DETERMINACY EXERCISES
DAY 1

PROBLEM 1. We say a tree $T \subseteq X^{<\omega}$ is **pruned** if it contains no terminal nodes. Show that for all trees T with $[T] \neq \emptyset$, there is a unique pruned tree S with $[S] = [T]$.

PROBLEM 2. We say a tree T is **ranked** if there is a function $\rho : T \to \alpha$ for some ordinal α, so that $s \supsetneq t$ implies $\rho(s) < \rho(t)$, for $s, t \in T$. Show a tree T is ranked if and only if $[T] = \emptyset$.

PROBLEM 3. Show the Axiom of Choice is equivalent to the determinacy of all games of length 2.

PROBLEM 4. Show:

1. If Player I has a winning strategy in $G(A)$, then $|A| = |2^\omega|$.
2. Suppose $A \subseteq \omega^\omega$, and that there is no surjection $f : A \to \omega^\omega$. Show $G(A)$ is determined.

PROBLEM 5. Show:

1. If X, Y are sets with $|X| = |Y|$, then AD_X if and only if AD_Y.
2. AD_2 is equivalent to AD.

PROBLEM 6. Using the Axiom of Choice, show there are

1. A set $A \subseteq \omega^\omega$ for which $G(A)$ is determined but $G(\omega^\omega \setminus A)$ is not.
2. Sets $A, B \subseteq \omega^\omega$ for which $G(A)$ and $G(B)$ are both determined but $G(A \cup B)$ is not.

PROBLEM 7. For a given set x, let $\mathcal{P}_{\mathrm{wo}}(x)$ denote the collection of subsets of x that can be well-ordered. Under the Axiom of Choice this is the same as $\mathcal{P}(x)$, but without the Axiom of Choice the two may well be different. Nonetheless, show without appealing to the Axiom of Choice that there is no injection $f : \mathcal{P}_{\mathrm{wo}}(x) \to x$.