{M}_{\mathrm{II}}$, but codes an illfounded linear order in $\mathcal{M}_{\mathrm{I}}$.

Call this condition $(*)$ (and notice $(*)$ is $\Sigma^0_{1+\beta+2}$). Otherwise, I wins just in case

1. $(\forall \beta \le \alpha)(\forall x \in \mathcal{P}^{\beta+1}(\omega)^{\mathcal{M}_{\mathrm{II}}})$ if x witnesses disagreement at β, then
   (a) $(\exists \beta' \le \alpha)(\exists y \in \mathcal{P}^{\beta'+1}(\omega)^{\mathcal{M}_{\mathrm{II}}})$
   y witnesses disagreement at β' and $(\mathrm{rank}_L(y) < \mathrm{rank}_L(x))^{\mathcal{M}_{\mathrm{II}}}$), or
   (b) $(\exists k,\gamma) R_\alpha(k,\gamma,x)$
   $\wedge (\forall k,\gamma)[R_\alpha(k,\gamma,x) \to (\exists k',\gamma') R_\alpha(k',\gamma',x) \wedge \langle k',\gamma'\rangle <_{\mathrm{Lex}} \langle k,\gamma\rangle]$,
   and
2. $\mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}} \subseteq \mathcal{M}_{\mathrm{I}}$ implies
   (a) $\mathrm{Th}(\mathcal{M}_{\mathrm{II}}) \in \mathcal{M}_{\mathrm{I}}$, or
   (b) $\mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}} \subseteq \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}}$.

That this game is $\Pi^0_{1+\alpha+3}$ is by now a routine computation. We claim I has no winning strategy if θ_α does not exist. For suppose σ is such; we can assume by absoluteness that $\sigma \in L$, so let μ be the least admissible ordinal with $\sigma \in L_\mu$, and let f_{II} be the theory of L_μ. Let $\mathcal{M}_{\mathrm{I}}$ be the model given by $f_{\mathrm{I}} = \sigma * f_{\mathrm{II}}$. If $\mathcal{M}_{\mathrm{I}}$ is wellfounded, then it has ordinal height strictly less than μ, since $\mathrm{Th}(\mathcal{M}_{\mathrm{I}}) \notin \mathcal{M}_{\mathrm{I}}$. But then $\mathrm{Th}(\mathcal{M}_{\mathrm{I}}) \in \mathcal{M}_{\mathrm{II}}$ is a witness to failure of (1) (with $\beta = 0$).

So $\mathcal{M}_{\mathrm{I}}$ must be illfounded, and $\mathrm{Th}(\mathcal{M}_{\mathrm{I}}) \notin \mathcal{M}_{\mathrm{I}}$ again implies $\mathrm{wfo}(\mathcal{M}_{\mathrm{I}}) \leq \mu$. It can't be the case that $\mathrm{wfo}(\mathcal{M}_{\mathrm{I}}) = \mu$, for then (2) fails (since $\mathcal{M}_{\mathrm{I}}$ can be computed by applying σ to $\mathrm{Th}(\mathcal{M}_{\mathrm{II}})$, so this latter real cannot belong to $\mathcal{M}_{\mathrm{I}}$). So $\mathrm{wfo}(\mathcal{M}_{\mathrm{I}}) < \mu$. By admissibility (and failure of $(\mathsf{T})_\alpha$), there is a largest cardinal in $L_{\mathrm{wfo}(\mathcal{M}_{\mathrm{I}})}$. So we must have $L_{\mathrm{wfo}(\mathcal{M}_{\mathrm{I}})} \models$ "ω_β is the largest cardinal", for some $\beta \leq \alpha + 1$. If $\beta = \alpha + 1$, then failure of $(\mathsf{T})_\alpha$ in $L_{\mathrm{wfo}(\mathcal{M}_{\mathrm{I}})}$ implies this model projects to its ω_α; so there is a subset of $\mathcal{P}^\alpha(\omega)$ in $L_{\mathrm{wfo}(\mathcal{M}_{\mathrm{I}})+1}$ that codes a wellorder isomorphic to $\mathrm{wfo}(\mathcal{M}_{\mathrm{I}})$, and this must be rank_L-minimal witnessing disagreement at α; but then (1) fails. Similarly, if $\beta < \alpha + 1$, then there is some least level above $\mathrm{wfo}(\mathcal{M}_{\mathrm{I}})$ projecting to ω_β of $L_{\mathrm{wfo}(\mathcal{M}_{\mathrm{I}})}$, and an element x of $\mathcal{P}^{\beta+1}(\omega)$ can be found to witness failure of (1). But this contradicts our assumption that σ was winning for Player I.

All that's left is to show that II doesn't win if θ_α doesn't exist. So suppose τ is a winning strategy in L; have I play $\mathcal{M}_{\mathrm{I}} = L_\nu$, the least admissible level of L containing τ. As before, we must have that $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}}) \leq \nu$.

Since II wins, $\mathcal{M}_{\mathrm{II}}$ must be illfounded (if $\mathcal{M}_{\mathrm{II}}$ is wellfounded then (1) holds vacuously and II holds via (2b) if $\mathcal{M}_{\mathrm{I}} = \mathcal{M}_{\mathrm{II}}$ and (2a) otherwise). It follows that $\mathcal{M}_{\mathrm{II}}$ has countable codes for nonstandard ordinals; if these belong to M_I then I wins via condition $(*)$, a contradiction. So it must be that $\mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}} \not\subseteq \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}}$, hence (2) holds vacuously. Now $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}})$ has a largest cardinal, say ω_β for some $\beta \leq \alpha + 1$. If $\beta \leq \alpha$, then by overspill, there are nonstandard ordinals of $\mathcal{M}_{\mathrm{II}}$ coded by subsets of $\mathcal{P}^\beta(\omega)$. Since II wins the game (so in particular $(*)$ fails), these cannot be coded by any element of $\mathcal{P}^{\beta+1}(\omega)$ in $\mathcal{M}_{\mathrm{I}}$. We thus obtain codes witnessing disagreement at β, and by overspill, there is no $<_L^{\mathcal{M}_{\mathrm{II}}}$-least such; this witnesses (1) via (1a), a contradiction. If $\beta = \alpha + 1$, on the other hand, then I wins the game via (1b) (here making use of Lemma 6.10). This contradiction completes the proof. ⊣

As before, relativizing to real parameters x produces a boldface result in a slightly weaker theory.

THEOREM 6.11 (KPI_0). *For all reals x and ordinals $\alpha < \omega_1^x$, $\Sigma^0_{1+\alpha+3}(x)$-DET if and only if $\theta_\alpha(x)$, the least ordinal so that $L_{\theta_\alpha(x)}[x] \models \Pi_1$-$\mathsf{RAP}_\alpha$, exists.*

THEOREM 6.12 (KP). *$\boldsymbol{\Sigma}^0_{1+\alpha+3}$-DET if and only if $\theta_\alpha(x)$ exists for all $x \subseteq \omega$.*

It is interesting to note that game trees on $\mathcal{P}^{\alpha+1}(\omega), \mathcal{P}^\alpha(\omega)$ appear to be crucial on both sides of the argument, though they are used in very different ways. Though $(\mathsf{T})_\alpha$ and Π_1-RAP_α are equivalent in levels of L, it is not clear whether this equivalence is provable in a more general setting, say, that of $\mathsf{BST} + \mathsf{DC}$. We are further led to wonder whether the (ostensibly weaker) axioms $(\mathsf{T})_\alpha$ could replace Π_1-RAP_α as the essential ingredient in the proof of Lemma 4.4.

§7. Borel determinacy and inductive definitions. For a pointclass Γ, $o(\Gamma)$ is defined to be the supremum of lengths of inductive definitions obtained by iterating Γ operators; $o(\Gamma\text{-mon})$ is the supremum of lengths of *monotone* inductive definitions (see [9] for full definitions).

The simplest winning strategies in games below Σ^0_3 can often be obtained by iterating an operator that gathers "sure winning positions", and this is reflected in the tight connection between the lengths of monotone inductive definitions and the location in L where winning strategies are first constructed. For example, $o(\Game\Sigma^0_1\text{-mon}) = o(\Pi^1_1\text{-mon}) = \omega_1^{\mathrm{CK}}$, and by the results of Solovay, winning strategies in Σ^0_2 games are constructed by $o(\Game\Sigma^0_2) = o(\Sigma^1_1\text{-mon})$ in L (for Player I) or in the next admissible (for Player II). Welch [18] has conjectured that a similar result holds for $o(\Game\Pi^0_3\text{-mon})$ and Σ^0_3 determinacy.

It is natural to ask whether $o(\Game\Sigma^0_{1+\alpha+3}\text{-mon})$ is related to the ordinals θ_α in this way. We content ourselves with some coarse bounds that follow easily from arguments given above. For simplicity, we restrict to the case Σ^0_4; analogous bounds hold for the higher pointclasses.

Proposition 7.1. *Put $\kappa = {\omega_1}^{L_\theta}$. For $i \in \omega$, define α_i to be the least ordinal so that $L_{\alpha_i} \prec_{\Sigma_i} L_\kappa$. Then $\alpha_1 \leq o(\Game\Sigma^0_4) < o(\Game\Pi^0_4\text{-mon}) < \alpha_2$.*

Proof. If Player I wins a Σ^0_4 game, then there is a winning strategy for I in L_{α_1}. For a fixed parameter-free Σ_1-formula ψ, we define a modified version G_ψ of the game G of Section 5 by requiring Player II to play a minimal model of $V = L + \mathsf{KP} + \psi + (\forall\alpha)L_\alpha \not\models (\mathsf{T})$ (and putting no additional restrictions on Player I). Then Player II wins G_ψ if and only if $L_{\alpha_1} \models \psi$. So, the (set of codes for the) Σ_1-theory of L_θ is a $\Game\Sigma^0_4$ set of integers (indeed, it is a *complete* $\Game\Sigma^0_4$ set of integers; compare [18]), and furnishes a $\Game\Sigma^0_4$ prewellordering of ω of order type α_1; this establishes the first inequalitity.

The second inequality is a consequence Theorem A of [9].

Next notice that $o(\Game\Pi^0_4\text{-mon}) < \kappa$, since L_κ is a model of ZFC^-, and for $x \in \mathbb{R} \cap L_\kappa$, the statement that Player II wins some $\Sigma^0_4(x)$ game $G(A, \omega^{<\omega})$ is Π_1 over L_κ in the parameter x (it is equivalent to the statement "there is no β so that L_β is a model of KPI in which I wins $G(A, \omega^{<\omega})$"). It can easily be verified that being the fixed point of a monotone Π_1-inductive operator is $\mathbf{\Delta}_2$ in models of ZFC^-. So the existence of a fixed point is Σ_2, and reflects to L_{α_2}.

This establishes $o(\Game\Pi^0_4\text{-mon}) \leq \alpha_2$. By the existence of a universal $\Game\Pi^0_4$-mon-monotone inductive definition (see Section 3 of [9]), the inequality is strict. $\dashv$

Note that by the Third Periodicity Theorem (see [13]), if Player I wins a Σ^0_4 game, then there exists a winning strategy that is $\Game\Sigma^0_4$. Indeed, we see that such a strategy can be computed from the Σ_1-theory of L_θ, a $\Game\Sigma^0_4$ set. As one would expect, winning strategies for the Π^0_4 player are rather more complicated, and needn't belong to $\Game\Pi^0_4$; we have seen here that at best, they are $\mathbf{\Delta}_1$-definable over L_θ in parameters.

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